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# Coating by Cubes

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## 1. Introduction

Let $P_0, P_1, \ldots, P_n$ be convex $d$–polytopes in $d$–dimensional Euclidean space with pairwise disjoint interiors. We say that $P_0$ is *coated* by $P_1, \ldots, P_n$ if $P_0 \subset \mathrm{int}\big(\bigcup_{i=0}^{n} P_i\big)$, where $\mathrm{int}(\cdot)$ stands for the interior of the corresponding set. Coating occurs very often in a very natural way. For example, in each tiling every tile is coated by its neighbors. Thus, if we take an arbitrary triangulation of $\mathbf{E}^d$, then the number of neighbors of any tile is at least as large as the minimum number of $d$–simplices that can coat a $d$–simplex in $\mathbf{E}^d$. In this connection the following problem is a rather very basic question.

**Problem 1.** Find the minimum number of $d$–simplices that can coat a $d$–simplex in $\mathbf{E}^d$.

The answer to the above question is obviously three in $\mathbf{E}^2$. In general, we know only the following.

**Proposition.** *Every $d$-simplex can be coated by $(2d-1)$ $d$–simplices in $\mathbf{E}^d$, where $d \geq 2$.*

Since the number of facets of a $d$–cube in $\mathbf{E}^d$ is $2d$, the number of $d$–cubes that can coat a fixed $d$–cube is at least $2d$. The following theorem formulates a sharper statement under some conditions.

**Theorem.** *Let $P_0$ be a $d$–cube of edgelength $\lambda$ with edges parallel to the coordinate–axes of $\mathbf{E}^d$. Moreover, let $P_1, \ldots, P_n$ be a collection of unit $d$–cubes with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that $P_0$ is coated by $P_1, \ldots, P_n$.*
(1) *If $0 < \lambda < 1$, then $n \geq 2^d$, where equality can be achieved for any $0 < \lambda < 1$ and $d \geq 1$.*
(2) *If $\lambda = k$ is a positive integer, then $n \geq 2(k+1)^d - 2k^d$, where equality can be achieved for any $k \geq 1$ and $d \geq 1$.*

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As a result we get the following:

**Corollary.** *The minimum number of the translates of a $d$–cube that can coat a given $d$–cube in $\mathbf{E}^d$ is at least $2^d$, where $d \geq 1$. If all $d$-cubes are translates of each other, then $2^d$ can be replaced by $2^{d+1}-2$.*

**Problem 2.** Prove or disprove that the minimum number of $d$–cubes that can coat a $d$–cube in $\mathbf{E}^d$ is $2^{d-1}+2$, where $d \geq 2$.

## 2. Proof of the Proposition

We prove the statement by induction on the dimension $d$. As the claim is obviously true for $d=2$ we may assume that it is true for any $d' < d$ with $d \geq 3$. Thus, let $S$ be a $d$–simplex in $\mathbf{E}^d$ with vertices $v_1, v_2, \ldots, v_{d+1}$. Moreover, let $H$ be the hyperplane in $\mathbf{E}^d$ spanned by the vertices $v_1, v_2, \ldots, v_d$ and let $S_0$ be the $(d-1)$–simplex with vertices $v_1, v_2, \ldots, v_d$. By induction there are $(d-1)$–simplices $S_1, S_2, \ldots, S_{2d-3}$ that coat $S_0$ in $H$. Let $v$ be a point in $\mathbf{E}^d$ such that $v_{d+1}$ is the relative interior point of the segment $v_1 v$ and let $v'$ be a point in $\mathbf{E}^d$ that is strictly separated from $v$ by $H$. Then it is easy to see that the $d$–simplices $\operatorname{conv}(S_1 \cup \{v\})$, $\operatorname{conv}(S_2 \cup \{v\}), \ldots, \operatorname{conv}(S_{2d-3} \cup \{v\})$, $\operatorname{conv}\{v_2, v_3, \ldots, v_{d+1}, v\}$ and $\operatorname{conv}(S_0' \cup \{v'\})$ coat the $d$–simplex $S$, where $S_0'$ is a simplex in $H$ containing $S_0$ in its relative interior. This completes the proof of the Proposition.

## 3. Proof of the Theorem

*Proof of (1).* In the following proof we assume only that the edgelengths of the $d$–cubes $P_1, \ldots, P_n$ are larger than $\lambda$.

At first, remove the $d$–cubes of the collection $P_1, \ldots, P_n$ that are disjoint from $P_0$. Let $P_1, \ldots, P_n$ denote the system left. Obviously, $P_1, \ldots, P_n$ still coat $P_0$. We are going to show that $n = 2^d$. Recall that an orthant in $\mathbf{E}^d$ is the closure of a connected component of the complement of $d$ pairwise orthogonal hyperplanes of $\mathbf{E}^d$.

**Lemma 1.** *Each $d$-cube $P_i$, $1 \leq i \leq n$ can be replaced by an orthant $O_i$ with $P_i \subset O_i$ such that the edges of the orthants $O_1, \ldots, O_n$ are parallel to the coordinate–axes of $\mathbf{E}^d$ and the interiors of the orthants $O_1, \ldots, O_n$ are pairwise disjoint.*

*Proof.* Take a $d$ cube $P_i$, $1 \leq i \leq n$. Let $v_i$ be the vertex of $P_i$ that lies closest to the $d$–cube $P_0$. Then let $O_i$ be the orthant with apex $v_i$ and with edges parallel to the coordinate–axes of $\mathbf{E}^d$ and with $P_i \subset O_i$. We are going to show that each $O_i$ is disjoint from the interiors of the $d$–cubes $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$ and then we prove that the interiors of the orthants $O_1, \ldots, O_n$ are pairwise disjoint indeed. In order to do so we need the following:

**Lemma 2.** *Let $H$ be the hyperplane of any facet of $P_i$ that does not contain $v_i$. Then $H \cap P_0 = \emptyset$.*

*Proof.* (Indirect) Assume that $H \cap P_0 \neq \emptyset$. Then take the orthogonal projection of $v_i$ onto $H$. This is a vertex say $v_i'$ of $P_i$. Moreover, let $w_i$ be the point of $P_0$ that is closest to $v_i$

and let $w_i'$ be the orthogonal projection of $w_i$ onto $H$. Obviously, as $H \cap P_0 \neq \emptyset$ we have $w_i' \in P_0$. Finally, as the edgelength of $P_0$ is smaller than the edgelength of $P_i$ we get that $\mathrm{dist}(v_i, w_i) > \mathrm{dist}(v_i', w_i')$. Thus, $\mathrm{dist}(v_i, P_0) > \mathrm{dist}(v_i', P_0)$, a contradiction. $\square$

Now imagine a $d$–cube $P_j$, $j \neq i$ with $\mathrm{int}\, P_j \cap \mathrm{int}\, O_i \neq \emptyset$. Recall that $\mathrm{int}\, P_j \cap \mathrm{int}\, P_i = \emptyset$. Then obviously, there exists a facet of $P_i$ the hyperplane $H$ of which separates $\mathrm{int}\, P_j$ from $\mathrm{int}\, P_i$. As $\mathrm{int}\, P_j \cap \mathrm{int}\, O_i \neq \emptyset$ therefore $v_i \notin H$. Hence, Lemma 2 implies that $H \cap P_0 = \emptyset$. Now, recall that $P_i \cap P_0 \neq \emptyset$ and $P_j \cap P_0 \neq \emptyset$. Consequently, $H$ (that separates $P_i$ from $P_j$) must intersect (the convex set) $P_0$, a contradiction. Hence, we proved that $\mathrm{int}\, O_i \cap \mathrm{int}\, P_j = \emptyset$ for any $i \neq j \in \{1, \ldots, n\}$. In order to finish the proof of Lemma 1 we proceed as follows. Take $O_1$ and enlarge $P_1$ from $v_1$ by a very large factor obtaining the cube $P_1'$ the vertex $v_1$ of which is still the closest vertex to $P_0$. As a result of the previous arguments $P_1', P_2, \ldots, P_n$ coat $P_0$. Then enlarge $P_2, P_3, \ldots, P_n$ after each other in order to get a coating system of $P_0$ using rather large $d$–cubes. Keep doing this to see that the orthants $O_1, \ldots, O_n$ have pairwise disjoint interiors. This completes the proof of Lemma 1. $\square$

Apply Lemma 1 to get a system $\{O_1, \ldots, O_n\}$ of orthants with $P_i \subset O_i$ and with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that the orthants $O_1, \ldots, O_n$ have pairwise disjoint interiors, where $1 \leq i \leq n$. Obviously, no two of the orthants $O_1, \ldots, O_n$ are translates of each other and they coat $P_0$. Thus, $n \leq 2^d$. Take the $2^d - 1$ orthants with edges parallel to the coordinate–axes of $\mathbf{E}^d$ that share the same vertex of $P_0$ as an apex and are disjoint from the interior of $P_0$. Then it is easy to see that any $O_i$ intersects the interior of $2^l$ orthants out of $2^d - 1$. Hence, there must be an $O_i$ that intersects the interior of one orthant out of $2^d - 1$ implying that its apex $v_i$ is a vertex of $P_0$. Thus, $n \geq 2^d$ and so $n = 2^d$.

We are left with the proof of showing the existence of $2^d$ orthants $O_1, \ldots, O_{2^d}$ that coat $P_0$. As a result of the previous arguments we look for $2^d$ orthants with the property that the apex of each orthant is a vertex of $P_0$ and each vertex of $P_0$ is an apex of exactly one orthant. We prove the existence of such orthants by induction on the dimension $d$. They obviously exist in case $d = 2$. So assume that if $P_0'$ is a $(d-1)$–cube of edgelength $\lambda$ with edges parallel to the coordinate–axes of $\mathbf{E}^{d-1}$, then there are $2^{d-1}$ orthants in $\mathbf{E}^{d-1}$ say, $O_1', \ldots, O_{2^{d-1}}'$ that coat $P_0'$ in $\mathbf{E}^{d-1}$. Also, assume that $\mathbf{E}^{d-1}$ is a hyperplane of $\mathbf{E}^d$. Then for each orthant $O_i'$, $1 \leq i \leq 2^{d-1}$ in $\mathbf{E}^{d-1}$ we assign two orthants of $\mathbf{E}^d$ say, $+O_i$ and $-O_i$ such that the distinct orthants $+O_i$ and $-O_i$ share the $(d-1)$–dimensional orthant $O_i'$ as a facet in common. Let $F_1'$ and $F_2'$ be two opposite (i.e., disjoint) facets of $P_0'$. Without loss of generality we may assume that the apexes of the orthants $O_1', \ldots, O_{2^{d-2}}'$ belong to $F_1'$ and the apexes of the orthants $O_{2^{d-2}+1}', \ldots, O_{2^{d-1}}'$ belong to $F_2'$. Finally, let $e_1$ be the vector of length $\lambda$ with $e_1 + F_1' = F_2'$ and let $e_d$ be a vector of length $\lambda$ orthogonal to $\mathbf{E}^{d-1}$. Without loss of generality we may assume that the $d$–dimensional orthants $+O_1, \ldots, +O_{2^{d-1}}$ lie in that closed half–space of $\mathbf{E}^d$ bounded by the $\mathbf{E}^{d-1}$ into which $e_d$ points.

Then take the following $2^d$ orthants in $\mathbf{E}^d$:

$$
\begin{aligned}
&e_d + (-O_1), e_d + (-O_2), \ldots, e_d + (-O_{2^{d-2}});\\
&e_1 + e_d + (+O_1), e_1 + e_d + (+O_2), \ldots, e_1 + e_d + (+O_{2^{d-2}});\\
&-e_1 + (-O_{2^{d-2}+1}), -e_1 + (-O_{2^{d-2}+2}), \ldots, -e_1 + (-O_{2^{d-1}});
\end{aligned}
$$

$$+O_{2^{d-2}+1}, +O_{2^{d-2}+2}, \ldots, +O_{2^{d-1}}.$$

If $P_0$ is the $d$–cube $\text{conv}(P'_0 \cup (e_d + P'_0))$, then using the induction hypothesis that $P'_0$ is coated by the $(d-1)$–dimensional orthants $O'_1, \ldots, 0'_{2^{d-1}}$ in $\mathbf{E}^{d-1}$ it is easy to see that the above $2^d$ $d$–dimensional orthants coat $P_0$ in $\mathbf{E}^d$. This completes the proof of (1). □

*Proof of (2).* At first, we show that if $P_0$ has an integer edgelength say $k \geq 1$, then $P_0$ can be coated by $2(k+1)^d - 2k^d$ unit $d$–cubes with edges parallel to the coordinate axes of $\mathbf{E}^d$, i.e., parallel to the edges of $P_0$. We prove this by induction on the dimension $d$. The claim is obviously true for the case $d = 1$. So assume that it is true for every $d' < d$ and take a $d$–cube $P_0$ of $\mathbf{E}^d$ with integer edgelength $k \geq 1$. Let $H_0$ be a supporting hyperplane of $P_0$ that intersects $P_0$ in a facet $F_0$. Then let $H_l$ be the translate of $H_0$ by the vector of length $l$ orthogonal to $H_0$ that intersects $P_0$ in a $(d-1)$–cube $F_l = H_l \cap P_0$ of edgelength $k$, where $l = 1, \ldots, k$. By induction each $F_l$ can be coated by $2(k+1)^{d-1} - 2k^{d-1}$ unit $(d-1)$–cubes in $H_l$, where $l = 0, 1, \ldots, k$. Thus, if we place $k(2(k+1)^{d-1} - 2k^{d-1})$ unit $d$–cubes between the consecutive hyperplanes $H_i, H_{i+1}$, $0 \leq i \leq k-1$ properly, then we are left with the problem to coat $P_0$ along the facets $F_0$ and $F_k$ only. This can be done easily by $2(k+1)^{d-1}$ unit $d$–cubes. Thus, $P_0$ is coated by $k(2(k+1)^{d-1} - 2k^{d-1}) + 2(k+1)^{d-1} = 2(k+1)^d - 2k^d$ unit $d$–cubes in $\mathbf{E}^d$ finishing the construction.

At second, notice that each unit $d$–cube of the above construction has a $(d-1)$–dimensional intersection with $P_0$. The following easy lemma is the key to prove the claim (2) completely.

**Lemma 3.** *Let $P_0$ be a $d$-cube of $\mathbf{E}^d$ with integer vertices and with edges parallel to the coordinate–axes of $\mathbf{E}^d$. We assign to $P_0$ each orthant of $\mathbf{E}^d$ that has an integer apex belonging to $P_0$ and the edges of which are parallel to the coordinate–axes of $\mathbf{E}^d$ such that the interior of the orthant is disjoint from $P_0$. If $P$ is a unit $d$-cube of $\mathbf{E}^d$ with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that $P \cap P_0$ is $(d-1)$-dimensional, then the number of the orthants assigned to $P_0$, each of whose interior intersects $P$ and each of whose apex belongs to $P$, is always $2^{d-1}$.*

*Proof.* We leave the rather easy proof to the reader. □

To complete the proof of (2) assume that $P_0$ is coated by the unit $d$–cubes $P_1, \ldots, P_n$ in $\mathbf{E}^d$. Without loss of generality we may assume that $P_0$ is a $d$–cube of $\mathbf{E}^d$ with integer vertices and with edges parallel to the coordinate–axes of $\mathbf{E}^d$. Assign to $P_0$ all orthants described in Lemma 3. As $P_i \cap P_0$ is at most $(d-1)$–dimensional $(1 \leq i \leq n)$ it is easy to see (using Lemma 3) that the number of the orthants assigned to $P_0$ the interior of each of which intersects $P_i$ and the apex of each of which belongs to $P_i$ is at most $2^{k-1}$. Thus, a very simple counting argument implies that $n$ is at least as large as the number of unit $d$–cubes in the above construction, i.e., $2(k+1)^d - 2k^d$. This completes the proof of (2). □

## 4. Proof of the Corollary

Without loss of generality we may assume that $P_0$ is a $d$–cube of edge length $\lambda$ with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that it is coated by the unit $d$–cubes $P_1, \ldots, P_n$

of $\mathbf{E}^d$ the edges of which are par allel to the coordinate–axes of $\mathbf{E}^d$. If $\lambda \leq 1$, then the claim follows from the Theorem in a straight way. So, we are left with the case, when $\lambda > 1$. Then take two opposite facets say, $F$ and $F'$ of $P_0$. Obviously, $F$ ($F'$, resp.) is a $(d-1)$–cube of edge length $\lambda$ that is covered by some $d$–cubes of the collection $P_1, \ldots, P_n$ each of which has edge length smaller than $\lambda$. Thus, the number of $d$–cubes of the collection $P_1, \ldots, P_n$ that cover $F$ ($F'$, resp.) is obviously at least $2^{d-1}$. Hence, $n \geq 2^{d-1} + 2^{d-1} = 2^d$. $\square$

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