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On Affine Subspaces that Illuminate a Convex Set

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Abstract. Let $\mathcal{K}$ be a convex body in E^d and let $0 \leq l \leq d-1$. Then let $I_l(\mathcal{K})$ be the smallest number of affine subspaces of dimension l lying in $E^d \setminus \mathcal{K}$ that illuminate $\mathcal{K}$. We give three equivalent definitions of $I_l(\mathcal{K})$ extending an earlier result of Boltjanskii and Soltan. The main result of the paper is that any convex body in E^d, $d \geq 3$ can be illuminated by two $(d-2)$–dimensional affine subspaces. In particular, any 3–dimensional convex body can be illuminated by two lines in E^3. Also, we give a class of the convex bodies in E^d that can be illuminated by two $\lfloor \frac{d}{2} \rfloor$–dimensional affine subspaces.

1. Introduction.

Let $\mathcal{K}$ be a closed convex set with non-empty interior in the d–dimensional Euclidean space E^d, $d \geq 1$. In this paper, we define and investigate the quantities $I_l(\mathcal{K}), I'_l(\mathcal{K})$ and $H_l(\mathcal{K})$ assigned to $\mathcal{K}$ which are natural generalizations of the quantities first introduced in [6], [10] and [11]. (In fact, $I_l(\mathcal{K})$ was already defined and partially discussed in [1], [2], [3], [4] and [5].)

(1) We say that a point $P \in E^d \setminus \mathcal{K}$ illuminates the boundary point Q of $\mathcal{K}$ if the (open) ray emanating from Q having direction vector $\overrightarrow{PQ}$ meets the interior of $\mathcal{K}$. Let $L \subset E^d \setminus \mathcal{K}$ be an affine subspace in E^d of dimension l, $0 \leq l \leq d-1$. Then L illuminates the boundary point Q of $\mathcal{K}$ if there exists a point P of L that illuminates Q. Moreover, we say that *the affine subspaces* $L_1, L_2, \ldots, L_n \subset E^d \setminus \mathcal{K}$ *illuminate* $\mathcal{K}$ if every boundary point of $\mathcal{K}$ is illuminated by at least one of the affine subspaces $L_1, L_2, \ldots, L_n$. Finally, let $I_l(\mathcal{K})$ be the smallest number of affine subspaces of dimension l lying in $E^d \setminus \mathcal{K}$ that illuminate $\mathcal{K}$, where $0 \leq l \leq d-1$. Obviously, $I_0(\mathcal{K}) \geq I_1(\mathcal{K}) \geq \ldots \geq I_{d-1}(\mathcal{K})$.

(2) We say that the unit vector v of the $(d-1)$–dimensional unit sphere $\mathcal{S}^{d-1}$ centered at the origin O in E^d illuminates the boundary point Q of $\mathcal{K}$ if the (open) ray emanating from Q having direction vector v intersects the interior of $\mathcal{K}$. Let $\mathcal{H}S^l$ be an l–dimensional open great–hemisphere of $\mathcal{S}^{d-1}$, where $0 \leq l \leq d-1$. Then $\mathcal{H}S^l$ illuminates the boundary point Q of $\mathcal{K}$ if there exists a unit vector v of $\mathcal{H}S^l$ that illuminates Q. Furthermore, we

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say that *the l–dimensional open great–hemispheres* $\mathcal{HS}_1^l, \mathcal{HS}_2^l, \ldots, \mathcal{HS}_n^l$ of $\mathcal{S}^{d-1}$ *illuminate* $\mathcal{K}$ if every boundary point of $\mathcal{K}$ is illuminated by at least one of the great–hemispheres $\mathcal{HS}_1^l, \mathcal{HS}_2^l, \ldots, \mathcal{HS}_n^l$. Finally, let $I_l'(\mathcal{K})$ be the smallest number of l–dimensional open great–hemispheres of $\mathcal{S}^{d-1}$ that illuminate $\mathcal{K}$, where $0 \leq l \leq d-1$. Obviously, $I_0'(\mathcal{K}) \geq I_1'(\mathcal{K}) \geq \ldots \geq I_{d-1}'(\mathcal{K})$.

(3) Let L be an affine subspace in E^d of dimension l, $0 \leq l \leq d-1$. The *l–th order circumscribed cylinder of $\mathcal{K}$ generated by L* is by definition the union of translates of L that intersect $\mathcal{K}$. Then let $H_l(\mathcal{K})$ be the smallest number of translates of the interiors of l–th order circumscribed cylinders of $\mathcal{K}$ the union of which covers $\mathcal{K}$, where $0 \leq l \leq d-1$. Obviously, $H_0(\mathcal{K}) \geq H_1(\mathcal{K}) \geq \ldots \geq H_{d-1}(\mathcal{K})$.

In this paper we first prove the following statement which can be viewed as a generalization of the theorem in [7] which deals with the case $l = 0$.

Theorem 1. (i) *Let $\mathcal{K}$ be a closed convex set with non-empty interior in E^d. Then $I_l(\mathcal{K}) \geq I_l'(\mathcal{K})$ and $H_l(\mathcal{K}) \geq I_l'(\mathcal{K})$ for any $0 \leq l \leq d-1$.*
(ii) *Let $\mathcal{K}$ be a convex body in E^d, i.e., a compact convex set with non–empty interior in E^d. Then $I_l(\mathcal{K}) = H_l(\mathcal{K}) = I_l'(\mathcal{K})$ for any $0 \leq l \leq d-1$.*

Theorem 1 gives three equivalent ways to study the quantity $I_l(\mathcal{K})$ of a convex body $\mathcal{K}$ in E^d. It is not hard to prove (see [2] or [3]) that if $\mathcal{K}$ is an arbitrary convex body in E^d, then $I_l(\mathcal{K}) \geq \lceil \frac{d+1}{l+1} \rceil$ for any $0 \leq l \leq d-1$. Moreover, equality is attained (not only) for a smooth $\mathcal{K}$. However, the following problem which is just the opposite question seems to be a natural generalization of the Hadwiger–Levi's covering problem ([5], [9], [10]) and seems to be much more difficult.

Conjecture 1. *If $\mathcal{C}$ is a d–dimensional affine cube in E^d and $\mathcal{K}$ is an arbitrary convex body in E^d, then $I_l(\mathcal{C}) \geq I_l(\mathcal{K})$ for any $0 \leq l \leq d-1$.*

Remark 1. It is easy to see that $I_{d-1}(\mathcal{K}) = 2$ for any convex body $\mathcal{K}$ in E^d. On the other hand, the problem of estimating $I_0(\mathcal{K})$ from above is an equivalent formulation of Hadwiger–Levi's covering problem (see [1], [5], [6], [9] and [10]). It is conjectured that $I_0(\mathcal{K}) \leq 2^d$ for any convex body in E^d. There are several partial results concerning this conjecture. We mention here only two of them. It is proved (see [1]) that $I_0(\mathcal{K}) \leq 8$ for any convex polyhedron $\mathcal{K}$ in E^3 that has an affine symmetry. For large d the current best upper estimate on $I_0(\mathcal{K})$ is due to C. A. Rogers (see [8] where this result is mentioned however, only for centrally symmetric convex bodies in E^d). Let $\mathcal{K}$ be an arbitrary convex body in E^d. If $d \geq 3$, then

$$I_0(\mathcal{K}) \leq \frac{Vol(\mathcal{K}-\mathcal{K})}{Vol(\mathcal{K})}(d \ln d + d \ln(\ln d) + 5d),$$

where it is known that the d–dimensional volumes $Vol(\mathcal{K}-\mathcal{K}), Vol(\mathcal{K})$ of the convex bodies $\mathcal{K}-\mathcal{K}, \mathcal{K}$ in E^d satisfy the inequality $2^d \leq Vol(\mathcal{K}-\mathcal{K})/Vol(\mathcal{K}) \leq \binom{2d}{d}$. After all, the Hadwiger–Levi's covering problem remains open for $d \geq 3$.

Remark 2. Conjecture 1 in a way, also suggests to investigate the d–dimensional affine cubes separately from the other convex bodies. Strangely enough we do not have a complete understanding of the cube's behavior in this regard. Namely, $I_0(\mathcal{C}) = 2^d$ for any d–dimensional affine cube $\mathcal{C}$ in E^d moreover, if $d+1 = 2^p$, $p \geq 1$, then $I_1(\mathcal{C}) = \frac{2^d}{d+1}$ (see [4]). Though, in this paper we gain some more information about the affine cubes (see Theorem 3) several questions still remain open. Finally, we mention here that one can prove the inequality

$$I_l(\mathcal{C}) \geq \left\lceil 2^d / \sum_{i=0}^{l} \binom{d}{i} \right\rceil.$$

In this note we prove Conjecture 1 for $l = d-2$. Namely, we have

Theorem 2. *Let $\mathcal{K}$ be a convex body in E^d, $d \geq 3$. Then $\mathcal{K}$ can be illuminated by two $(d-2)$–dimensional affine subspaces, i.e., $I_{d-2}(\mathcal{K}) = 2$.*

In fact, one can get a stronger result for the following special but, still "large" classes of convex bodies. The *inner cone* of the boundary point P of the convex body $\mathcal{K}$ in E^d is the open convex cone of E^d with apex P that is the union of open rays emanating from P and intersecting the interior of $\mathcal{K}$. Finally, an *orthant* in E^d is an open connected component of the complement of d pairwise orthogonal hyperplanes in E^d. Then we have

Theorem 3. *Let $\mathcal{P}$ be a convex polytope in E^d such that the inner cone of any boundary point of $\mathcal{P}$ contains some orthant of E^d, $d \geq 3$. Then $\mathcal{P}$ can be illuminated by two $\lfloor \frac{d}{2} \rfloor$–dimensional affine subspaces, i.e.,*

$$I_{\lfloor \frac{d}{2} \rfloor}(\mathcal{P}) = I_{\lfloor \frac{d}{2} \rfloor + 1}(\mathcal{P}) = \ldots = I_{d-1}(\mathcal{P}) = 2.$$

A *zonotope* of E^d is a convex polytope that represents the vector sum of finitely many closed segments in E^d. A compact convex set in E^d is called a *zonoid* if it is the limit (in the sense of the Hausdorff metric) of some covergent sequence of zonotopes of E^d. [4] contains the following observation. If $\mathcal{C}$ is a d–dimensional affine cube in E^d and $\mathcal{Z}$ is an arbitrary d–dimensional zonoid in E^d, then $I_l(\mathcal{C}) \geq I_l(\mathcal{Z})$ for any $0 \leq l \leq d-1$. Thus, Theorem 3 implies

Corollary 1. *Let $\mathcal{Z}$ be a d–dimensional zonoid in E^d, $d \geq 3$. Then $\mathcal{Z}$ can be illuminated by two $\lfloor \frac{d}{2} \rfloor$–dimensional affine subspaces, i.e.,*

$$I_{\lfloor \frac{d}{2} \rfloor}(\mathcal{Z}) = I_{\lfloor \frac{d}{2} \rfloor + 1}(\mathcal{Z}) = \ldots = I_{d-1}(\mathcal{Z}) = 2.$$

2. Proof of Theorem 1.

As the following proof is a rather natural extension of the proof of the case $l = 0$ given in [7] we give an outline only.

(i) In order to prove the inequality $I_l(\mathcal{K}) \geq I'_l(\mathcal{K})$ assume that the origin O of E^d lies in the interior of $\mathcal{K}$ and let L be an affine subspace in $E^d \setminus \mathcal{K}$ of dimension l, $0 \leq l \leq d-1$. Then let $\mathcal{HS}^l = \{v \in \mathcal{S}^{d-1} | \lambda v = \overrightarrow{OP}$ for some $\lambda < 0$ and $P \in L\}$. It is easy to see that any boundary point of $\mathcal{K}$ that is illuminated by L is also illuminated by $\mathcal{HS}^l$.

To prove the inequality $H_l(\mathcal{K}) \geq I'_l(\mathcal{K})$ assume that $\mathcal{C}_L$ is an l–th order circumscribed cylinder of $\mathcal{K}$ generated by the affine subspace L in E^d of dimension l, $0 \leq l \leq d-1$ moreover, let $\mathcal{C}_L + t$ be a translate of $\mathcal{C}_L$ by the vector t in E^d. Without loss of generality we may assume that L is a linear subspace in E^d, i.e., $O \in L$. Let $v \in \mathcal{S}^{d-1}$ be the unit vector with $\|t\| v = -t$, where $\|\cdot\|$ stands for the norm of the corresponding vector. Finally, let $\mathcal{HS}^l$ be the l–dimensional open great–hemisphere of $\mathcal{S}^{d-1}$ with center v lying in the linear subspace spanned by v and L. Then it is easy to show that any boundary point of $\mathcal{K}$ that is covered by the interior of $\mathcal{C}_L + t$ is also illuminated by $\mathcal{HS}^l$.

(ii) Having (i) it is sufficient to prove the inequalities $I'_l(\mathcal{K}) \geq I_l(\mathcal{K})$ and $I'_l(\mathcal{K}) \geq H_l(\mathcal{K})$. Just as in the case $l = 0$ (see [7]) as the boundary of $\mathcal{K}$ is compact rather standard compactness arguments yield the above inequalities.

3. Proof of Theorem 2.

Theorem 1 implies that $I'_{d-2}(\mathcal{K}) = I_{d-2}(\mathcal{K})$ for any convex body $\mathcal{K}$ in E^d. Thus, it is sufficient to prove that $I'_{d-2}(\mathcal{K}) = 2$. The following proof consists of 3 steps.

STEP 1. It is sufficient to prove the claim in E^3. Namely, assume that any convex body in E^{d-1}, $d \geq 4$ can be illuminated by two $(d-3)$–dimensional open great–hemispheres of $\mathcal{S}^{d-2}$ in E^{d-1}. Then let $\mathcal{K}$ be a convex body in E^d and assume for convenience that $\mathcal{S}^{d-2}$ is a great–sphere of $\mathcal{S}^{d-1}$. Let H be the hyperplane of E^d spanned by $\mathcal{S}^{d-2}$. Finally, let $\mathrm{Pr}(\mathcal{K})$ be the orthogonal projection of $\mathcal{K}$ onto H. By assumption, there are two $(d-3)$–dimensional open great–hemispheres $\mathcal{HS}_1^{d-3}$ and $\mathcal{HS}_2^{d-3}$ of $\mathcal{S}^{d-2}$ that illuminate $\mathrm{Pr}(\mathcal{K})$ in H. Let $\mathcal{HS}_1^{d-2}$ ($\mathcal{HS}_2^{d-2}$, resp.) be the $(d-2)$–dimensional open great–hemisphere of $\mathcal{S}^{d-1}$ the orthogonal projection of which onto H is the convex hull of $\mathcal{HS}_1^{d-3}$ ($\mathcal{HS}_2^{d-3}$, resp.). It is easy to check that $\mathcal{HS}_1^{d-2}$ and $\mathcal{HS}_2^{d-2}$ illuminate $\mathcal{K}$ in E^d.

STEP 2. Let $\mathcal{K}$ be a convex body in E^3 and assume that P_1 and P_2 are strictly antipodal points of the orthogonal projection $\mathrm{Pr}(\mathcal{K})$ of $\mathcal{K}$ onto the plane H of E^3 (i.e. there are distinct parallel supporting lines l_1 and l_2 of $\mathrm{Pr}(\mathcal{K})$ in H such that $l_1 \cap \mathrm{Pr}(\mathcal{K}) = \{P_1\}$ and $l_2 \cap \mathrm{Pr}(\mathcal{K}) = \{P_2\}$). Let l_{12} denote the line passing through P_1 and P_2 moreover, let H_{12} be the plane passing through l_{12} and being orthogonal to H in E^3. If H_{12}^+ denotes one of the closed half spaces bounded by H_{12} in E^3, then there exists an open great–hemicircle $\mathcal{HS}^1$ of $\mathcal{S}^2$ in E^3 such that the set of the boundary points of $\mathcal{K}$ which are not illuminated by $\mathcal{HS}^1$ is $H_{12}^+ \cap \mathrm{bd}\, \mathcal{K} \cap \mathrm{bd}\, \mathcal{C}_H$, where $\mathcal{C}_H$ denotes the 1st–order circumscribed cylinder of $\mathcal{K}$ generated by a line orthogonal to H and $\mathrm{bd}(\cdot)$ denotes the boundary of the corresponding set (Fig. 1). In order to see this let the center of $\mathcal{HS}^1$ be the unit vector v of $\mathcal{S}^2$ that

Figure 1: Finding $\mathcal{HS}^1$.

is parallel to l_1 and points into the interior of H_{12}^+ moreover, let the diameter of $\mathcal{HS}^1$ be orthogonal to the plane H.

STEP 3. Let the line segment P_1P_2 be the diameter of the convex body $\mathcal{K}$ in E^3. Obviously, P_1 and P_2 are strictly antipodal points of $\mathcal{K}$, i.e., there are parallel supporting planes H_1 and H_2 of $\mathcal{K}$ in E^3 such that $H_1 \cap \mathcal{K} = \{P_1\}$ and $H_2 \cap \mathcal{K} = \{P_2\}$. Let H_3 be a plane in E^3 orthogonal to P_1P_2 and let $\mathrm{Pr}_3(\mathcal{K})$ be the orthogonal projection of $\mathcal{K}$ onto H_3. Moreover, let the point $X_1 = X_2 \in H_3$ be the orthogonal projection of the points P_1 and P_2 onto H_3. Then it is easy to show that there are strictly antipodal points X_3 and X_4 of $\mathrm{Pr}_3(\mathcal{K})$ in H_3 such that the points X_1, X_3 and X_4 are pairwise disjoint (Fig. 2). Then let H be the plane in E^3 that contains the line segment P_1P_2 and is parallel to the line segment X_3X_4. (It might happen that X_3X_4 lies in H.) Let $l_1 = H \cap H_1$ and $l_2 = H \cap H_2$. Finally, let H_4 be a plane in E^3 that is orthogonal to l_1 (as well as to l_2) and let $\mathrm{Pr}_4(\mathcal{K})$ be the orthogonal projection of $\mathcal{K}$ onto H_4. Obviously, the orthogonal projections Y_1 and Y_2 of the points P_1 and P_2 onto H_4 are strictly antipodal points of $\mathrm{Pr}_4(\mathcal{K})$ in H_4. Let H^+ and H^- be the two closed halfspaces bounded by H in E^3. Then STEP 2 implies that there are two open great–hemicircles $\mathcal{HS}_1^1$ and $\mathcal{HS}_2^1$ of $\mathcal{S}^2$ in E^3 such that the set of the boundary points of $\mathcal{K}$ which are not illuminated by $\mathcal{HS}_1^1$ ($\mathcal{HS}_2^1$, resp.) lies in H^+ (H^-, resp.). Moreover, as $l_1 \cap \mathcal{K} = \{P_1\}$ and $l_2 \cap \mathcal{K} = \{P_2\}$ it is easy to see that the set of the boundary points of $\mathcal{K}$ which are not illuminated by $\mathcal{HS}_1^1$ is disjoint from the set of the boundary points of $\mathcal{K}$ which are not illuminated by $\mathcal{HS}_2^1$. Thus, $\mathcal{HS}_1^1$ and $\mathcal{HS}_2^1$ illuminate $\mathcal{K}$. This completes the proof of Theorem 2.

Figure 2: The construction of $\mathcal{H}S_1^1$ and $\mathcal{H}S_2^1$ that illuminate $\mathcal{K}$.

4. Proof of Theorem 3.

We prove the following stronger statement.

Theorem 4. *Let $\mathcal{P}$ be a convex polytope in E^d such that the inner cone of any boundary point of $\mathcal{P}$ contains some orthant of E^d, $d \geq 3$. Then there are an l–dimensional and a $(d-l-1)$–dimensional open great–hemispheres $\mathcal{HS}^l$ and $\mathcal{HS}^{d-l-1}$ of $\mathcal{S}^{d-1}$ in E^d such that $\mathcal{P}$ is illuminated by $\mathcal{HS}^l$ and $\mathcal{HS}^{d-l-1}$ for any $0 \leq l \leq d-1$.*

Proof. Let $V_1, V_2, \dots, V_n$ be the vertices of $\mathcal{P}$ and let $\mathcal{Q}_i$ be the orthant with apex O the translate of which by the vector $\overrightarrow{OV_i}$ lies in the inner cone of V_i, $1 \leq i \leq n$. Obviously, $\mathcal{Q}_i \cap \mathcal{S}^{d-1}$ is a $(d-1)$–dimensional open spherical ortho–simplex with d vertices say, $V_{i1}, V_{i2}, \dots, V_{id}$ for any $1 \leq i \leq n$. Let $[V_{i1}V_{i2}\dots V_{id}]$ be the set $\mathcal{Q}_i \cap \mathcal{S}^{d-1}$ for all $1 \leq i \leq n$. If l is given, then we look for $\mathcal{HS}^l$ and $\mathcal{HS}^{d-l-1}$ with the property that $[V_{i1}V_{i2}\dots V_{id}] \cap (\mathcal{HS}^l \cup \mathcal{HS}^{d-l-1}) \neq \phi$ for all $1 \leq i \leq n$.

We start with an arbitrary l–dimensional open great–hemisphere $\mathcal{HS}^l$ of $\mathcal{S}^{d-1}$. Then let $(\mathcal{HS}^l)^* = \{X \in \mathcal{S}^{d-1} | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle < 0 \text{ for all } Y \in \mathcal{HS}^l\}$, where $\langle , \rangle$ stands for the usual inner product of E^d. It is easy to see that $(\mathcal{HS}^l)^*$ is a $(d-l-1)$–dimensional open great–hemisphere of $\mathcal{S}^{d-1}$ moreover, if E^l and E^{d-l-1} are the linear subspaces of E^d containing the great–spheres of $\mathcal{S}^{d-1}$ on which $\mathcal{HS}^l$ and $(\mathcal{HS}^l)^*$ sit, then E^l and E^{d-l-1} are totally orthogonal (Fig. 3). We say that the *orthant $\mathcal{Q}_i$ lies in general position with respect to E^l (E^{d-l-1}, resp.)* if choosing any $d-l$ ($l+1$, resp.) unit vectors out of $\{\overrightarrow{OV_{i1}}, \overrightarrow{OV_{i2}}, \dots, \overrightarrow{OV_{id}}\}$ they together with E^l (E^{d-l-1}, resp.) span E^d. It is not hard to show that there exists an $\mathcal{HS}^l$ of $\mathcal{S}^{d-1}$ such that the orthants $\mathcal{Q}^i$, $1 \leq i \leq n$ lie in general position with respect to E^l as well as E^{d-l-1}. Now assume that $[V_{i1}V_{i2}\dots V_{id}] \cap (\mathcal{HS}^l \cup (\mathcal{HS}^l)^*) = \phi$ for some

Figure 3: Finding $\mathcal{HS}^{d-l-1}$.

$1 \leq i \leq n$. We seek for a contradiction. The sets $[V_{i1}V_{i2}\dots V_{id}]$ and $\mathcal{HS}^l$ ($[V_{i1}V_{i2}\dots V_{id}]$ and $(\mathcal{HS}^l)^*$, resp.) can be separated by a hyperplane of E^d passing through O i.e. there exists a unit vector $n_1 \in \mathcal{S}^{d-1}$ ($n_2 \in \mathcal{S}^{d-1}$, resp.) such that

$$\langle n_1, \overrightarrow{OX}\rangle \geq 0 \text{ for all } X \in [V_{i1}V_{i2}\dots V_{id}], \tag{1}$$

$$\langle n_1, \overrightarrow{OX}\rangle \leq 0 \text{ for all } X \in \mathcal{HS}^l \text{ and} \tag{2}$$

$$\langle n_2, \overrightarrow{OX}\rangle \geq 0 \text{ for all } X \in [V_{i1}V_{i2}\dots V_{id}], \tag{3}$$

$$\langle n_2, \overrightarrow{OX}\rangle \leq 0 \text{ for all } X \in (\mathcal{HS}^l)^*. \tag{4}$$

(2) and (4) immediately imply that $n_1 \in \mathrm{cl}(\mathcal{HS}^l)^*$ and $n_2 \in \mathrm{cl}\mathcal{HS}^l$, where $\mathrm{cl}(\cdot)$ stands for the closure of the corresponding set. (Note that $((\mathcal{HS}^l)^*)^* = \mathcal{HS}^l$.) So, $\langle n_1, n_2\rangle \leq 0$. However, (1) and (3) in a rather easy way imply that $n_1 \in \mathrm{cl}[V_{i1}V_{i2}\dots V_{id}]$ and $n_2 \in \mathrm{cl}[V_{i1}V_{i2}\dots V_{id}]$. Hence, $\langle n_1, n_2\rangle = 0$. Having this and the fact that $n_1 \in \mathrm{cl}(\mathcal{HS}^l)^*$ and $n_2 \in \mathrm{cl}\mathcal{HS}^l$ it is easy to show that either $n_1 \in E^{d-l-1}$ or $n_2 \in E^l$. Without loss of generality we may assume that $n_1 \in E^{d-l-1}$. As $[V_{i1}V_{i2}\dots V_{id}] \cap (\mathcal{HS}^l)^* = \phi$ we can in fact find a hyperplane H of E^d containing E^{d-l-1} that separates $[V_{i1}V_{i2}\dots V_{id}]$ from $(\mathcal{HS}^l)^*$. As $n_1 \in \mathrm{cl}[V_{i1}V_{i2}\dots V_{id}] \cap E^{d-l-1} \subset \mathrm{cl}[V_{i1}V_{i2}\dots V_{id}] \cap H$ there exists a face say, $[V_{i1}V_{i2}\dots V_{ik}] \subset H$ of $[V_{i1}V_{i2}\dots V_{id}]$ in $\mathcal{S}^{d-1}$ the relative interior of which contains n_1 for some $1 \leq k \leq d-1$. Thus, $n_1 \in (\mathrm{lin}\{\overrightarrow{OV_{i1}}, \overrightarrow{OV_{i2}}, \dots, \overrightarrow{OV_{ik}}\}) \cap E^{d-l-1}$ and $(\mathrm{lin}\{\overrightarrow{OV_{i1}}, \overrightarrow{OV_{i2}}, \dots, \overrightarrow{OV_{ik}}\}) \cup E^{d-l-1} \subset H$ contradict to the fact that $\mathcal{Q}_i$ lies in general position with respect to E^{d-l-1}. This completes the proof of Theorem 4.

5. Proof of Remark 2.

Let $\mathcal{Q}_j$, $1 \leq j \leq 2^d$ be the open orthants in E^d generated by d pairwise orthogonal hyperplanes in E^d that all pass through the origin O. Moreover, let $(O \notin)L$ be an arbitrary affine subspace in E^d of dimension l, where $0 \leq l \leq d-1$. In order to prove Remark 2 it is sufficient to show that the number of orthants in $\{\mathcal{Q}_j | 1 \leq j \leq 2^d\}$ that are intersected by L is at most $\sum_{i=0}^{l}\binom{d}{i}$. Finally, notice that the cell-complex $\{\mathcal{Q}_1 \cap L, \mathcal{Q}_2 \cap L, \dots, \mathcal{Q}_{2^d} \cap L\}$ is generated by d hyperplanes in $L = E^l$. Thus, it is a rather known exercise to show that the number of l-dimensional cells in the above cell-complex is at most $\sum_{i=0}^{l}\binom{d}{i}$.

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