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Maximum Convex Hulls of Connected Systems of Segments and of Polyominoes

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Abstract. In this note we prove the following two isoperimetric-type theorems. The d-dimensional volume of the convex hull of any connected system of finitely many segments in $\mathbf{R}^d$ with total length 1 which are parallel to the standard co-ordinate axes is at most $\frac{1}{d^d d!}$. Moreover, the area of the convex hull of any facet-to-facet connected system of n unit squares in $\mathbf{R}^2$, that is, a polyomino with n cells [7], is at most $n + \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$.

1. Introduction

A still open problem of [2] is the following. Consider finitely many segments in the Euclidean plane that form a connected subset of $\mathbf{R}^2$. Without changing the lengths of these segments find a connected system the area of which is the maximum. It is conjectured [2] that the extremal arrangement is the polygonal path of the segments which is inscribed in a hemicircle. For some partial results and related problems see [1], [2], [3], [5], [6], [8] and [9]. However, if we make some assumptions on the positions of the segments in the connected system then one can prove the following statement in any dimension.

Theorem 1. *The d-dimensional volume of the convex hull of any connected system of finitely many segments in $\mathbf{R}^d$ which are parallel to the standard co-ordinate axes of $\mathbf{R}^d$ such that the length of the connected system is 1 is at most $\frac{1}{d^d d!}$.*

An extremal configuration consists of d edges of length $\frac{1}{d}$ that are parallel to the standard co-ordinate axes of $\mathbf{R}^d$ and meet in the origin. Obviously, there are extremal configurations other than this d-simplex when $d \geq 2$.

Theorem 1 naturally leads to the following discrete problem which is somewhat related to the problem of [4] as well. A polyomino in $\mathbf{R}^d$ is a facet-to-facet connected system of finitely many unit d-cubes (*cells*) having pairwise disjoint interiors such that from any unit d-cube one can move to any other using a sequence of unit d-cubes of this system where consecutive d-cubes have a facet in common.

Conjecture. *The d-dimensional volume of the convex hull of any polyomino with n cells is at most*

$$1+\sum_{I\subset\{1,\dots,d\}}\frac{1}{|I|!}\prod_{i\in I}\left\lfloor\frac{n-2+i}{d}\right\rfloor.$$

An extremal configuration for the conjectured value would consist of d pairwise orthogonal linear arms of $\lfloor\frac{n-2+i}{d}\rfloor$ cubes ($i=1,\dots,d$) joined to a central cube.

In this note we prove the above conjecture in the plane.

Theorem 2. *The area of the convex hull of any polyomino with n cells in the plane is at most $n+\frac{1}{2}\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor$.*

2. Proof of Theorem 1

Let $e_1,\dots,e_d$ denote the standard basis vectors of $\mathbf{R}^d$. Whenever we consider a connected system of finitely many segments in $\mathbf{R}^d$ we assume that the segments have only endpoints in common. The following is the key observation for the proof of Theorem 1.

Lemma 1. *The d-dimensional volume of the convex hull of any connected system of n segments in $\mathbf{R}^d$ each of which is parallel to one of the basis vectors $e_1,\dots,e_d$ such that the total length of the segments parallel to e_i equals l_i for $i=1,\dots,d$ is at most $\frac{1}{d!}l_1\cdots l_d$.*

Proof. We prove the statement by double induction on d and n. As the claim is obviously true for $d=1$ we may assume that Lemma 1 is proved for each $d'<d$ with $d\geq 2$ and all n in that dimension. Furthermore, the d-dimensional volume of the convex hull of any connected system of $n\leq d-1$ segments in $\mathbf{R}^d$ is equal to 0, so we may assume that in dimension d Lemma 1 is proved for any connected system of $n'<n$ segments which are parallel to the basis vector $e_1,\dots,e_d$ where $n\geq d\geq 2$.

We consider a connected system S of n segments $s_1,\dots,s_n$ in $\mathbf{R}^d$. If this connected system as a graph — with segments as edges and their endpoints as vertices — contains a cycle, then removing a proper segment does not change the convex hull and we are done by induction on n. Thus, we may assume that the graph of S is a tree. Consequently, there exists a segment, say s_n, that is parallel to e_d and such that after removing it from S the obtained system S' (consisting of the segments $s_1,\dots,s_{n-1}$) is still connected. Hence, if $Pr(S')$ denotes the projection of S' parallel to e_d onto a hyperplane of $\mathbf{R}^d$ orthogonal to e_d, then by the induction hypothesis on d for the $(d-1)$-dimensional volume of the convex hull $\mathrm{conv}(Pr(S'))$ we have the inequality

$$\mathrm{Vol}_{d-1}\Big(\mathrm{conv}(Pr(S'))\Big)\leq\frac{1}{(d-1)!}l_1\cdots l_{d-1}\,.$$

Also, by the induction hypothesis on n we have

$$\mathrm{Vol}_d\Big(\mathrm{conv}(S')\Big) \leq \frac{1}{d!} l_1 \cdots l_{d-1}(l_d - s_n)\,.$$

Finally, observe that

$$\mathrm{Vol}_d\Big(\mathrm{conv}(S)\Big) \leq \mathrm{Vol}_d\Big(\mathrm{conv}(S')\Big) + \frac{1}{d} s_n \mathrm{Vol}_{d-1}\Big(\mathrm{conv}(Pr(S'))\Big)\,.$$

Thus, the above three inequalities immediately yield the desired inequality

$$\mathrm{Vol}_d\Big(\mathrm{conv}(S)\Big) \leq \frac{1}{d!} l_1 \cdots l_{d-1} l_d\,.$$

This completes the inductive proof of Lemma 1. □

Having Lemma 1 the proof of Theorem 1 needs to observe only the trivial fact that the maximum value of $\frac{1}{d!} l_1 \cdots l_d$ on the set $l_1 + \cdots + l_d = 1$, $l_i \geq 0$ $(i = 1, \ldots, d)$ is attained for $l_1 = \cdots = l_d = \frac{1}{d}$. □

3. Proof of Theorem 2

The lower bound for the maximum area $A(n)$ of the convex hull of any polyomino with n square cells in the plane follows from configurations as in Figure 1,

$$A(n) \geq n + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor\,.$$

Figure 1

To prove that this is an upper bound we use induction. For an extremal polyomino with n squares we prove that one square can be deleted such that the area of the convex hull is diminished by at most $1 + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor$. Then the inductive step follows from

$$A(n) \leq A(n-1) + 1 + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor\,.$$

The following lemma will be used.

Lemma 2. *The height or the width of any polyomino with n squares is at most* $\lfloor \frac{n+1}{2} \rfloor$.

Proof. We consider the skeleton graph of a polyomino with n squares (the centers of the squares as vertices and unit edges between the squares with a common side). A spanning tree of this graph with h vertical and w horizontal edges has $h+w+1=n$ vertices, so that $\min(h,w) \le \lfloor \frac{n-1}{2} \rfloor$. Since the height and the width of the polyomino are at most $h+1$ and $w+1$, respectively, Lemma 2 is proved. □

For the proof of Theorem 2 we may assume that the height of a given extremal polyomino Π with n cells is at most $\lfloor \frac{n+1}{2} \rfloor$. We distinguish three cases concerning the squares of the leftmost column L of Π.

(1) There exists a square S in L which has only its right neighbour: Let P and Q be those vertex points of the convex hull of Π which follow and precede clockwise the left upper and lower vertex points of S, respectively. Because of the right neighbour of S the heights of P and Q cannot be smaller or larger than the heights of the upper or lower sides of S, respectively. Since S has only its right neighbour, deletion of S does not destroy the connectivity and decreases the area of the convex hull by at most the areas of S and of the two triangles determined by the upper side of S and P so as the lower side of S and Q (Figure 2). The sum of the heights of these triangles is at most $\lfloor \frac{n-1}{2} \rfloor$.
Thus we have obtained a polyomino with $n-1$ cells which implies

$$A(n-1) \ge A(n) - 1 - \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor .$$

Figure 2

(2) There exists a square S in L which has only one neighbour, above or below S: Only the uppermost and the lowermost square in L could have this property since otherwise S could be deleted without change of the convex hull. We may assume that S has a lower neighbour. In L from S we step down square by square of Π until we come to the first square M which has a right neighbour. Such a square M exists since otherwise Π would be linear, that is, with minimum area of the convex hull. Let N be that square of Π which determines the first vertex point of the convex hull of Π that clockwise follows those vertex points determined by S. We distinguish three subcases.

(2.1) The square N is not below the level of S: We add a new square T as upper neighbour of N, and thereafter we delete S as shown in Figure 3. The first step increases the area of the convex hull by at least $1 + \frac{1}{2}b$ whereas the second step decreases it by at most $\frac{1}{2}b$ where b denotes the distance of the centers of the columns containing S and N. This implies a contradiction to the maximality of Π.

Figure 3

(2.2) The square N is below the level of S and above the level of M: Let R be the square between S and M that is on the level of N. All squares in L above R are moved horizontally to the column of N (see Figure 4). This does not decrease the area of the convex hull. However, the obtained polyomino fulfills the condition of case (2.1).

Figure 4

(2.3) The square N is not above the level of M: If all squares above M in L are moved by one unit to the right as in Figure 5, the area of the convex hull does not decrease for the obtained polyomino Π_1. We note that M in Π_1 is the uppermost square and has a right neighbour.

(2.3.1) The lowermost square in L has a right neighbour. Then either case 1 or case 3 can be used for Π_1.

(2.3.2) The lowermost square in L has only an upper neighbour. By reflection we obtain an uppermost square in L with only a lower neighbour. The reflected polyomino can be handled by case (1) or by case (2.1), (2.2), or (2.3.1) since the mirror square of M is the lowermost square which now has a right neighbour.

Figure 5

(3) Each square in L has at least two neighbours: Denote by S the uppermost square in L and by M the next square in L below S that has a right neighbour.

(3.1) Squares S and M are not neighboured: All squares between S and M in L can be moved by one unit to the right without change of the convex hull and without destruction of the connectivity of Π. Then the obtained polyomino can be settled by case (1).

(3.2) Squares S and M are neighboured: Then S can be deleted without loss of the connectivity. The area of the convex hull decreases by less than $1 + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor$.

This completes the inductive proof of Theorem 2. $\square$

4. Remarks

Besides the question for a proof of the above mentioned conjecture for $d \geq 3$ it may be of interest to ask for the number of different polyominoes with n cells and maximum area of the convex hull. For example, there are 9 different extremal polyominoes for $n = 7$ as shown in Figure 6.

Figure 6

Another class of problems which also are related to the problems in [4] and in [8] is the question for the maximum area of the convex hull of all connected edge-to-edge packings of n congruent regular k-gons in the plane.

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