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The Cubic Representation of a Submanifold

Bang-yen Chen and **Wei-Eihn Kuan**
Department of Mathematics, Michigan State University
East Lansing, Michigan 48824–1027, U.S.A.

Abstract. Let $f : M \to E^m$ be an immersion of a manifold in a Euclidean space. Then the tensor product map $f^3 := f \otimes f \otimes f : M \to E^m \otimes E^m \otimes E^m \cong E^{m^3}$ is called the cubic representation of the submanifold. The purpose of this article is to investigate the geometry of the cubic representation of a spherical submanifold. Several theorems in this respect are obtained.

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1. Introduction

Let V and W be two vector spaces over the field of real numbers, $\mathbf{R}$. Then we have the notion of the direct sum $V \oplus W$ and the notion of the tensor product $V \otimes W$. If V and W are vector spaces equipped with inner products $< \, , \, >_V$ and $< \, , \, >_W$, respectively, then $V \oplus W$ and $V \otimes W$ are inner product spaces with their inner products given respectively by

$$
\begin{aligned}
&< v \oplus w, x \oplus y > = < v, x >_V + < w, y >_W, \\
&< v \otimes w, x \otimes y > = < v, x >_V < w, y >_W, \quad v, x \in V, \quad w, y \in W.
\end{aligned} \tag{1.1}
$$

Let E^m denote the Euclidean m-space with the usual inner product. With respect to the inner products defined above, $E^m \oplus E^{m'}$ and $E^m \otimes E^{m'}$ are isometric to $E^{m+m'}$ and $E^{mm'}$, respectively. By applying these algebraic notions, one has the notions of *direct sum map* $f \oplus h : M \to E^m \oplus E^{m'}$ and of *tensor product map* $f \otimes h : M \to E^m \otimes E^{m'}$ associated with any two given maps $f : M \longrightarrow E^m$ and $h : M \longrightarrow E^{m'}$ of M defined as follows:

$$
\begin{aligned}
&(f \oplus h)(p) = f(p) \oplus h(p) \in E^m \oplus E^{m'}, \\
&(f \otimes h)(p) = f(p) \otimes h(p) \in E^m \otimes E^{m'}, \quad p \in M.
\end{aligned} \tag{1.2}
$$

An immersion $f : M \to E^m$ is said to be *transversal at a point* $p \in M$ if the position vector $f(p)$ is not tangent to M at p, *i.e.*, $f(p) \notin f_*(T_pM)$. An immersion f is called

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transveral if it is transversal at every point of M and it is called *spherical* if $f(M)$ is contained in a hypersphere of E^m centered at the origin of E^m. It is obvious that every spherical immersion is transversal.

Denote by $\mathcal{R}(M)$ the set of all transversal immersions of a manifold M into Euclidean spaces. Then $\oplus$ and $\otimes$ are binary operations on $\mathcal{R}(M)$. The set $\mathcal{R}(M)$ together with the binary operations $\oplus$ and $\otimes$ forms a commutative, cancellative semiring, which is called the *semiring of immersions* of M (cf. [5,8]). The geometry of $\mathcal{R}(M)$ has been investigated in [4-8].

If $f : M \longrightarrow E^m$ is a transversal immersion, then, for any natural number k, the tensor product map $f^k := f \otimes \cdots \otimes f$ (k-times) is also a transversal immersion. The immersions f^2 and f^3 are called the *quadric* and *cubic* representations of the submanifold, respectively. It is easy to see that the quadric and the cubic representations of a spherical submanifold are also spherical.

It is fundamental to ask the following question:

To what extent do the quadric and the cubic representations of a submanifold determine the submanifold?

This question has been studied rather extensively for the quadric representation (cf. for instances, [1,3,5,6,7,9,12]). On the contrary, for the cubic representation, this question has just been barely touched in [5,6].

The main purpose of this article is to provide a more extensive study of this problem for the cubic representation. In §4 we give some basic lemmas concerning the cubic representation of a spherical submanifold. In §§5-7 we mainly establish some classification theorems for the cubic representation of a spherical submanifold.

2. Preliminaries

Let $f : M \to N$ be an isometric immersion of an n-dimensional Riemannian manifold M into another Riemannian manifold N. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and of N, respectively. The *second fundamental form* σ of f is then given by

$$\sigma(X, Y) = \tilde{\nabla}_X(f_*Y) - f_*(\nabla_X Y), \tag{2.1}$$

The *mean curvature vector field* H of f is defined by $H = \frac{1}{n}\,\mathrm{Tr}\,\sigma$. For any vector field X tangent to M and ξ normal to M, we have the Weingarten formula:

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

where A denotes the shape operator and D the normal connection.

The covariant derivative of the second fundamental form σ is defined by

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.3}$$

A normal vector field ξ of f is said to be *parallel* if $D\xi = 0$, identically. An immersion $f : M \longrightarrow E^m$ is called *totally umbilical* if its second fundamental form satisfies $\sigma(X, Y) = \mu g(X, Y)H$ for some function μ on M; and it is called *totally geodesic* if its second fundamental form vanishes identically.

If $f : M \longrightarrow E^m$ is an isometric immersion of M in the Euclidean m-space, we have

$$\Delta f = -nH. \tag{2.4}$$

A smooth map $x : M \to E^m$ of a Riemannian manifold M in E^m is said to be *of finite type* (respectively, *k–type*) if x can be decomposed as a sum of finitely many (respectively, k, not counting a constant function,) eigenfunctions of the Laplacian Δ of M from different eigenspaces, *i.e.,* we have

$$x = x_0 + x_1 + \ldots + x_k \tag{2.5}$$

where x_0 is a constant vector and $\Delta x_i = \lambda_{t_i} x_i$, $i = 1, 2, ..., k$, where $\lambda_{t_1}, \ldots, \lambda_{t_k}$ are k different eigenvalues of the Laplacian of M which acts on vector functions componentwise. In that case we have $P(\Delta)(x - x_0) = 0$, where

$$P(T) = \prod_{i=1}^{k} (T - \lambda_{t_i}). \tag{2.6}$$

If x cannot be represented as a finite sum, it is said to be *of infinite type.* The *type number* is a natural and important invariant associated with an arbitrary map or an immersion of a Riemannian manifold in a Euclidean space. Many interesting results in this respect have been obtained by various authors (see, for instances, [3,13] for more details).

In the following, we denote by $S^{m-1}(r)$ the hypersphere of E^m with radius r centered at the origin. We simply denote $S^{m-1}(1)$ by S^{m-1}. A spherical k-type immersion is called *mass-symmetric* if its spectral decomposition takes the following form:

$$x = x_1 + \ldots + x_k$$

for some eigenfunctions $x_1, \ldots, x_k$ of Δ associated with k nonzero eigenvalues.

We recall two results concerning cubic representation from [5] and [6].

Theorem A. *Let $f \in \mathcal{R}(M)$ be an isometric immersion of a Riemannian manifold M in E^m. Then the cubic representation f^3 of M is a conformal immersion if and only if f is spherical.*

Theorem B. *Let $f : M \longrightarrow S^{m-1} \subset E^m$ be a spherical isometric immersion of an n-dimensional Riemannian manifold M. Then*
(a) *the cubic representation f^3 of M is of ℓ-type with $\ell \geq 2$;*
(b) *the cubic representation is of 2-type if and only if M is isometric to an open portion of S^n and the induced isometric immersion $f' : M \to S^{m-1}$ is totally geodesic.*

Theorem B completely classifies spherical submanifolds with 2-type cubic representation.

3. Cubic representations

Let $f : M \to S^{m-1}(r) \subset E^m$ be a spherical isometric immersion of an n-dimensional Riemannian manifold M in E^m and $e_1, \dots, e_n$ an orthonormal local frame field of the tangent bundle of M. We have

$$\Delta f^3 = -n \sum^{\text{sym}} H \otimes f \otimes f - 2 \sum^{\text{sym}} e_i \otimes e_i \otimes f, \tag{3.1}$$

where we have identified e_i with $f_* e_i$ and $\sum^{\text{sym}}$ denotes the symmetric sum. In particular, we have

$$\sum^{\text{sym}} H \otimes f \otimes f = H \otimes f \otimes f + f \otimes H \otimes f + f \otimes f \otimes H$$

and

$$\sum^{\text{sym}} e_i \otimes e_i \otimes f = e_i \otimes e_i \otimes f + e_i \otimes f \otimes e_i + f \otimes e_i \otimes e_i.$$

From (3.1) and a long computation, we obtain

$$\begin{aligned} \Delta^2 f^3 = &- n \sum^{\text{sym}} \Delta H \otimes f \otimes f - 2n \sum^{\text{sym}} A_H e_i \otimes e_i \otimes f \\ &+ 4n \sum^{\text{sym}} D_{e_i} H \otimes e_i \otimes f + 4n \sum^{\text{sym}} H \otimes e_i \otimes e_i \\ &+ 2n^2 \sum^{\text{sym}} H \otimes H \otimes f - 2 \sum^{\text{sym}} A_{\sigma(e_i,e_j)} e_i \otimes e_j \otimes f \\ &+ 4 \sum^{\text{sym}} \sigma(e_i,e_j) \otimes \sigma(e_i,e_j) \otimes f + 8 \sum^{\text{sym}} \sigma(e_i,e_j) \otimes e_i \otimes e_j. \end{aligned} \tag{3.2}$$

From (3.2) and a long computation, we obtain

$$\begin{aligned} \Delta^3 f^3 = &- 12n \sum^{\text{sym}} H \otimes \sigma(e_i,e_j) \otimes \sigma(e_i,e_j) - 6n^3 H \otimes H \otimes H \\ &- 48 \sum \sigma(e_i,e_j) \otimes \sigma(e_i,e_k) \otimes \sigma(e_j,e_k) \\ &- 16n \sum^{\text{sym}} D_{e_i} H \otimes \sigma(e_i,e_j) \otimes e_j - 12n^2 \sum^{\text{sym}} D_{e_i} H \otimes e_i \otimes H \\ &- 24 \sum^{\text{sym}} (\bar\nabla_{e_k} \sigma)(e_i,e_j) \otimes \sigma(e_i,e_j) \otimes e_k \\ &+ 12 \sum^{\text{sym}} A_{\sigma(e_i,e_j)} e_i \otimes \sigma(e_j,e_k) \otimes e_k + 8 \sum^{\text{sym}} A_{\sigma(e_i,e_j)} e_k \otimes \sigma(e_i,e_j) \otimes e_k \\ &+ 16 \sum^{\text{sym}} A_{\sigma(e_i,e_j)} e_k \otimes \sigma(e_i,e_k) \otimes e_j + 6n \sum^{\text{sym}} H \otimes A_{\sigma(e_i,e_j)} e_j \otimes e_i \\ &+ 4 \sum^{\text{sym}} \sigma(e_k, A_{\sigma(e_i,e_j)} e_i) \otimes e_j \otimes e_k + 4n \sum^{\text{sym}} \sigma(e_j, A_H e_i) \otimes e_i \otimes e_j \\ &+ 4n \sum^{\text{sym}} A_H e_i \otimes \sigma(e_i,e_j) \otimes e_j + 6n^2 \sum^{\text{sym}} A_H e_i \otimes e_i \otimes H \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&- 8n \sum^{\text{sym}} D_{e_j} D_{e_i} H \otimes e_i \otimes e_j + 8n \sum^{\text{sym}} A_H e_j \otimes \sigma(e_i, e_j) \otimes e_i \\
&+ 6n \sum^{\text{sym}} \Delta H \otimes e_i \otimes e_i + 8 \sum^{\text{sym}} \Delta\sigma(e_i, e_j) \otimes e_i \otimes e_j \\
&+ 4n \sum^{\text{sym}} (\nabla_{e_j} A_H) e_i \otimes e_i \otimes e_j + 4 \sum^{\text{sym}} (\nabla_{e_k} A_{\sigma(e_i,e_j)}) e_i \otimes e_j \otimes e_k \\
&+ 8n \sum^{\text{sym}} A_{D_{e_i} H} e_j \otimes e_i \otimes e_j + \text{ terms involving at least one } f.
\end{aligned}
$$

Because $f : M \to E^m$ is spherical, f can also be regarded as a vector field defined along M which is normal to S^{m-1} as well as normal to M.

4. Some basic lemmas

Let $f : M \to S^{m-1} \subset E^m$ be a spherical isometric immersion and $f' : M \to S^{m-1}$ the isometric immersion induced from f. Denote by H' and σ' the mean curvature vector and the second fundamental form of f', respectively. We have

$$H = H' - f, \tag{4.1}$$

$$\sigma(X, Y) = \sigma'(X, Y) - g(X, Y) f. \tag{4.2}$$

If the cubic representation of M is mass-symmetric and of 3-type, then there exist real numbers $\alpha_1, \alpha_2, \alpha_3$ such that

$$\Delta^3 f^3 + \alpha_1 \Delta^2 f^3 + \alpha_2 \Delta f^3 + \alpha_3 f^3 = 0. \tag{4.3}$$

In the following, by a S-normal vector of M in E^m we mean a vector normal to M, but tangent to S^{m-1}.

Consider the *S-normal*$\otimes$ *S-normal* $\otimes$ *S-normal*–components of (4.3), with help of (3.1), (3.2), (3.3), (4.1) and (4.2), we obtain

$$
\begin{aligned}
&2n \sum^{\text{sym}} H' \otimes \sigma'(e_i, e_j) \otimes \sigma'(e_i, e_j) + n^3 H' \otimes H' \otimes H' \\
&+ 8 \sum_{i,j,,k} \sigma'(e_i, e_j) \otimes \sigma'(e_i, e_k) \otimes \sigma'(e_j, e_k) = 0.
\end{aligned} \tag{4.4}
$$

Lemma 1. *Let $f : M \longrightarrow S^{m-1} \subset E^m$ be a spherical isometric immersion of an n-dimensional Riemannian manifold M. If the cubic representation of M is mass-symmetric and of 3-type, then, for any S-normal vector ξ of M perpendicular to H', we have*

(a) $\mathrm{Tr}(\mathrm{A}_\xi^3) = 0$,

(b) $6\,\mathrm{Tr}(\mathrm{A}_{\mathrm{H}'})\mathrm{Tr}(\mathrm{A}_{\mathrm{H}'}^2) + (\mathrm{Tr}(\mathrm{A}_{\mathrm{H}'}))^3 + 8\,\mathrm{Tr}(\mathrm{A}_{\mathrm{H}'}^3) = 0$,

(c) $\mathrm{Tr}(\mathrm{A}_{\mathrm{H}'})\mathrm{Tr}(\mathrm{A}_\xi^2) + 4\mathrm{Tr}(\mathrm{A}_\xi^2)\mathrm{Tr}(\mathrm{A}_{\mathrm{H}'}) = 0$,

(d) $\mathrm{Tr}(\mathrm{A}_{\mathrm{H}'})\mathrm{Tr}(\mathrm{A}_\xi \mathrm{A}_{\mathrm{H}'}) + 2\,\mathrm{Tr}(\mathrm{A}_\xi \mathrm{A}_{\mathrm{H}'}^2) = 0$.

Proof. By taking the inner product of (4.4) with $\xi\otimes\xi\otimes\xi$, we obtain formula (a). Similarly, by taking the inner product of (4.4) with $H'\otimes H'\otimes H'$, $\xi\otimes\xi\otimes H'$ and $\xi\otimes H'\otimes H'$, respectively, we obtain formulas (b), (c) and (d), respectively. $\square$

Lemma 2. *Let $f: M\longrightarrow S^{m-1}\subset E^m$ be a spherical isometric immersion of a 2-dimensional Riemannian manifold M. If the cubic representation of M is mass-symmetric and of 3-type, then the induced immersion $f': M\longrightarrow S^{m-1}$ is a minimal immersion.*

Proof. If the induced isometric immersion $f': M\longrightarrow S^{m-1}$ is not minimal, then there is a point $p\in M$ such that the mean curvature vector H' is nonzero at p. Choose an orthonormal basis e_1, e_2 of T_pM such that e_1, e_2 are eigenvectors of the shape operator $A_{H'}$ at p with eigenvalues, say κ_1,κ_2. Then $\mathrm{Tr}(\mathrm{A}'_{\mathrm{H}})=|\mathrm{H}'|^2$; moreover, from formula (b) of Lemma 1, we get

$$(\kappa_1+\kappa_2)(15\kappa_1^2+15\kappa_2^2-6\kappa_1\kappa_2)=0$$

at p. Therefore, $\kappa_1+\kappa_2=0$ at p which is a contradiction. $\square$

Lemma 3. *Let $f: M\longrightarrow S^{m-1}\subset E^m$ be a minimal isometric immersion of an n-dimensional Riemannian manifold M into S^{m-1}. If the cubic representation of M is mass-symmetric and of 3-type, then we have*

$$\sum_{i,j,k}\sigma'(e_i,e_j)\otimes\sigma'(e_i,e_k)\otimes\sigma'(e_j,e_k)=0.$$

Proof. This lemma follows immediately from (4.4). $\square$

5. Cubic representation which is mass-symmetric and of 3-type

Theorem 1. *Let $f: M\longrightarrow S^{m-1}\subset E^m$ be an isometric immersion such that M minimal in S^{m-1}. If the cubic representation of M is mass-symmetric and of 3-type, then*
(a) *M has constant scalar curvature;*
(b) *The second fundamental form of $f': M\longrightarrow S^{m-1}$ satisfies*

$$\sum_{i,j=1}^{n}\{(\bar\nabla_X\sigma')(e_i,e_j)\otimes\sigma'(e_i,e_j)+\sigma'(e_i,e_j)\otimes(\bar\nabla_X\sigma')(e_i,e_j)\}=0 \tag{5.1}$$

for any vector X tangent to M.

Proof. Assume the cubic representation of M is mass-symmetric and of 3-type. By taking the *S-normal* $\otimes$ *S-normal* $\otimes$ *tangent*–components of (4.3), with the help of (3.1), (3.2), (3.3), (4.1) and (4.2), we obtain (5.1).

Let $e_{n+1},\dots,e_{m-1}$ be an orthonormal local frame field of the normal bundle of M in S^{m-1}. We put

$$h^r_{ij}=<\sigma'(e_i,e_j),e_r>,\quad h^r_{ij,k}=<(\bar\nabla_{e_k}\sigma)(e_i,e_j),e_r>, \tag{5.2}$$
$$i,j,k=1,\dots,n,\quad r=n+1,\dots,m-1.$$

Then (5.1) yields

$$\sum_{i,j=1}^{n} h^r_{ij,k} h^r_{ij} = 0, \quad k = 1, \ldots, n, \quad r = n+1, \ldots, m-1. \tag{5.3}$$

From (5.3), we may conclude that the length $||\sigma'||$ of the second fundamental form σ' is constant. Since $f' : M \longrightarrow S^{m-1}$ is a minimal isometric immersion, the equation of Gauss implies that M has constant scalar curvature. □

By a *Clifford torus* in S^{m-1} we mean a surface given by the product two plane circles with radius $\frac{1}{\sqrt{2}}$ which lies in a totally geodesic 3-sphere S^3 of S^{m-1}.

The following result completely classifies spherical surfaces whose cubic representation is mass-symmetric and of 3-type.

Theorem 2. *Let $f : M \longrightarrow S^{m-1} \subset E^m$ be an isometric immersion of a 2-dimensional Riemannian manifold M in $S^{m-1} \subset E^m$. Then the cubic representation of M is mass-symmetric and of 3-type if and only if M is an open portion of the flat torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ and $f : M \to S^{m-1}$ is a Clifford immersion.*

Proof. If M is an open portion of the flat torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ and $f : M \to S^{m-1}$ is a Clifford immersion, then, up to rotations of E^m about the origin, f is given by

$$f(u,v) = \frac{1}{\sqrt{2}}(\cos\sqrt{2}u, \sin\sqrt{2}u, \cos\sqrt{2}v, \sin\sqrt{2}v, 0, \ldots, 0). \tag{5.4}$$

From (5.4), it is easy to verify that f^3 is mass-symmetric and of 3-type (whose coordinate functions belong to the three eigenspaces of Δ with eigenvalues 2, 10 and 18).

Conversely, if $f : M \longrightarrow S^{m-1} \subset E^m$ is an isometric immersion of a 2-dimensional Riemannian manifold whose cubic representation is mass-symmetric and of 3-type, then Lemma 2 implies that the induced isometric immersion $f' : M \longrightarrow S^{m-1}$ is minimal. Therefore, by Thereom 1, we obtain (5.1). Moreover, because f' is minimal and M is 2-dimensional, the first normal space $\operatorname{Im}\sigma'$ is of dimension ≤ 2 at each point of M. We put

$$M_i = \{p \in M : \dim(\operatorname{Im}\sigma') = i \ \text{ at } \ p\}, \quad i = 0, 1, 2.$$

Assume that the interior $int(M_2)$ of M_2 is non-emptpy. Let U be a connected component of $int(M_2)$. On U, we can choose an orthonormal local frame field $e_1, e_2, e_3, \ldots, e_{m-1}$ such that e_1, e_2 are tangent vectors and the shape operators of U in S^{m-1} are given by

$$A_3 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad A_4 = \begin{pmatrix} \mu & \eta \\ \eta & -\mu \end{pmatrix}, \quad A_5 = \cdots = A_{m-1} = 0, \tag{5.5}$$

for some non-zero functions λ, μ on U.

By taking the inner product of (5.1) with $e_3 \otimes e_r$ for $r = 5, \ldots, m-1$, we have

$$\sum_{i,j=1}^{2} h^3_{ij} h^r_{ij,k} = 0. \tag{5.6}$$

By using (5.5) and (5.6) and the fact: $h^r_{11,k}+h^r_{22,k}=0$, we obtain

$$h^r_{11,k}=h^r_{22,k}=0,\quad k=1,2,\quad r=5,\dots,m-1. \tag{5.7}$$

On the other hand, by taking the inner product of (5.1) with $e_4\otimes e_r$ for $r=5,\dots,m-1$ and by applying (5.7), we obtain $h^r_{12,k}=0$, $k=1,2$. Hence

$$h^r_{ij,k}=0,\quad i,j,k=1,2,\quad r=5,\dots,m-1. \tag{5.8}$$

Furthermore, by using (2.3), (5.2), (5.5) and (5.8), we get

$$\begin{aligned}0=h^r_{ij,k}&=<(\bar\nabla_{e_k}\sigma')(e_i,e_j),e_r>=<(D_{e_k}\sigma')(e_i,e_j),e_r>\\&=-<\sigma'(e_i,e_j),D_{e_k}e_r>,\quad i,j,k=1,2,\quad r=3,\dots,m-1\end{aligned}$$

which implies that the first normal spaces form a parallel subbundle of the normal bundle. Hence, by applying a result of Erbacher, U lies in a totally geodesic 4-sphere S^4 of S^{m-1}. Since U has constant Gauss curvature according to Theorem 1, U is a minimal surface with constant Gauss curvature which lies fully in S^4. Therefore, by a result of [10], U has constant Gaussian curvature $\frac{1}{3}$ which is immersed as an open portion of a Veronese surface in S^4. In particular, the functions λ and μ in (5.5) are non-zero constants. Hence, by continuity, $M_2=M$, and M has constant Gaussian curvature $\frac{1}{3}$, immersed as an open portion of a Veronese surface in a totally geodesic 4-sphere S^4 of S^{m-1}. Therefore, up to rigid motions of E^m, f is given by

$$\left(\frac{1}{\sqrt3}yz,\frac{1}{\sqrt3}zx,\frac{1}{\sqrt3}xy,\frac{1}{2\sqrt3}(x^2-y^2),\frac{1}{6}(x^2+y^2-2z^2)\right), \tag{5.9}$$

where

$$x=\sqrt3\sin\phi\cos\theta,\ y=\sqrt3\sin\phi\sin\theta,\ z=\sqrt3\cos\phi. \tag{5.10}$$

By using (5.9) and (5.10) and a direct computation, one can prove that the cubic representation of M is of 3-type, but it is *not* mass-symmetric. Therefore $int(M_2)$ is empty. Hence, M has flat normal connection in S^{m-1}. Thus, by a theorem of [2], either M is immersed as an open submanifold of a totally geodesic 2-sphere S^2 of S^{m-1} or M is immersed as a Clifford torus in a totally geodesic 3-sphere S^3. The first case is impossible, since the cubic representation of a totally geodesic S^2 is of 2-type according to Theorem B. □

6. Cubic representation of Cartan hypersurface

E. Cartan has investigated the hypersurface in $S^4\subset E^5$ defined by the equation:

$$2x_5^3+3(x_1^2+x_2^2)x_5-6(x_3^2+x_4^2)x_5+3\sqrt3(x_1^2-x_2^2)x_4+6\sqrt3x_1x_2x_3=2. \tag{6.1}$$

Cartan proved that this hypersurface in S^4 is a homogeneous Riemannian manifold given by $SO(3)/(\mathbf{Z}_2\times\mathbf{Z}_2)$ and that the principal curvatures of the hypersurfaces are equal to $\sqrt3, 0-\sqrt3$. This hypersurface is now known as the *Cartan hypersurface.*

Theorem 3. *The cubic representation of the Cartan hypersurface in $S^4 \subset E^5$ is mass-symmetric and of 5-type.*

Proof. Let M be the Cartan hypersurface in $S^4 \subset E^5$. Then there exist orthonormal frame fields e_1, e_2, e_3, e_4 such that e_4 is a unit normal vector field of M in S^4 and e_1, e_2, e_3 are eigenvectors of A_4 with eigenvalues given by $\sqrt{3}, 0 - \sqrt{3}$, repsectively.

By applying the fundamental equations, we may prove that the connection forms ω_B^A of the Cartan hypersurface with respect to e_1, e_2, e_3, e_4 are given by

$$\begin{aligned} &\omega_1^2 = \omega^3, \quad \omega_1^3 = \frac{1}{2}\omega^2, \quad \omega_2^3 = \omega^1, \\ &\omega_1^4 = -\sqrt{3}\omega^1, \quad \omega_2^4 = 0, \quad \omega_3^4 = \sqrt{3}\omega^3. \end{aligned} \tag{6.2}$$

By using (2.1), (2.2), (2.4), (3.1), (6.2) and long computation, we obtain

$$\Delta f^3 = 9f^3 - 2\sum^{\text{sym}} e_i \otimes e_i \otimes f, \tag{6.3}$$

$$\begin{aligned} \Delta^2 f^3 = {} & 117f^3 - 48\sum^{\text{sym}} e_i \otimes e_i \otimes f + 12\sum^{\text{sym}} e_2 \otimes e_2 \otimes f \\ & + 24\sum^{\text{sym}} e_4 \otimes e_4 \otimes f - 8\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_1 \otimes e_1 + 8\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_3 \otimes e_3, \end{aligned} \tag{6.4}$$

$$\begin{aligned} \Delta^3 f^3 = {} & 1845f^3 - 1218\sum^{\text{sym}} e_i \otimes e_i \otimes f \\ & + 708\sum^{\text{sym}} e_2 \otimes e_2 \otimes f + 1128\sum^{\text{sym}} e_4 \otimes e_4 \otimes f \\ & - 520\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_1 \otimes e_1 + 520\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_3 \otimes e_3. \end{aligned} \tag{6.5}$$

$$\begin{aligned} \Delta^4 f^3 = {} & 34281f^3 - 36384\sum^{\text{sym}} e_i \otimes e_i \otimes f \\ & + 30936\sum^{\text{sym}} e_2 \otimes e_2 \otimes f + 44016\sum^{\text{sym}} e_4 \otimes e_4 \otimes f \\ & - 24464\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_1 \otimes e_1 + 24464\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_3 \otimes e_3. \end{aligned} \tag{6.6}$$

$$\begin{aligned} \Delta^5 f^3 = {} & 777825f^3 - 1233858\sum^{\text{sym}} e_i \otimes e_i \otimes f \\ & + 1240008\sum^{\text{sym}} e_2 \otimes e_2 \otimes f + 1683984\sum^{\text{sym}} e_4 \otimes e_4 \otimes f \\ & - 1031056\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_1 \otimes e_1 + 1031056\sqrt{3}\sum^{\text{sym}} e_4 \otimes e_3 \otimes e_3. \end{aligned} \tag{6.7}$$

Put

$$P(t) = (t-3)(t-9)(t-15)(t-17)(t-39). \tag{6.8}$$

Then, by (6.3)-(6.7), we may prove that $P(\Delta)f^3 = 0$; morevoer, $P(t)$ is the monic polynomial of least degree satisfying this property. Thus, $P(t)$ is the minimal polynomial of the cubic representation of the Cartan hypersurface. Consequently, by applying a result of [3], the cubic representation is mass-symmetric and of 5-type. □

By applying Theorem 3, we can give the following non-existence theorem.

Theorem 4. *There exists no (3-dimensional) minimal hypersurface in $S^4 \subset E^5$ whose cubic representation is mass-symmetric and of 3-type.*

Proof. Let $f : M \to S^4 \subset E^5$ be a minimal hypersurface of S^4. If the cubic representation of M is mass-symmetric and of 3-type, then, by Theorem 1, M has constant scalar curvature. Because M is minimal in S^4, the second fundamental form of M in S^4 has constant length which is non-zero according to Theorem B.

Choose an orthonormal frame e_1, e_2, e_3 such that the shape operator A_4 satisfies $A_4 e_i = \kappa_i e_i$, $i = 1, 2, 3$. Since $\kappa_1 + \kappa_2 + \kappa_3 = 0$, Lemma 3 implies that one of $\kappa_1, \kappa_2, \kappa_3$ is zero. Because $||\sigma'||$ is a nonzero constant, the other two principal curvatures are given by λ and $-\lambda$ for some non-zero constant λ. Thus, M is a minimal isoparametric hypersurfce in S^4 with three distinct principal curvatures. According to the classification theorem of Cartan for such isoparametric hypersurfaces M is given by (6.1), up to rotations of E^5. Thus, M is a Cartan hypersurface. By Theorem 3, M is of 5-type which is a contradiction. □

Remark 1. There exist hypersurfaces in S^4 whose cubic representation is of 3-type. For instance, the cubic representation of a small hypersphere in $S^{n+1} \subset E^{n+2}$ is of 3-type (although it is not mass-symmetric).

Remark 2. Although the eigenvalues of the Laplacian of Cartan's hypersurface have been computed in [11], it is interesting to point out that from the minimal polynomial (6.8) of the cubic representation we know that 3, 9, 15, 17 and 39 are eigenvalues of the Laplacian of the Cartan hypersurface. In general, it is quite difficult to determine the eigenvalues of the Laplacian of a Riemannian manifold. However, the method of using minimal polynomials provides us a new way to obtain information on $Spec(M)$ of a submanifold.

7. Minimial surfaces with 3-type cubic representation

In this section we give the following classification theorem.

Theorem 5. *Let $f : M \longrightarrow S^{m-1} \subset E^m$ be an isometric immersion of a 2-dimensional Riemannian manifold such that M is minimal in S^{m-1}. Then the cubic representation of M is of 3-type if and only if*

(a) *M is flat and $f : M \to S^{m-1}$ is locally a Clifford immersion, or*

(b) *M is of constant Gaussian curvature $\frac{1}{3}$ and f is locally a Veronese immersion.*

Proof. Let $f : M \longrightarrow S^{m-1} \subset E^m$ be an isometric immersion of a 2-dimensional Riemannian manifold which immerses M as a minimal surface in S^{m-1}. If the cubic representation of M is of 3-type, then there exist real numbers $\alpha_1, \alpha_2, \alpha_3$ and a constant vector $c \in E^m \otimes E^m \otimes E^m$ such that

$$\Delta^3 f^3 + \alpha_1 \Delta^2 f^3 + \alpha_2 \Delta f^3 + \alpha_3 (f^3 - c) = 0. \tag{7.1}$$

By taking the covariant derivative of (7.1) with resepct to a tangent vector X of M, we have

$$\tilde{\nabla}_X(\Delta^3 f^3) + \alpha_1 \tilde{\nabla}_X(\Delta^2 f^3) + \alpha_2 \tilde{\nabla}_X(\Delta f^3) + \alpha_3 \tilde{\nabla}_X(f^3) = 0. \tag{7.2}$$

Because $f' : M \longrightarrow S^{m-1}$ is minimal, the *S-normal* $\otimes$ *S-normal* $\otimes$ *S-normal*–components of (7.2) yields

$$\begin{aligned} 0 = 2 &\sum^{\text{sym}} (\bar{\nabla}_X \sigma')(e_i, e_j) \otimes \sigma'(e_i, e_k) \otimes \sigma'(e_j, e_k) \\ &+ \sum^{\text{sym}} (\bar{\nabla}_{e_k} \sigma')(e_i, e_j) \otimes \sigma'(e_i, e_k) \otimes \sigma'(e_j, X), \end{aligned} \tag{7.3}$$

for any X tangent to M, where we have used formulas (2.1), (2.2), (3.1), (3.2), (3.3), (4.1) and (4.2).

Let ξ be any normal vector of M in S^{m-1}. By taking the inner product of (6.3) with $\xi \otimes \xi \otimes \xi$, we obtain

$$0 = \sum_{i=1}^{2} \{< (\bar{\nabla}_X \sigma')(A_\xi e_i, A_\xi e_i), \xi > + < (\bar{\nabla}_{A_\xi X} \sigma')(A_\xi e_i, e_i), \xi >\}. \tag{7.4}$$

Since the cubic representation is of 3-type, the immersion $f' : M \longrightarrow S^{m-1}$ is non-totally geodesic. Assume $\sigma' \neq 0$ at a point $p \in M$. We choose an orthonormal basis $e_3, \ldots, e_{m-1}$ of the normal space of M in S^{m-1} at p such that $e_r \in \operatorname{Im} \sigma'$ and $e_t \in (\operatorname{Im} \sigma')^\perp$, for some suitable $r, t \in \{3, \ldots, m-1\}$. For a fixed $e_r \in \operatorname{Im} \sigma'$, let e_1, e_2 be orthonormal eigenvectors of A_r with non-zero eigenvalues, say $\lambda, -\lambda$. Since $n = 2$, (7.4) implies $h^r_{11,k} = h^r_{22,k} = 0,\ k = 1, 2$; from which we get

$$h^r_{ij,k} = 0, \quad i, j, k = 1, 2, \tag{7.5}$$

for any $e_r \in \operatorname{Im} \sigma'$ with the help of Codazzi equation.

On the other hand, by taking the inner product of (7.3) with $e_r \otimes e_r \otimes e_t$ where $e_r \in \operatorname{Im} \sigma'$, $e_t \in (\operatorname{Im} \sigma')^\perp$, we obtain

$$0 = \sum_{i=1}^{2} < (\bar{\nabla}_X \sigma')(A_r e_i, A_r e_i), e_t > + \sum_{i=1}^{2} < (\bar{\nabla}_{A_r X} \sigma')(A_r e_i, e_i), e_t > . \tag{7.6}$$

Since $A_r(e_1) = \lambda e_1$, $A_r(e_2) = -\lambda e_2$, $\lambda \neq 0$ and dim $M = 2$, (7.6) and the Cadazzi equation yield

$$h^t_{ij,k} = 0, \quad i, j, k = 1, 2, \tag{7.7}$$

for any $e_t \in (\operatorname{Im} \sigma')^{\perp}$. (7.5) and (7.6) show that M has parallel second fundamental form in S^{m-1}, *i.e.*, $\bar{\nabla}\sigma' = 0$, identically. Since M is minimal in S^{m-1}, this implies either M is flat and $f : M \to S^{m-1}$ is locally a Clifford immersion, or M is of constant Gaussian curvature $\frac{1}{3}$ and f is locally a Veronese immersion.

The converse can be verified by direct computation. □

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