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On Alperin's Fusion Theorem

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The purpose of this note is to give a graph theoretic interpretation of Alperin's Fusion Theorem and its variations by Alperin, Gorenstein and Goldschmidt (see [1], section 2 of [2] and [3]) and then to use this interpretation for a short proof of a result (Theorem 1) which includes all of these theorems.

Let C be a non–empty set and R a symmetric relation on C. Then $\Gamma = (C, R)$ is called a *graph* with vertex set C; and two elements $a, b \in C$ are said to be *adjacent* if $(a, b) \in R$. An $(n+1)$–tuple $(c_0, \ldots, c_n)$ of vertices is a *path* of Γ, if c_i is adjacent to c_{i+1} for $i = 0, \ldots, n-1$.

The group G *operates* (transitively) on Γ if G operates (transitively) on C and R is invariant under the induced operation of G on $C \times C$. For $c \in C$ the stabilizer of c in G is denoted by G_c.

Lemma 1. *Let $\Gamma = (C, R)$ be a graph, $c \in C$ and G be a group which operates transitively on Γ. Suppose that there exist $h \in G$ and a non–empty subset A of G such that $A, A^h \subseteq G_c$. Then the following two properties are equivalent:*

(a) *There exists $n \geq 1$ and a path $(c_0, \ldots, c_n)$ such that $c_0 = c$, $c_n = ch^{-1}$ and $A \subseteq G_{c_k}$ for $k = 1, \ldots, n$.*

(b) *There exists $n \geq 1$ and elements $y_1, \ldots, y_n \in G$ such that*

(b$_1$) *$h = y_1 \cdots y_n s$ for a suitable $s \in G_c$,*

(b$_2$) *c is adjacent to cy_k for $k = 1, \ldots, n$,*

(b$_3$) *$A^{y_1 \cdots y_k} \subseteq G_c$ for $k = 1, \ldots, n$.*

Proof. (a) $\Longrightarrow$ (b):
Since G is transitive on C there exist $x_1, \ldots, x_n \in G$ such that $c_k = cx_1 \cdots x_k$ for $k = 1, \ldots, n$. We define $x_0 = 1$ and

$$y_k^{-1} = x_k^{(x_0 \cdots x_{k-1})^{-1}} \text{ for } k = 1, \ldots, n.$$

It follows that $x_1 \cdots x_k = y_k^{-1} \cdots y_1^{-1}$ for $k = 1, \ldots, n$. In particular, $cy_n^{-1} \cdots y_1^{-1} = c_n = ch^{-1}$ and thus $h = y_1 \cdots y_n s$ for some $s \in G_c$.

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Since $A \subseteq G_{c_k} = G_c^{x_1\cdots x_k} = G_c^{y_k^{-1}\cdots y_1^{-1}}$ we also get $A^{y_1\cdots y_k} \subseteq G_c$. In addition, $(c_{k-1}, c_k) \in R$ implies $(c, c(x_0\cdots x_{k-1})x_k(x_0\cdots x_{k-1})^{-1}) = (c, cy_k^{-1}) \in R$. Hence, (b_1) – (b_3) are satisfied.

(b) $\Longrightarrow$ (a):
Define $c_0 = c$ and $c_k = cy_k^{-1}\cdots y_1^{-1}$ for $k = 1,\ldots,n$. Then

$$c_n = cy_n^{-1}\cdots y_1^{-1} = csy_n^{-1}\cdots y_1^{-1} = ch^{-1}.$$

From (b_2) we get that $(cy_k^{-1}, c) \in R$. Hence

$$(cy_k^{-1}\cdots y_1^{-1}, cy_{k-1}^{-1}\cdots y_1^{-1}) = (c_k, c_{k-1}) \in R,$$

and $(c_0,\ldots,c_n)$ is a path of Γ.

Since by (b_3) $A \subseteq G_c^{(y_1\cdots y_k)^{-1}}$ and $c_k = c(y_1\cdots y_k)^{-1}$ we also get that $A \subseteq G_{c_k}$. $\square$

Lemma 2. *Suppose that Γ and G satisfy (b) of Lemma 1 for some $c \in C$, $h \in G$ and $A \subseteq G_c$. Then for every choice of $b_1,\ldots,b_n \in G_c$ the elements $g_i = (b_iy_i)^{b_i\cdots b_1}$, $i = 1,\ldots,n$, satisfy (b_1) – (b_3) in place of $y_1,\ldots,y_n$.*

Proof. Define $z_i = b_iy_i$, $b_i \in G_c$. Then $y_i = b_i^{-1}z_i$ and $y_1\cdots y_k = b_1^{-1}z_1\cdots b_k^{-1}z_k = g_1\cdots g_kb_1^{-1}\cdots b_k^{-1}$. Now (b_1) – (b_3) are easy to check. $\square$

Remark. Property (b) in Lemma 1 (or its seemingly stronger variation in Lemma 2) can be viewed as a graph theoretic version of Alperin's Fusion Theorem. The crucial observation is that this version is equivalent to a very natural graph theoretic condition:

($*$) Any set of elements (automorphisms) fixing two vertices also fixes a path joining these two vertices.

In the following let G be a finite group and p a prime which divides $|G|$. A map W from the set of p–subgroups of G into itself is said to be a *conjugacy functor*, if for every p–subgroup P of G:

(i) $W(P) \leq P$, and $W(P) \neq 1$ if $P \neq 1$.
(ii) $W(P)^g = W(P^g)$ for every $g \in G$.

For every pair of p–subgroups (P_0, P) with $P_0 \leq P$ we define $W_0(P_0,P) = P_0 = H_0(P_0,P)$ and

$$H_i(P_0,P) = N_P(W_{i-1}(P_0,P)) \text{ and } W_i(P_0,P) = W(H_i(P_0,P)).$$

Because of (ii) there exists an integer n such that $H_n(P_0,P) = P$. The next Lemma and its Corollary are trivial.

Lemma 3. *Let (P_0,P) be a pair of p–subgroups with $P_0 \leq P$, set $N_0 = P_0$ and $N_i = N_G(W_{i-1}(P_0,P))$ for $i \geq 1$ and let $k \geq 0$ be an integer such that $H_i(P_0,P) \in Syl_p(N_i)$ for $i = 0,\ldots,k$. Then there exists a p–subgroup T containing $H_{k+1}(P_0,P)$ such that $H_{k+1}(P_0,T) \in Syl_p(N_{k+1})$.*

Corollary 1. *Let P_0 be a p–subgroup. Then there exists $P \in Syl_p(G)$ such that $P_0 \le P$ and*

$$(*) \qquad H_i(P_0, P) \in Syl_p(N_G(W_{i-1}(P_0, P))) \text{ for every } i \ge 1.$$

Definition. Let $P_0 \le P \in Syl_p(G)$. Then P_0 is *well–placed* in P (with respect to W) if $(*)$ holds for (P_0, P).

A proper subgroup M of G is *strongly p–embedded* if p divides $|M|$ and

$$M \cap M^g \text{ is a } p'\text{–group for every } x \in G \setminus M.$$

Let $P \in Syl_p(G)$ and $\mathcal{F}(P)$ be a set of subgroups of P. Then $\mathcal{F}(P)$ is a *conjugacy family* (for P in G), if for every $g \in G$ and non–empty subsets $A, A^g \subseteq P$ there exist subgroups $Q_1, \ldots, Q_n \in \mathcal{F}(P)$ and p–elements $g_i \in N_G(Q_i)$ such that:

(I) $g = g_1 \cdots g_n b$ for suitable $b \in N_G(P)$, and

(II) $A^{g_1 \cdots g_i} \subseteq Q_i$ for $i = 1, \ldots, n$.

Let $\mathcal{T}(P)$ be the set of all subgroups P_0 of P such that

(i) P_0 is well–placed in P with respect to W, and

(ii) $N_G(P_0)/P_0$ contains a strongly p–embedded subgroup.

Note that $P \notin \mathcal{T}(P)$ and $N_G(P)$ acts on $\mathcal{T}(P)$ by conjugation. Note further that $\mathcal{T}(P) \neq \emptyset$ since the maximal Sylow intersections contained in P are in $\mathcal{T}(P)$. We now define a graph on which G operates.

Let $C = \{P^g \mid g \in G\}$ and R be the set of all pairs (P^x, P^y) such that there exist $\tilde{x} \in N_G(P)x$ and $\tilde{y} \in N_G(P)y$ with

(1) $P \cap P^{\tilde{x}\tilde{y}^{-1}} = Q \in \mathcal{T}(P)$, and

(2) $\tilde{x}\tilde{y}^{-1} \in O^{p'}(N_G(Q))$.

Then R is a symmetric relation on C, and G operates by conjugation on the graph $\Gamma = (C, R)$. In particular, the stabilizer of the vertex P^g in G is $N_G(P^g)$. Next we show that Γ and G satisfy property (a) of Lemma 1.

Lemma 4. *Let $g \in G$ and $D = P \cap P^g$. Then there exists a path joining P and P^g which is fixed by D.*

Proof. Let (P, P^g) be a counterexample such that $|D|$ is maximal. Set $M = N_G(D)$ and $P_1 = N_P(D)$. Note that $P \neq D$ and thus $P_1 \neq D$.

Let $C_0 = \{T \in C \mid D \le T\}$ and Γ_0 be the induced subgraph of Γ with vertex set C_0. Then Γ_0 is not connected since (P, P^g) is a counterexample. Let C_1 be the vertex set of the connected component of Γ_0 which contains P. The maximality of $|D|$ gives:

$$(*) \; T \in C_1 \text{ for every } T \in C_0 \text{ with } T \cap P \neq D.$$

By Lemma 3 there exists $\tilde{P} \in C_0$ such that D is well–placed in $\tilde{P}$ and $P_1 \le \tilde{P}$. Hence $(*)$ implies $\tilde{P} \in C_1$, and we may assume that $\tilde{P} = P$. With a symmetric argument we may assume that D is well placed in P^g and $g \in O^{p'}(M)$.

Now let $C_0^* = \{P^m \mid m \in M\}$ and Γ_0^* be the induced subgraph of Γ_0 with vertex set C_0^*, and let M_0 be the stabilizer in M of the connected component of Γ_0^* which contains

P. Clearly $P_1 \leq M_0$ and thus p divides $|M_0/D|$. Moreover, M_0 is a proper subgroup of M since $g \in M$ and (P, P^g) is a counterexample. Let $x \in M \setminus M_0$ and $D_0 \in Syl_p(M_0 \cap M_0^x)$. Then P^x is not in the same connected component of Γ_0^* as P. Hence, $(*)$ yields $D = D_0$, and $M_0 \cap M_0^x/D$ is a p'–group. It follows that $D \in \mathcal{T}(P)$ and $(P, P^g) \in R$, a contradiction. $\square$

Theorem 1. *Let $P \in Syl_p(G)$. Then $\mathcal{T}(P)$ is a conjugacy family for P in G.*

Proof. Set $B = N_G(P)$. Let A be a non–empty subset of G and $h \in G$. Suppose that $A, A^h \subseteq P$. We have to show:

There exist subgroups $Q_i \in \mathcal{T}(P)$ and p–elements $g_i \in N_G(Q_i)$, $i = 1, \ldots, n$, such that

(i) $h = g_1 \cdots g_n b$, $b \in N_G(P)$, and

(ii) $A^{g_1 \cdots g_i} \subseteq Q_i$ for $i = 1, \ldots, n$.

We may assume that $h \notin B$. Note that $A \subseteq P \cap P^{h^{-1}}$. Hence, by Lemma 4 property (a) of Lemma 1 is satisfyied. From the equivalent property (b) we conclude that there exist $Q_i \in \mathcal{T}(P)$ and $y_i \in BO^{p'}(N_G(Q_i))$, $i = 1, \ldots, n$ such that (i) and (ii) hold. In particular $y_i = b_i^{-1} z_i$ for suitable $b_i \in B$ and $z_i \in O^{p'}(N_G(Q_i))$. Set $\tilde{b}_i = b_1 \cdots b_i$. Note that $Q_i^{\tilde{b}_i} \in \mathcal{T}(P)$. Now Lemma 2 shows that there exist $g_i \in O^{p'}(N_G(Q_i^{\tilde{b}_i}))$ satisfying (i) and (ii). $\square$

The next result indicates that the conjugacy family $\mathcal{T}(P)$ is in a certain sense minimal among the conjugacy families for P. For any conjugacy family $\mathcal{F}(P)$ set $\mathcal{F}(G) = \bigcup_{g \in G} \mathcal{F}(P)^g$.

Theorem 2. *Let $\mathcal{F}_1(P), \ldots, \mathcal{F}_r(P)$ be the distinct conjugacy families for P in G. Then $\bigcap_{i=1}^r \mathcal{F}_i(G) = \mathcal{T}(G)$.*

Proof. Clearly $\bigcap_{i=1}^r \mathcal{F}_i(G) \subseteq \mathcal{T}(G)$ since by Theorem 1 $\mathcal{T}(P)$ is a conjugacy family. Thus, it suffices to show that for every $Q \in \mathcal{T}(P)$ and $k \in \{1, \ldots, r\}$ there exists $x \in G$ such that $Q^x \in \mathcal{F}_k(P)$.

Assume that $Q \in \mathcal{T}(P)$ is a counterexample such that $|Q|$ is maximal. Let $M = N_G(Q)$ and $\overline{M} = M/Q$. Then there exists a strongly p–embedded subgroup $\overline{M}_0$ in $\overline{M}$ containing $\overline{M \cap P}$. Pick $h \in M \setminus M_0$. Then $P \cap P^h = Q$ since $N_{P \cap P^h}(Q) = Q$; in particular $Q, Q^{h^{-1}} \subseteq P$. Since $\mathcal{F}_k(P)$ is a conjugacy family there exist $Q_i \in \mathcal{F}_k(P)$ and p-elements $y_i \in N_G(Q_i)$, $i = 1, \ldots, n$ such that

(i) $h^{-1} = y_1 \cdots y_n b$, $b \in N_G(P)$, and

(ii) $Q^{y_1 \cdots y_i} \leq Q_i \leq P$.

Let R_0 be the set of all pairs (P^x, P^y) for which there exist $\tilde{x} \in N_G(P)x$, $\tilde{y} \in N_G(P)y$ and $K \in \mathcal{F}_k(P)$ such that

$(*)$ $\tilde{x}\tilde{y}^{-1} \in O^{p'}(N_G(K))$.

Let $\Lambda(C, R_0)$ be the corresponding graph. Define $P_0 = P$ and $P_i = P^{y_i^{-1} \cdots y_1^{-1}}$ for $i = 1, \ldots, n$. Then (i) implies that $P_n = P^h$, and (ii) gives

(iii) $Q \leq Q_{i+1}^{y_{i+1}^{-1} \cdots y_1^{-1}} \leq P_i \cap P_{i+1}$ for $i = 1, \ldots, n-1$.

Moreover, $(P_i, P_{i+1}) \in R_0$ since $y_{i+1} \in O^{p'}(N_G(Q_{i+1}))$. It follows that $(P_0, \ldots, P_n)$ is a path of $\Lambda(C, R_0)$.

Assume first that $P_i \cap P_{i+1} \neq Q$ for every $i = 1, \ldots, n-1$. Then also $N_{P_i \cap P_{i+1}}(Q) \neq Q$. On the other hand, if T_1 and T_2 are p-subgroups of M such that Q is properly contained in $T_1 \cap T_2$ and $T_1 \leq M_0$, then also $T_2 \leq M_0$ since $\overline{M}_0$ is strongly p-embedded in $\overline{M}$. It follows that $P^h \cap M \leq M_0$ and $h \in M_0$, a contradiction.

Thus there exists $j \in \{1, \ldots, n-1\}$ such that $P_j \cap P_{j+1} = Q$. But now (iii) implies that $Q^{y_1 \cdots y_{j+1}} = Q_{j+1} \in \mathcal{F}_k(P)$. □

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