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The Dissection of Five-Dimensional Simplices into Orthoschemes

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Abstract. In this paper the dissection of five-dimensional simplices into orthoschemes is shown. Firstly, some general methods for dissecting n-dimensional Euclidean simplices are described. For this, a description of simplices by graphs is given. All methods for cutting a simplex are investgated with the help of these graphs. The dissection of the five-dimensional Euclidean simplices is thoroughly investigated and it is shown that each of them is decomposable into orthoschemes. Furthermore, the validity of the propositions in spaces of constant curvature is checked and it is proved that every five-dimensional spherical or hyperbolic simplex can be dissected into orthoschemes.

0. Introduction

The calculation of the volume of an arbitrary polyhedron in spaces of constant curvature is based on the dissection of these polyhedrons into simplices. In the Euclidean space there is a simple formula for the volume of a n-dimensional simplex. But in non-Euclidean spaces a further dissection into more simple pieces is desirable. One possible class is the class of the so-called orthoschemes (or orthogonal simplices). Orthoschemes can be characterized as special simplices with a totally orthogonal edge path. They are a generalization of a triangle with a right angle and had been investigated by Schläfli [11] in the middle of the last century in context with investigations of the volume of spherical simplices. To be exact, simplex and orthoscheme can be defined as follows:
Let $\mathbb{E}^n$ be the n-dimensional Euclidean space.

Definition. A subset $S \subseteq \mathbb{E}^n$ is called n-dimensional *simplex* if and only if S is the convex hull of $n+1$ points $P_1, \ldots, P_{n+1}$ which are not contained in a $(n-1)$-dimensional hyperplane. The set of all n-dimensional simplices is denoted by $\mathfrak{S}^n$. A simplex $S \in \mathfrak{S}^n$

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can be represented by a definite vertex P_1 and the n linearly independent edge vectors $a_i := P_{i+1} - P_i$ for $i = 1, \ldots, n$:

$$\begin{aligned} S &=< P_1, a_1, \ldots, a_n > \\ &:= \{X : X = P_1 + \sum_{i=1}^{n} \alpha_i a_i \quad \text{with} \quad 1 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0\}. \end{aligned}$$

Definition. A simplex $T \in \mathfrak{S}^n$ is called an *orthoscheme* if and only if there is an suitable numbering of the vertices of T such that $T =< P_1, a_1, \ldots, a_n >$ with $(a_i,\, a_j) = 0$ for $i \neq j$. The sequence of edge vectors $a_1, \ldots, a_n$ is called a *totally orthogonal edge path.*

In his book "Vorlesungen über Inhalt, Oberfläche und Isoperimetrie" written in 1957 [8] the Swiss mathematician Hugo Hadwiger conjectured that an arbitrary simplex is decomposable into a finite number of orthoschemes. But whether or not the set of orthoschemes is sufficient to put together all simplices and, consequently, all polyhedrons, remains an unsolved question, even in the Euclidean space.

For the dimension $n = 2$ the dissection of a triangle without a right angle into two triangles with right angle is always possible. The two pieces arise by dropping the perpendicular from a suitable vertex to the opposite edge. The only problem is that the foot of the perpendicular is not outside the edge, but this is always possible. Dealing with this problem in case of the dimension three is not so simple. There exist tetrahedrons with the property that the foot of the perpendicular from an arbitrary vertex to the opposite facet is always outside the tetrahedron. In 1960, H. Chr. Lenhardt [10] was able to prove that an arbitrary tetrahedron is decomposable into 12 orthoschemes or less. J. Böhm [1] has proved that a tetrahedron which is decomposable into 12 orthoschemes but not in less really exists. Therefore, the number 12 can't be decreased as upper bound for the maximal minimum number of orthoschemes. For the four-dimensional case one possibility of dissection is given by Charsischwili in [5]. 730 is obtained as an upper bound of the minimum number of orthoschemes. In 1982 this dissection method was extended by J. Böhm and H. Schwulow [4] to include non-Euclidian simplices. In the papers [9], [12] the upper bound is finally reduced to 640, 610 and 500.

In this paper the hypothesis of H. Hadwiger is proved in five-dimensional spaces of constant curvature. For this, a new graphtheoretical method is chosen with the hope, that this method will be successful even in higher dimensions.

Firstly, the dissection of an n-dimensional Euclidian simplex S by a hyperplane is investgated. The hyperplane is chosen in a way that $n-1$ of the $n+1$ vertices of the simplex (for instance $P_3, \ldots, P_{n+1}$) are in this hyperplane. The n-th point which determines the hyperplane is an arbitrary point of the edge $\overline{P_1 P_2}$. The value of the dihedral angles (that is the angle between to facets of the simplex) is thoroughly investigated. Propositions are proved in what way the dihedral angles of the two subsimplices are linked with the dihedral angles of the original simplex S and with the exact position of the hyperplane. Conditions are given in particular for this when it is possible to find such a hyperplane that a determined angle of every subsimplex is a right angle. In this investigation an elegant representation of the anlge between two hyperplane depending on the barycentric coordinates of these hyperplanes with respect to the simplex S is used. These investigations are exactly desribed in [12].

1. Graph-theoretical description of simplices

For the investigation of whether or not a simplex is decomposable into orthoschemes, the number and mutual position of the obtuse dihedral angles are of interest. For a clear representation of all dihedral angles the following graphtheoretical description of simplices is suited.

The vertices of a complete graph $G = (V, E)$ are assigned to the $n+1$ facets of a simplex S. The edges of G can be divided into edges of three subgraphs G_+, G_0 and G_-.

An edge $\{a, b\}$ belongs to $G_+ = (V, E_+)$ if the dihedral angle between the two facets of S corresponding to a and b is acute. The edge $\{a, b\}$ belongs to $G_0 = (V, E_0)$ if the dihedral angle between the facets a and b is right and $\{a, b\}$ belongs to $G_- = (V, E_-)$ if this angle is obtuse. In the following figures the shown notation is always used.

Now such graphs are assigned to arbitrary n-dimensional simplices. Then the following characterizition of the simplices by its acute dihedral angles holds which was proved by M. Fiedler [6] in the nineteen fifties.

Theorem. *For every n-dimensional Simplex $S \in \mathfrak{S}^n$ the subgraph G_+ representing the acute edges is connected, and conversely, for every decomposition G_+, G_0 and G_-, for which G_+ is connected, an n-dimensional simplex S exists which corresponds to this decomposition.*

Consequently, every n-dimensional simplex has at least n acute dihedral angles. When a simplex S is dissected into two subsimplices S_1 and S_2 by a hyperplane, like mentioned above, two graphs G_1 and G_2 can be assigned to these subsimplices. It is possible to say how the graphs of the two subsimplices are linked with the graph of the original simplex. For this, the results which has been got with the help of the barycentric coordinates are transformed to the graphtheoretical language. Conditions are given in particular for this when the simplex S can be dissected into two subsimplices, such that for the graphs G^1 and G^2 a pre-assigned edge belongs to G^1_0 and to G^2_0, respectively.

2. The dissection of the five-dimensional Euclidean simplices

For the dissection of the five-dimensional simplices the graphtheoretical methods are used. With the help of the Fiedler's theorem a thorough classification of the simplices with respect to the number and mutual position of the obtuse dihedral is possible. This distinction of cases is used and one succeeds in finding a dissection for every type of simplex. A first upper bound for the maximal minimum number is given. The investigations yield the very large number of 12.5 millions orthoschemes. But this number is only a very rough estimation, the exact number is much less for special simplices and probably it is possible to reduce this bound in general.

The dissection of the simplices is carried out step by step. At first, very simple types of simplex are dissected into orthoschemes, more complex types are reduced to these simple types. Three considerable methods of dissection are used, the first method is cutting the simplex into two subsimplices by a hyperplane such that each of the two subsimplices has a definite right dihedral angle. The second method is cutting an obtuse angle into a right and an acute angle. The third method is dropping the perpendicular from a vertex to the opposite facets or within faces of lower dimension. It is possible to describe dropping the perpendicular with graphtheoretical methods, because it is equivalent to cutting the simplex by more than one hyperplane.

The first simplices thoroughly investigated are the simplices with a graph shown in Figure 1.

Figure 1

All edges which are not drawn can be chosen arbitrarily always on the condition that the subgraph G_+ is connected. Simplices with such a graph have an orthogonal edge path of length 3. They are nearly orthoschemes and it is easy to show that they can be dissected into 5 orthoschemes or less. The next group are the simplices with an orthogonal edge path of length 2. They have a graph shown in Figure 2.

Now there are already six edges not uniquely determined and it is much more difficult to show that these simplices can be dissected into 64 orthoschemes or less. Different constructions are necessary here, especially repeated dropping perpendiculars within faces of lower dimension. In order to dissect arbitrary five-dimensional simplices into orthoschemes a theorem is used which also holds in n-dimensional spaces, every n-dimensional simplex has an insphere. By dropping the perpendiculars to the facets from the centre of the

Figure 2

insphere, a dissection into 30 (in the five-dimensional case) simplices with an orthogonal edge path of length 1 arises.

Figure 3

In order to show that every five-dimensional simplex can be dissected into orthoschemes it is sufficient to dissect the simplices with an orthogonal edge path of length 1 into simplices with an orthogonal edge path of length 2 or 3. It is necessary to distinguish between two important cases.

First case. The vertex 2 doesn't belong to an edge of G_-. We have the possibilities which are shown in Figure 4.

Figure 4

Now it is easy to show that these simplices are decomposable into no more than four simplices with an orthogonal edge path of length 2, that is into no more than $4 \cdot 64 = 256$ orthoschemes.

Second case. The much more important case is that the vertex 2 belongs to at least one edge of G_- (Fig. 5).

Figure 5

Now all these simplices can be dissected into no more than four simplices with the following graph (Fig. 6):

Figure 6

The dissection of these simplices into orthoschemes is the most important part. Again six edges are not uniquely determined here. Each of them could belong to G_+, G_0 or

G_-. That are $3^6 = 729$ thinkable cases. Since G_+ must be connected only 201 cases remain. By interchanging the numbers 4 and 6 of the facets the graph G remains the same. Therefore, the number of cases is reduced a little. But in general all these cases must be considered and with the help of the mentioned methods it is possible to find a dissection into orthoschemes for each type of simplex. In the worst case one gets the large number of 175.212 orthoschemes.

In order to dissect an arbitrary five-dimensional Euclidean simplex into orthoschemes the perpendiculars from the centre of the insphere are dropped to the facets and a dissection into 30 simplices with an orthogonal edge path of length 1 arises. Each of these simplices can be dissected into four simplices with a graph in Figure 6 in the worst case. Each of these 120 pieces could have the worst type and is decomposable into 175.212 orthoschemes. But this "worst" case can't occur, thorough investigations which types occur simultaneously yield the number of 12.598.800 orthoschemes.

3. Examples

In order to illustrate the different methods of dissection two examples are given here. The first simplex has the following graph, shown in Figure 7.

Figure 7

This simplex has a graph shown in Figure 6. It can be dissected into two simplices with an orthogonal edge path of length 2, by cutting the simplex into two subsimplices in the following way.

It holds that $\{6,3\} \in K_+$ and $\{6,4\} \in K_+$. Therefore, a hyperplane α exists, dissecting S into S_1 and S_2 with $4 \subseteq S_1$ and $3 \subseteq S_2$ such that the graphs G^1 and G^2 satisfy the condition $\{6^1,\alpha\} \in K_0^1$ and $\{6^2,\alpha\} \in K_0^2$. The facet of S_1 and S_2 which newly arises is denoted by α, and all factes which are cut by the hyperplane are denoted into 6^1 and 6^2 for instance. Propositions detailed proved in [13] yield the following statements:

$\{1,3\} \in K_0$ and $\{1,4\} \in K_0 \Rightarrow \{1^1,\alpha\} \in K_0^1$ and $\{1^2,\alpha\} \in K_0^2$.

$\{2,3\} \in K_+$ and $\{2,4\} \in K_0 \Rightarrow \{2^1,\alpha\} \in K_+^1$ and $\{2^2,\alpha\} \in K_-^2$.

$\{5,3\} \in K_0$ and $\{5,4\} \in K_+ \Rightarrow \{5^1,\alpha\} \in K_-^1$ and $\{5^2,\alpha\} \in K_+^2$.

Since G_+^1 and G_+^2 must be connected graphs it holds that $\{4,\alpha\} \in K_+^1$ and $\{3,\alpha\} \in K_+^2$. G^1 and G^2 have the graphs shown in Figure 8.

Figure 8

Both S_1 and S_2 have an orthogonal edge path of length 2, which can easily be checked and thorough investgations, see [13], show that each of them is decomposable into no more than 128 orthoschemes, see [13], and as a consequence S is decomposable into 256 orthoschemes.

The second example illustrates the method of dropping perpendiculars within faces of lower dimension. Let S be a simplex with the graph in Figure 9.

Figure 9

This simplex S has also a graph of the type in Figure 6. $\{3, 6\} \in K_-$ holds. Therefore, a hyperplane α exists which dissects S into S_1 and S_2 with $6 \subseteq S_1$ and $3 \subseteq S_2$ such that G^1 and G^2 satisfy the condition $\{6, \alpha\} \in K_0^1$ and $\{3, \alpha\} \in K_+^2$. As in the first example it is possible to derive the following statements from propositions in [12] and [13], respectively.

$\{1, 3\} \in K_0$ and $\{1, 6\} \in K_0 \Rightarrow \{1^1, \alpha\} \in K_0^1$ and $\{1^2, \alpha\} \in K_0^2$.

$\{2, 3\} \in K_+$ and $\{2, 6\} \in K_0 \Rightarrow \{2^1, \alpha\} \in K_+^1$ and $\{2^2, \alpha\} \in K_-^2$.

$\{4, 3\} \in K_+$ and $\{4, 6\} \in K_0 \Rightarrow \{4^1, \alpha\} \in K_+^1$ and $\{4^2, \alpha\} \in K_-^2$.

The graph G_+^1 must be connected, hence $\{5^1, \alpha\} \in K_+^1$ and consequently $\{5^2, \alpha\} \in K_-^2$. The graphs G^1 and G^2 are shown in Figure 10.

The simplex S_1 has a very simple graph and it can be easily shown that it is decomposable into 6 orthoschemes. The dissection of the simplex S_2 is much more difficult and can only be sketched. $\{3, 2^2\}$, $\{3, \alpha\}$, $\{3, 5^2\}$, $\{3, 4^2\} \in K_+^2$ hold. Therefore, it is possible to drop the perpendicular from $Y := \alpha \cap \overline{P_3\ P_6}$ within the facet 1^2 to the face $1^2 \cap 3$ and the foot X of the perpendicular is in the simplex S_2. The simplex S_2 is dissected into four simplices S_{21}, S_{22}, S_{23} and S_{24}. Thorough investigations, see [13], especially about the

Figure 10

dihedral angles of lower order, show that these simplices have the four following graphs (Figure 11).

Figure 11

Obviously, all these four simplices have a graph of the type shown in Figure 5, that is they can easily be dissected into simplices with an orthogonal edge path of length 2. Strictly speaking, S_{21} can be dissected into 256 orthoschemes, S_{22} and S_{23} into 192 orthoschemes and S_{24} into 64 orthoschemes. Consequently, S is decomposable into no more than $6 + 256 + 192 + 192 + 64 = 710$ orthoschemes.

These two examples show the methods used to dissect all five-dimensional simplices into orthoschemes. In a similiar way all different types of simplices are decomposed. These investigations are desribed in detail in [13].

4. The dissection of non-Euclidean simplices

With regard to the calculation of the volume the dissection of simplices into orthoschemes in non-Euclidean spaces is desirable. The idea to dissect spherical simplices is the representation of an n-dimensional spherical simplex as intersection of $n+1$ hyperplanes with an n-dimensional sphere. With the help of this representation it is easy to see that the relations of the dihedral angles remain the same as in the Euclidean space when a simplex is cut by a hyperplane. Therefore, all propositions found with the help of the barycentric coordinates hold in the spherical space, too. The main problem of the dissection of spherical simplices is the fact that Fiedler's theorem doesn't hold in the spherical space. A simple counter-example is a triangle with three right angles. Similiar examples exist in every dimension. A first step in order to dissect the spherical simplices is the following theorem proved by J. Böhm, [2].

Theorem. *Every n-dimensional spherical simplex can be dissected into simplices with edges of a length less than $\pi/2$.*

But even for these simplices Fiedler's theorem doesn't hold, but one can prove the following fact: Every spherical simplex S can be dissected into simplices with the property that for their graph G^i the subgraph G^i_+ is connected. Further it is possible to show that the property "G_+ is connected" is preserved in all considered dissections.

Now the whole investigations for five-dimensional Euclidean simplices can be applied to the spherical simplices and it follows: Every five-dimensional spherical simplex can be dissected into orthoschemes.

In the hyperbolic space the relations are not so difficult. With the help of a suitable model of the hyperbolic space the validity of all propositions concerning the dihedral angles can be proved. Furthermore it can be shown that the important part of Fiedler's theorem is true even in the hyperbolic space.

Let S^H be a hyperbolic simplex. A graph $G^H = (E, K)$ is assigned to S^H as in the Euclidean space. Then the subgraph G^H_+ of the acute dihedral angles is a connected graph.

All propositions necessary for the dissection hold in the hyperbolic space and one gets: Each five-dimensional hyperbolic simplex is decomposable into orthoschemes.

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