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Centroaffinely Homogeneous Surfaces in $\mathbf{R}^3$

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Abstract. A nondegenerate centroaffine surface in $\mathbf{R}^3$ is called homogeneous, if for any two points p and q on the surface there exists a centroaffine transformation in $\mathbf{R}^3$ which takes the surface to itself and takes p to q. In this paper all centroaffinely homogeneous surfaces in $\mathbf{R}^3$ are determined up to centroaffine transformations.

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1. Introduction

In centroaffine differential geometry we study the properties of hypersurfaces in $\mathbf{R}^{n+1}$ that are invariant under the centroaffine group Γ in $\mathbf{R}^{n+1}$. Γ is the subgroup of the affine transformation group in $\mathbf{R}^{n+1}$ which keeps the origin $0 \in \mathbf{R}^{n+1}$ invariant. It is well-known that for any hypersurface immersion $x : M \to \mathbf{R}^{n+1}$ such that x is transversal to $x_*(TM)$ there exists a Γ-invariant symmetric bilinear form G on TM. x is then called a nondegenerate centroaffine hypersurface if G is nondegenerate.

An interesting class of centroaffine hypersurfaces in $\mathbf{R}^{n+1}$ is that of the so-called centroaffinely homogeneous hypersurfaces. A hypersurface in $\mathbf{R}^{n+1}$ is called centroaffinely homogeneous, if for any two points p and q on the hypersurface there exists a centroaffine transformation in $\mathbf{R}^{n+1}$ which takes the hypersurface to itself and takes p to q. Any homogeneous proper equiaffine sphere is evidently centroaffinely homogeneous if we move its center to the origin. So we have plenty examples of centroaffinely homogeneous hypersurfaces in $\mathbf{R}^{n+1}$, which can be found in H. Guggenheimer [2], K. Nomizu and T. Sasaki [4],

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F.Dillen and L. Vrancken [1]. Some examples of centroaffinely homogeneous hypersurfaces which are not equiaffine spheres were given by A. M. Li and the second author of this paper in [3].

Our purpose in this paper is to classify the centroaffinely homogeneous surfaces in $\mathbf{R}^3$. We obtain the following classification theorem:

Theorem: *Let $x: M \to \mathbf{R}^3$ be a nondegenerate centroaffinely homogeneous surface. Then x is centroaffinely equivalent to one of the following surfaces in $\mathbf{R}^3$:*

(i) $x_1^2 + x_2^2 + x_3^2 = 1$;

(ii) $x_1^\alpha x_2^\beta x_3^\gamma = 1,\ \alpha \neq 0,\ \beta > 0,\ \gamma > 0,\ \alpha + \beta + \gamma \neq 0$;

(iii) $exp(-\alpha \arctan \frac{x_1}{x_2})(x_1^2 + x_2^2)^\beta x_3^\gamma = 1,\ \alpha^2 + \beta^2 \neq 0,\ \gamma > 0,\ 2\beta + \gamma \neq 0$;

(iv) $x_3 = -x_1(\alpha \log x_1 + \beta \log x_2),\ \beta > 0,\ \alpha + \beta \neq 0$;

(v) $x_2 x_3 = \pm x_1^2 + x_2^\alpha,\ \alpha \neq 1, 2$;

where α, β, γ are constants.

2. The centroaffine surfaces in $\mathbf{R}^3$

Let $x: M \to \mathbf{R}^3$ be an immersion of surface and $[\ ,\ ,\ ]$ the standard determinant in $\mathbf{R}^3$. x is said to be a centroaffine surface if x is always transversal to the tangent plane $x_*(TM)$ at each point of M in $\mathbf{R}^3$. We define a symmetric bilinear form G on TM by

$$G = -\sum_{i,j=1}^{2} \frac{[e_1(x), e_2(x), e_i e_j(x)]}{[e_1(x), e_2(x), x]}\ \theta^i \otimes \theta^j, \tag{2.1}$$

where $\{e_1, e_2\}$ is a local basis for TM with the dual basis $\{\theta^1, \theta^2\}$. Note that G is globally defined. A centroaffine surface x is said to be nondegenerate if G is nondegenerate. We call G the centroaffine metric of x. From the equations

$$e_i e_j(x) = \sum_k \Gamma_{ij}^k e_k(x) - G_{ij} x,\ i, j = 1, 2, \tag{2.2}$$

we get the so-called induced connection $\nabla = \{\Gamma_{ij}^k\}$. It is clear that the centroaffine metric and the induced connection ∇ are invariant under the centroaffine transformation.

Let $\widehat{\nabla}$ be the Levi-Civita connection of G. We define the 3-form C by

$$\Gamma_{ij}^k - \widehat{\Gamma}_{ij}^k = C_{ij}^k,\ \ C_{ijk} = \sum_m G_{km} C_{ij}^m,\ \ i, j, j, m = 1, 2. \tag{2.3}$$

We know that $C = \sum C_{ijk} \theta^i \theta^j \theta^k$ is the Fubini-Pick form for x which is totally symmetric. The Chebyshev vector field and the Chebyshev form are defined by

$$T = \sum_j T^j e_j = \frac{1}{2} \sum_i G^{ik} C_{ik}^j e_j, \tag{2.4}$$

$$T^\sharp = \sum_{ij} G_{ij} T^i \theta^j. \tag{2.5}$$

By (2.2) and (2.3) we have $2T^\sharp = \mathrm{d}(\log |[e_1, e_2, x]|)$ for an orthonormal basis $\{e_1, e_2\}$ ([8]). As is well-known T and $T^\sharp$ are centroaffine invariants.

Let $x : M \to \mathbf{R}^3$ be a centroaffinely homogeneous surface. Since G and T are centroaffine invariants, by homogeneity, we know that $G(T,T) = constant$. Thus we have the following three cases: (i) $T \equiv 0$; (ii) $G(T,T) = 0$ but $T \neq 0$; (iii) $G(T,T) \neq 0$.

If $T \equiv 0$, x is a proper homogeneous equiaffine sphere in $\mathbf{R}^3$ centered at $0 \in \mathbf{R}^3$. From [2] and [4] we know that x is one of the following surfaces:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1; \\ x_1^2 + x_2^2 - x_3^2 &= \pm 1; \\ x_1 x_2 x_3 &= 1 \\ (x_1^2 + x_2^2) x_3 &= 1. \end{aligned}$$

From [3] we have the following examples of centroaffinely homogeneous surfaces not being equiaffine spheres:

Example 2.1: The surface defined by

$$x_1^\alpha x_2^\beta x_3^\gamma = 1, \ \ \alpha \neq 0, \ \beta > 0, \ \gamma > 0, \ \alpha + \beta + \gamma \neq 0, \tag{2.6}$$

is a homogeneous centroaffine surface in $\mathbf{R}^3$. Its group is

$$\left\{ \begin{pmatrix} e^{\gamma u} & 0 & 0 \\ 0 & e^{\gamma v} & 0 \\ 0 & 0 & e^{-\alpha u - \beta v} \end{pmatrix} \right\}$$

which acts on the point ${}^t(1,1,1)$, and

$$G = -\frac{(\alpha^2 + \alpha\gamma)}{(\alpha+\beta+\gamma)\gamma} \mathrm{d}u^2 - 2\frac{\alpha\beta}{(\alpha+\beta+\gamma)\gamma} \mathrm{d}u\mathrm{d}v - \frac{(\beta^2 + \beta\gamma)}{(\alpha+\beta+\gamma)\gamma} \mathrm{d}v^2;$$

$$\|T\|^2 = \frac{1}{4}\left(6 - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} - \frac{\beta}{\gamma} - \frac{\gamma}{\beta} - \frac{\gamma}{\alpha} - \frac{\alpha}{\gamma}\right).$$

Example 2.2: The surface defined by

$$exp\left(-\alpha \arctan \frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^\beta x_3^\gamma = 1, \ \ \alpha^2 + \beta^2 \neq 0, \ \gamma > 0, \ 2\beta + \gamma \neq 0, \tag{2.7}$$

is a homogeneous centroaffine surface in $\mathbf{R}^3$. Its group is

$$\left\{\begin{pmatrix} e^{\gamma u}\begin{pmatrix} \cos\gamma v & \sin\gamma v \\ -\sin\gamma v & \cos\gamma v \end{pmatrix} & 0 \\ 0 & e^{-2\beta u+\alpha v} \end{pmatrix}\right\}.$$

which acts on the point ${}^t(0,1,1)$, and

$$G = \frac{2\beta}{\gamma}\mathrm{d}u^2 + 2\frac{\alpha}{\gamma}\mathrm{d}u\mathrm{d}v + \frac{2\beta\gamma-\alpha^2}{(2\beta+\gamma)\gamma}\mathrm{d}v^2;$$

$$\|T\|^2 = \gamma^3(4\alpha^2\gamma + 8\beta^2\gamma - 4\beta^3 - \alpha^2\beta - 4\beta\gamma^2);$$

Example 2.3: The surface defined by

$$x_3 = -x_1(\alpha\log x_1 + \beta\log x_2), \quad \beta > 0, \ \alpha+\beta \neq 0,$$

is a homogenous centroaffine surface in $\mathbf{R}^3$. Its group is

$$\left\{\begin{pmatrix} e^u & 0 & 0 \\ 0 & e^v & 0 \\ -e^u(\alpha u+\beta v) & 0 & e^u \end{pmatrix}\right\},$$

which acts on the point ${}^t(1,1,0)$, and

$$G = \frac{\alpha}{\alpha+\beta}\mathrm{d}u^2 + 2\frac{\beta}{\alpha+\beta}\mathrm{d}u\mathrm{d}v - \frac{\beta}{\alpha+\beta}\mathrm{d}v^2;$$

$$\|T\|^2 = \frac{8\beta-\alpha}{4\beta}.$$

3. The centroaffinely homogeneous surfaces in $\mathbf{R}^3$ with $G(T,T)=0$ and $T\neq 0$

Let $x : M \to \mathbf{R}^3$ be a centroaffinely homogeneous surface in $\mathbf{R}^3$ with $G(T,T) = 0$ and $T \neq 0$. We define $e_1 = T$ and e_2 satisfying $G(e_1,e_2) = 1$, $G(e_2,e_2) = G(e_1,e_1) = 0$. Then we have the structure equations

$$\mathrm{d}x = \sum_i \theta_i e_i, \tag{3.1}$$

$$\mathrm{d}e_1 = \theta_{11}e_1 + \theta_{12}e_2 - \theta_2 x, \tag{3.2}$$

$$\mathrm{d}e_2 = \theta_{21}e_1 + \theta_{22}e_2 - \theta_1 x. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we get

$$\mathrm{d}\theta_1 = \theta_1 \wedge \theta_{11} + \theta_2 \wedge \theta_{21}, \tag{3.4}$$

$$\mathrm{d}\theta_2 = \theta_1 \wedge \theta_{12} + \theta_2 \wedge \theta_{22}. \tag{3.5}$$

Since

$$\mathrm{d}(\log|[e_1, e_2, x]|) = 2T^\sharp = 2T_1\theta_1 + 2T_2\theta_2 \quad and \quad G(T, e_i) = T^\sharp(e_i) = T_i,$$

we get

$$T = T_2 e_1 + T_1 e_2 = e_1, \text{ then } T_2 = 1 \; and \; T_1 = 0,$$

so we have

$$\mathrm{d}(\log|[e_1, e_2, x]|) = 2\theta_2. \tag{3.6}$$

From (3.1), (3.2) and (3.3), we get

$$\mathrm{d}\theta_{11} - \theta_{12} \wedge \theta_{21} + \theta_2 \wedge \theta_1 = 0, \tag{3.7}$$
$$\mathrm{d}\theta_{12} - \theta_{11} \wedge \theta_{12} - \theta_{12} \wedge \theta_{22} = 0, \tag{3.8}$$
$$-\mathrm{d}\theta_2 + \theta_{11} \wedge \theta_2 + \theta_{12} \wedge \theta_1 = 0, \tag{3.9}$$
$$\mathrm{d}\theta_{21} - \theta_{21} \wedge \theta_{11} - \theta_{22} \wedge \theta_{21} = 0, \tag{3.10}$$
$$\mathrm{d}\theta_{22} - \theta_{21} \wedge \theta_{12} + \theta_1 \wedge \theta_2 = 0, \tag{3.11}$$
$$-\mathrm{d}\theta_1 + \theta_{21} \wedge \theta_2 + \theta_{22} \wedge \theta_1 = 0. \tag{3.12}$$

From (3.5), (3.6) and (3.9),we have

$$\theta_1 \wedge \theta_{12} + \theta_2 \wedge \theta_{22} = 0, \quad \theta_{12} \wedge \theta_1 + \theta_{11} \wedge \theta_2 = 0. \tag{3.13}$$

Then by the Cartan's Lemma

$$\theta_{11} = \alpha\theta_1 + \nu\theta_2, \;\; \theta_{12} = \beta\theta_1 + \alpha\theta_2, \;\; \theta_{22} = \alpha\theta_1 + \delta\theta_2. \tag{3.14}$$

Let $\mathrm{d}\theta_1 = \rho\theta_1 \wedge \theta_2$, then (3.4) and (3.12) yield

$$\rho\theta_1 \wedge \theta_2 = \nu\theta_1 \wedge \theta_2 + \theta_2 \wedge \theta_{21},$$
$$\rho\theta_1 \wedge \theta_2 = -\delta\theta_1 \wedge \theta_2 + \theta_{21} \wedge \theta_2,$$

hence

$$\rho = \frac{1}{2}(\nu - \delta),$$
$$2\theta_{21} \wedge \theta_2 = (\nu + \delta)\theta_1 \wedge \theta_2,$$
$$\theta_{21} = \frac{1}{2}(\nu + \delta)\theta_1 + \varphi\theta_2. \tag{3.15}$$

Since x is homogeneous and $\{\theta_1, \theta_2\}$ is invariant under all centroaffine transformations which take the surface to itself, we know that all coefficients of (3.14) and (3.15) are constants. Moreover, (3.7), (3.14) and (3.15) yield

$$\alpha\nu - \beta\varphi = 1. \tag{3.16}$$

(3.8) and (3.14) yield

$$\beta\rho = 0. \tag{3.17}$$

(3.10), (3.14) and (3.15) yield

$$(\nu + \delta)\rho = 0. \tag{3.18}$$

(3.11), (3.14) and (3.15) yield

$$\beta\varphi - \alpha\delta + 1 = 0. \tag{3.19}$$

If $\rho \neq 0$, (3.17) and (3.18) become $\beta = 0$ and $\nu + \delta = 0$. (3.16) and (3.19) become $\alpha\nu = 1$ and $\alpha\delta = 1$. So $\nu = -\delta$ and $\nu = \delta$. But $\nu \neq 0$, $\delta \neq 0$. Therefore we obtain $\rho = 0$, ie. $\mathrm{d}\theta_1 = \mathrm{d}\theta_2 \equiv 0$. From (2.3), (3.14) and (3.15) we get $C_{ijk} = constant$. From [3] we know that x is one of the surfaces given by Example 2.1, 2.2 and 2.3.

4. The centroaffinely homogeneous surfaces in $\mathbf{R}^3$ with $G(T,T) \neq 0$

Let $x : M \to \mathbf{R}^3$ be a centroaffinely homogeneous surface in $\mathbf{R}^3$ with $G(T,T) \neq 0$. Since $\|T\|^2 = G(T,T) = constant \neq 0$, we define e_1 by

$$e_1 = \|T\|^{-1}T, \quad e_2 \perp e_1, \quad G(e_1, e_2) = \varepsilon_i\delta_{ij}, \tag{4.1}$$

where $\varepsilon_i = \pm 1$. Let $\{\theta_1, \theta_2\}$ be the dual basis of $\{e_1, e_2\}$. Then we have the structure equations

$$\mathrm{d}x = \sum_i \theta_i e_i, \tag{4.2}$$

$$\mathrm{d}e_i = \sum_j \theta_{ij} e_j - \varepsilon_i \theta_i x. \tag{4.3}$$

Differentiating (4.2) and (4.3) yield

$$\mathrm{d}\theta_i = \sum_j \theta_j \wedge \theta_{ji}, \tag{4.4}$$

$$\mathrm{d}\theta_i = \varepsilon_i \sum_j \varepsilon_j \theta_{ij} \wedge \theta_j, \tag{4.5}$$

$$\mathrm{d}\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj} - \varepsilon_i \theta_i \wedge \theta_j. \tag{4.6}$$

From $2T^\sharp = \mathrm{d}(\log|[e_1, e_2, x]|) = 2\sum_i T_i\theta_i = 2\varepsilon_1\|T\|\theta_1$, we have $\mathrm{d}\theta_1 = 0$. Then

$$\sum_j \theta_{j1}\wedge\theta_j = 0, \quad \sum_j \varepsilon_j\theta_{1j}\wedge\theta_j = 0. \tag{4.7}$$

By the Cartan's Lemma

$$\theta_{11} = \lambda\theta_1 + \mu\theta_2, \quad \theta_{21} = \mu\theta_1 + \varepsilon_1\varepsilon_2\beta\theta_2, \quad \theta_{12} = \varepsilon_1\varepsilon_2\mu\theta_1 + \alpha\theta_2. \tag{4.8}$$

Set $\mathrm{d}\theta_2 = \rho\theta_1\wedge\theta_2$. From (4.4) and (4.5) we get

$$\rho\theta_1\wedge\theta_2 = \theta_1\wedge\theta_{12} + \theta_2\wedge\theta_{22},$$
$$\rho\theta_1\wedge\theta_2 = \varepsilon_1\varepsilon_2\theta_{21}\wedge\theta_1 + \theta_{22}\wedge\theta_2.$$

Thus we obtain

$$\rho = \frac{1}{2}(\alpha - \beta), \tag{4.9}$$

$$\theta_{22} = \frac{1}{2}(\alpha + \beta)\theta_1 + \delta\theta_2. \tag{4.10}$$

By the homogeneity of the surface, we know that all $\lambda, \mu, \alpha, \beta, \delta, \rho$ here are constants.

From $\mathrm{d}\theta_{11} = \theta_{12}\wedge\theta_{21}$, we have $\rho\mu = \mu\beta - \mu\alpha = -2\mu\rho$, i.e.

$$\mu\rho = 0. \tag{4.11}$$

If $\rho = 0$, then $\mathrm{d}\theta_1 = \mathrm{d}\theta_2 = 0$. From (2.3), (4.8) and (4.10) we get $C_{ijk} = constant$ for all i, j, k. By [3] we know that x is one of the surfaces given by Example 2.1, 2.2 and 2.3.

Suppose $\rho \neq 0$. From (4.11) $\mu = 0$. So $\mathrm{d}\theta_{22} = \theta_{21}\wedge\theta_{12}$ yields $\delta = 0$. Thus we have

$$\theta_{11} = \lambda\theta_1, \quad \theta_{12} = \alpha\theta_2, \quad \theta_{21} = \varepsilon_1\varepsilon_2\beta\theta_2, \quad \theta_{22} = \frac{1}{2}(\alpha + \beta)\theta_1. \tag{4.12}$$

From $\mathrm{d}\theta_{12} = \theta_{11}\wedge\theta_{12} + \theta_{12}\wedge\theta_{22} - \varepsilon_1\theta_1\wedge\theta_2$ we get

$$\alpha^2 - \lambda\alpha + \varepsilon_1 = 0. \tag{4.13}$$

From $\mathrm{d}\theta_{21} = \theta_{21}\wedge\theta_{11} + \theta_{22}\wedge\theta_{21} + \varepsilon_2\theta_1\wedge\theta_2$ we get

$$\beta^2 - \lambda\beta + \varepsilon_1 = 0. \tag{4.14}$$

Since $\rho = \frac{1}{2}(\alpha - \beta) \neq 0$, we obtain

$$\lambda^2 - 4\varepsilon_1 > 0. \tag{4.15}$$

Therefore

$$\alpha = \frac{\lambda}{2} \pm \frac{\sqrt{\lambda^2 - 4\varepsilon_1}}{2}, \quad \beta = \frac{\lambda}{2} \mp \frac{\sqrt{\lambda^2 - 4\varepsilon_1}}{2}, \tag{4.16}$$

$$\rho^2 = (\frac{1}{2}(\alpha-\beta))^2 = \frac{1}{4}(\lambda^2 - \varepsilon_1) > 0.$$

Set $\theta_1 = \mathrm{d}u,\ \ \theta_2 = e^{\rho u}\mathrm{d}v,\ e_1 = x_u,\ e_2 = e^{-\rho u}x_v$. Then from (4.3) by a direct calculation we get

$$x_{uu} = \lambda x_u - \varepsilon_1 x. \tag{4.17}$$

$$x_{uv} = \alpha x_v, \tag{4.18}$$

$$x_{vv} = \varepsilon_1\varepsilon_2\beta e^{2\rho u}x_u - \varepsilon_2 e^{2\rho u}x. \tag{4.19}$$

From (4.17) and (4.16) we have

$$x = f(v)e^{\alpha u} + g(v)e^{\beta u}. \tag{4.20}$$

(4.20) and (4.18) yield

$$x_{uv} = \alpha f'(v)e^{\alpha u} + \beta g'(v)e^{\beta u} = \alpha f'(v)e^{\alpha u} + \alpha g'(v)e^{\beta u},$$

but $\alpha \neq \beta$, hence $g'(v) \equiv 0$ and $g(v) \equiv c$ is a constant vector in $\mathbf{R}^3$. Thus we have

$$x = f(v)e^{\alpha u} + ce^{\beta u}. \tag{4.21}$$

(4.21) and (4.19) yield

$$\begin{aligned} x_{vv} = f''(v)e^{\alpha u} &= \varepsilon_1\varepsilon_2\beta e^{2\rho u}(\alpha f(v)e^{\alpha u} + \beta ce^{\beta u}) - \varepsilon_2 e^{2\rho u}(f(v)e^{\alpha u} + ce^{\beta u}) \\ &= \alpha\beta\varepsilon_1\varepsilon_2 f(v)e^{(2\rho+\alpha)u} + \beta^2\varepsilon_1\varepsilon_2 ce^{\alpha u} - \varepsilon_2 f(v)e^{(2\rho+\alpha)u} - \varepsilon_2 ce^{\alpha u} \\ &= \varepsilon_1\varepsilon_2 c(\beta^2 - \varepsilon_1)e^{\alpha u}, \end{aligned}$$

so we get

$$f''(v) = \varepsilon_1\varepsilon_2(\beta^2 - \varepsilon_1)c, \quad \text{i.e.}$$

$$f(v) = \frac{1}{2}\varepsilon_1\varepsilon_2(\beta^2 - \varepsilon_1)cv^2 + av + b$$

for some constant vectors $a, b \in \mathbf{R}^3$. Therefore we obtain

$$\begin{aligned} x &= [\frac{1}{2}\varepsilon_1\varepsilon_2(\beta^2 - \varepsilon_1)cv^2 + av + b]e^{\alpha u} + ce^{\beta u} \\ &= (a, b, c)\begin{pmatrix} ve^{\alpha u} \\ e^{\alpha u} \\ \frac{1}{2}\varepsilon_1\varepsilon_2(\beta^2 - \varepsilon_1)v^2 e^{\alpha u} + e^{\beta u} \end{pmatrix}, \end{aligned} \tag{4.22}$$

where a, b and c are constant vectors in $\mathbf{R}^3$. Thus up to a centroaffine transformation we have

$$x = \left(ve^{\alpha u}\ ,\ e^{\alpha u}\ ,\ kv^2 e^{\alpha u} + e^{\beta u}\right) \tag{4.23}$$

for $k = \frac{1}{2}\varepsilon_1\varepsilon_2(\beta^2 - \varepsilon_1)$. It is a homogeneous surface with the group

$$\left\{ e^{\alpha u} \begin{pmatrix} e^{-\rho u} & v & 0 \\ 0 & 1 & 0 \\ 2kve^{-\rho u} & kv^2 & e^{-2\rho u} \end{pmatrix} \right\}$$

which acts on the point ${}^t(0,1,1)$. Its centroaffine metric is given by

$$G = \varepsilon_1 \mathrm{d}u^2 + \varepsilon_2 e^{(\alpha-\beta)u}\mathrm{d}v^2,$$

which has (nonzero) constant curvature. Its Chebyshev vector field satisfies

$$\|T\|^2 = \varepsilon_1 \frac{9}{16}(\alpha+\beta)^2.$$

Its Pick invariant

$$J = \frac{1}{2}\sum_{ijkl}^{2} G^{ij}C^l_{ik}C^k_{jl} = \frac{7}{8}\varepsilon_1(\alpha+\beta)^2.$$

Using the Cartesian coordinates (x_1, x_2, x_3) we can write the surface (4.23) as

$$x_2x_3 = kx_1^2 + x_2^{\varepsilon_1\beta^2+1}. \tag{4.24}$$

Since $k \neq 0$ and $\beta \neq 0$, it is centroaffinely equivalent of one of the surfaces given by (v) in the theorem in [1].

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