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Symmetry Breaking of the Cube Tiling and the Spatial Chess Board by D-Symbols

Emil Molnár
Technical University of Budapest, Department of Geometry
Egry J. út 1. H.22, H-1521 Budapest XI
molnar_e@dora.inflab.bme.hu

Abstract. The purpose of this paper is to popularize the method of D-symbols (Delone–Delaney–Dress symbols) for classifying periodic tilings $(\mathcal{T},\Gamma)$, where a group Γ acts on a d-dimensional combinatorial tiling $\mathcal{T}$ with prescribed transitivity properties. As a new result, we describe the equivariant classification of all cube tilings $(\mathcal{T},\Gamma)$ in Euclidean space $\mathbf{E}^3$, where the group Γ acts on the infinite face-to-face cube tiling $\mathcal{T}$ with 1 transitivity class (Γ-orbit) of faces. In Table 1–3 the 43 classes of these pairs $(\mathcal{T},\Gamma)$ are characterized by giving for each the number of the space group Γ, the D-symbol, a fundamental domain $\mathcal{F}$ in Fig.2, face identifying generators and defining relations for $\mathcal{F}$ and the Γ-conjugacy classes for stabilizers of solids (Γ^3), faces (Γ^2), edges (Γ^1) and vertices (Γ^0), respectively. Such classification problems seem to have applications, e.g. in the theory and praxis of solid state matter (Dirichlet–Voronoi tilings and coordination polyhedra). Therefore, we survey the general theory and indicate some problems as well.

1. Introduction

The chess board can be extended to an infinite square tiling with or without black and white coloring. Each of both can be characterized by the *group of its self-symmetries.* For the black and white tiling $\mathcal{T}_{ch}$ this group is $\Gamma_{ch} = \mathbf{p4m}$ *a plane crystallographic group,* which acts on $\mathcal{T}_{ch}$ with 2 (black resp. white) classes (Γ_{ch}-orbits) of tiles, 1 class of edges and 1 class of vertices (Fig.1.3.1). For the usual square tiling $\mathcal{T}_{sq}$ we have the isomorphic symmetry group $\Gamma_{sq} = \mathbf{p4m}$ (Fig.1.1.1) but Γ_{sq} has only 1 *tile orbit,* 1 *edge orbit* and 1 *vertex orbit.* All the 14 classes of pairs $(\mathcal{T},\Gamma)$ have been classified [6], [7], [11] where Γ acts on the infinite combinatorial square tiling $\mathcal{T}_{sq} = \mathcal{T}$ with 1 edge orbit. From these "isotoxal" square tilings we can choose particular *symmetry breakings* of $\mathcal{T}_{ch}$ or $\mathcal{T}_{sq}$ (Fig.1) and their *realizations* (C: convex, N: non-convex, M: marked).

Analogous classification problems arise from the metric cube tiling $\mathcal{T}_{cu}$ (with or without coloring) in Euclidean space $\mathbf{E}^3$. *In this paper all 43 classes of* $(\mathcal{T},\Gamma)$ *pairs are described* *where a space group* Γ *acts on a combinatorial cube tiling* $\mathcal{T}_{cu} = \mathcal{T}$ *with exactly 1 face class* (face orbit under Γ). In Tables 1–3 we see also particular symmetry breakings $(\mathcal{T}, \Gamma)$. For instance, we obtain 17 classes with exactly 1 Γ-orbit of solids, faces, edges, vertices, respectively (in Tables 1–2, (1)–(17)).

Furthermore we find 4 pairs $(\mathcal{T}, \Gamma)$ where the space group Γ acts on the cube tiling $\mathcal{T}$ with 2 solid orbits, 1 face orbit, 1 edge orbit and 1 vertex orbit (Table 3 $(1^\star)$–$(4^\star)$). These are the 4 symmetry breakings of the infinite spatial chess board with black and white coloring, in a restricted sense.

In this paper we intend to popularize *the method of D-symbols (Delaney symbols, generalized Schläfli symbols), developed for classifying d-dimensional tilings with prescribed transitivity properties.* A.W.M. Dress initiated this method [3] completing some ideas of M.S. Delaney [1]. The origins go back to the adjacency symbols of B.N. Delone (Delaunay) [2]. G. Grünbaum and G.C. Shephard applied adjacency symbols in various classifications in the Euclidean plane [7, Ch.6]. A. Dress and his coworkers R. Scharlau, R. Franz, D. Huson have developed a nice theory which is connected to the theory of generalized Coxeter groups (see [4], [5], [8], [9] for other references). They applied this theory to classifying 2-dimensional tilings. O. Delgado Friedrichs and D. Huson elaborated important and effective computer programs for Delaney symbols. I suggest using the name D-symbol in honour of Delone–Delaney–Dress.

Recently in [5] we have combined this theory with my *polyhedron algorithm which finds all the discontinuous transformation groups for combinatorially given d-dimensional fundamental polyhedron, together with their stabilizers for any m-face class* $0 \leq m \leq d-1$; $d \geq 2$ [15]. In the joint work [5] we have classified the 88 face transitive tilings of Euclidean space $\mathbf{E}^3$ by computer. The algorithm promises to be useful in other classifications in Euclidean and non-Euclidean spaces, e.g. *minimal presentations for space groups* (see [13], [17] or *polyhedron realizations for 3-manifolds* [14], etc.

We intend to illustrate the method with the familiar square and cube tilings and to give general ideas *how to find tilings with prescribed transitivity properties.* The results, appearing in Tables 1–3 and Fig.2, have been obtained by hand. These experiences of the author, however, were helpful in finding the general algorithm in [5] which will be sketched in Section 4.

2. The Concept of D-Symbol Illustrated by Square Tilings

Figure 1 shows the 14 equivariant classes of edge transitive symmetry breakings of the usual square tiling $\mathcal{T}$ which are considered combinatorially (or topologically, with Schläfli symbol $\{4,4\}$). That means $\mathcal{T}$ is an incidence structure of 2-dimensional faces (squares), 1-dimensional sides (curved edges) and 0-dimensional vertices (points). The *formal barycentric subdivision* of $\mathcal{T}$ leads to the simplex (triangle) set $\mathcal{C}$ where the (formal) midpoints of i-dimensional pieces of $\mathcal{T}$ (faces, edges, vertices) are labelled by $i \in I := \{0,1,2\}$ as are the opposite simplex sides as well. Thus, a flag of $\mathcal{T}$, i.e. a triple of incident 0-, 1-, 2-pieces, can uniquely be characterized by a labelled simplex of $\mathcal{C}$. Two simplices C_1 and C_2 of $\mathcal{C}$ are called *i-adjacent* iff they have a common i-side, i.e. only their i-vertices are different. We also say that the *involutive i-operation* σ_i carries C_1 onto $C_2 =: \sigma_i C_1$. That means σ_i

Figure 1

is first a permutation of simplices of $\mathcal{C}$ which assigns to any simplex of $\mathcal{C}$ the i-adjacent one.

(Later $\sigma_i C_1$ will be considered also as an image of C_1 under the "local topological reflection" σ_i in the i-facet of C_1. Then our (necessarily) strange notations

$$C^{\gamma_1\gamma_2} := (C^{\gamma_1})^{\gamma_2}, \qquad \sigma_j\sigma_i C := \sigma_j(\sigma_i C), \qquad (\sigma_j C)^{\gamma} = \sigma_j(C^{\gamma})$$

show how to get the corresponding simplex. Here γ's in the exponent are global mappings of $\mathcal{C}$, preserving the j-adjacencies, $j \in I = \{0,1,2\}$; σ_i is a local mapping, then it always relates to a fixed simplex C of $\mathcal{C}$, so that σ_i preserves the j-adjacencies of C and $\sigma_i C$. In this sense we say: if the transformations of $\mathcal{C}$ act from the right, then the σ_i-operations act on $\mathcal{C}$ from the left. Our convention is to be in accordance with our Tables 1–3 and with our earlier papers [13]–[18]. See Sect.4 for more details and draw pictures.)

At a k-vertex of any $C \in \mathcal{C}$ some simplices of $\mathcal{C}$ meet cyclically at their i- and j-sides ($i, j, k \in I$ are different). Let $2 \cdot m_{ij}(C)$ denote the number of these joining simplices ($m_{ij}(C) \in \mathbf{N}$ the set of natural numbers). Then we can write

$$(2.1) \qquad (\sigma_j\sigma_i)^{m_{ij}(C)} C = C$$

and the *symmetric matrix function* $M : \mathcal{C} \to \mathbf{N}_{I\times I}$, $C \mapsto M(C) = (m_{ij}(C))$ will be introduced to describe the combinatorial structure of $\mathcal{T}$.

In case of the square tiling $\mathcal{T} = \mathcal{T}_{sq}$

$$(2.2) \qquad \big(m_{ij}(C)\big) = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$$

is a constant matrix on $\mathcal{C}$ ($m_{ii} = 1$ means σ_i is involutive, $\{m_{01}, m_{12}\} = \{4,4\}$ is the Schläfli symbol, $m_{02} = 2$ since $|i-j| > 1$).

Now we assume that an isometry group Γ acts on $\mathcal{T}$ (as in our 14 cases in Fig.1) which is compatible with the barycentric subdivision $\mathcal{C}$ and the σ_i-operations as well. Γ *factors the simplices of* $\mathcal{C}$ *into* Γ*-orbits,* and it factors the matrix function as well. Edge-transitivity of Γ easily implies [11] that any Γ-orbit will be an element of a finite set $\mathcal{D} := \{D_1, D_2, \ldots, D_n\}$ (in Fig.1 $D_k \leftrightarrow k$, $n = 1,2,4$). From $n =: |\mathcal{D}|$ representative simplices of $\mathcal{D}$ a *fundamental domain* $\mathcal{F}$ for Γ can be glued by the σ_i-operations. Any σ_i-operation, defined on $\mathcal{C}$, induces a definition of σ_i on $\mathcal{D}$ in a natural way ($i \in I := \{0,1,2\}$). Furthermore, if $D := C^{\Gamma} := \{C^{\gamma} : \gamma \in \Gamma\}$ is the Γ-orbit of C in $\mathcal{C}$, then $m_{ij}(D) := m_{ij}(C)$ by (2.2) defines the matrix function

$$(2.3) \qquad \mathcal{M} : D \longrightarrow \mathbf{N}_{I\times I}, \quad D \longmapsto \mathcal{M}(D) = \big(m_{ij}(D)\big)$$

on the *D-diagram* (D-graph). $\mathcal{D}$ *and* $\mathcal{M}$ *together are called a D-symbol* $\mathcal{D}(\mathcal{M})$.

In our example Fig.1.3.1 $\mathcal{D}(\mathcal{M})$ determines the group $\Gamma = \mathbf{p4m}$ which has a fundamental domain $\mathcal{F}$ consisting of 2 simplices of $\mathcal{C}$. Thus the D-diagram $\mathcal{D}$ has 2 nodes (vertices). The σ_i-operations

$$(2.4) \qquad \sigma_0 : (1)(2) \cdots\cdots, \quad \sigma_1 : (1)(2) - - -, \quad \sigma_2 : (1,2) \text{ ——},$$

as involutive permutations, are indicated by the above i-lines as distinguished (colored) edges. These σ_i's also show the line reflections

$$m_0 \text{ and } m_0' \text{ (by } \sigma_0), \quad m_1 \text{ and } m_1' \text{ (by } \sigma_1) \tag{2.5}$$

in the sides of $\mathcal{F}$ as *generators for* $\Gamma = \mathbf{p4m}$. Any *partial D-symbol* $\mathcal{D}^k(\mathcal{M}) = \mathcal{D}_{ij}(\mathcal{M})$ of $\mathcal{D}(\mathcal{M})$ $(i,j,k \in I$ are different), obtained by cancelling the σ_k-operation, i.e. the k-lines, describes the simplicial neighbourhoods of the various k-vertices in $\mathcal{C}$ and the stabilizer subgroups Γ^k in Γ by a local fundamental domain $\mathcal{F}^k$. *We shall have precisely as many* Γ*-conjugacy classes for* Γ^k*, as we have* Γ*-orbits for* k*-vertices, as we have connected components of* $\mathcal{D}^k = \mathcal{D}_{ij}$ *in* $\mathcal{D}(\mathcal{M})$.

In our case 3.1 $\mathcal{D}(\mathcal{M})$ has 2 components of $\mathcal{D}^2 = \mathcal{D}_{01}$ each with $1 =: e_{01}$ node (D-simplex). As $m_{01}(D_1) = 4 = m_{01}(D_2)$, so we have the *defining relations* for our generators in (2.5):

$$(m_0m_1)^4 = 1 = (m_0'm_1')^4, \text{ since } m_{01}/e_{01} = 4.$$

Similarly, we have 1 component of $\mathcal{D}^1 = \mathcal{D}_{02}$, consisting of $2 =: e_{02}$ nodes, and $m_{02}(D_1) = 2 = m_{02}(D_2)$ involves the relation

$$m_0m_0' = 1 \text{ or } m_0 = m_0', \text{ since } m_{02}/e_{02} = 1.$$

We have 1 component of $\mathcal{D}^0 = \mathcal{D}_{12}$ consisting of $2 =: e_{12}$ nodes, thus $m_{12}(D_1) = 4 = m_{12}(D_2)$ implies the relation $(m_1m_1')^2 = 1$ because $m_{12}/e_{12} = 2$ is just the exponent. So we see, the D-symbol $\mathcal{D}(\mathcal{M})$ determines the group $\Gamma = \mathbf{p4m}$ by the presentation

$$\begin{aligned} \Gamma = \Big(m_0, m_0', m_1, m_1' \quad - \quad & 1 = m_0^2 = {m_0'}^2 = m_1^2 = {m_1'}^2 = \\ & = (m_0m_1)^4 = (m_0'm_1')^4 = m_0m_0' = (m_1m_1')^2\Big). \end{aligned} \tag{2.6}$$

We choose a common edge (connecting 0- and 1-vertices) of simplices D_1 and D_2 in the fundamental domain $\mathcal{F}$, with great freedom, and apply the isometries of Γ. So we get the simplices of $\mathcal{C}$ and the square tiling $(\mathcal{T}, \Gamma)$ determined by $\mathcal{D}(\mathcal{M})$ up to an equivariant deformation.

In Sect.4.3 we shall give a general definition of the *equivariant classification of d-dimensional tilings* $(\mathcal{T}, \Gamma)$*, and the isomorphism classification of D-symbols* $\mathcal{D}(\mathcal{M})$, and we will see that the two definitions correspond to each other (Th.4.1).

In the cases described in Fig.1 all the 14 square tilings $(\mathcal{T}_s, \Gamma)$ are *edge-transitive,* i.e. each D-symbol $\mathcal{D}(\mathcal{M})$ has a partial symbol $\mathcal{D}^1 = \mathcal{D}_{02}$ consisting of one component. Since any edge in $\mathcal{T}_{sq}$ belongs to 4 simplices in the barycentric subdivision $\mathcal{C}_{sq}$, a fundamental domain $\mathcal{F}$ contains at most 4 simplices and so does the D-set $\mathcal{D}$. Thus we can easily check by hand that our classification is complete.

We emphasize that combinatorially different fundamental domains $\mathcal{F}_1$ and $\mathcal{F}_2$ for the group Γ may provide the same (equivariant) tiling (see e.g. the cases 2.1–5 in Fig.1). This is in accordance with the fact that $\mathcal{F}_1$ may be rebuilt to $\mathcal{F}_2$ by cutting and gluing simplices at the labelled sides as the σ_i-operations dictate in the D-symbol $\mathcal{D}(\mathcal{M})$.

Summarizing, a D-symbol $\mathcal{D}(\mathcal{M})$ can geometrically be considered as an "amoeba-like" fundamental domain $\mathcal{F}$ with a numbered simplex subdivision and identifications, where labelled sides are distinguished by the σ_i-operation ($i \in I$), for the possible cutting and gluing procedures.

In Sect.4.3 we shall also define the *symmetry breakings of a given tiling $\mathcal{T}$* which will be related with the *D-morphisms* onto the D-symbol of ($\mathcal{T}$, $\Gamma = \mathrm{Aut}\,\mathcal{T}$).

Actually, we can say that all the 14 edge-transitive square tilings in Fig.1 can be obtained by symmetry breaking from the most symmetric one ($\mathcal{T}_{sq}$, $\Gamma_{sq} = \mathrm{Aut}\,\mathcal{T}_{sq} = \mathbf{p4m}$) under case 1.1.

3. Symmetry Breakings of the Cube Tilings in $\mathbf{E}^3$

The concept of D-symbol for any 3-dimensional tiling $(\mathcal{T}, \Gamma)$ can be formed in full analogy to Sect.2. We consider a Γ-compatible barycentric subdivision of $\mathcal{T}$ into the set $\mathcal{C}$ of labelled simplices. The σ_i-operations are introduced for $i \in I := \{0, 1, 2, 3\}$. The new σ_3-operation or 3-adjacency is denoted by $\sim\!\sim\!\sim$ lines in any D-diagram. Here the D-set $\mathcal{D} := \mathcal{C}/\Gamma$ is the set of Γ orbits of $\mathcal{C}$, and the σ_i-operations ($i \in I$) are again induced from $\mathcal{C}$.

$\mathcal{D}$ can be characterized by an "amoeba-like" fundamental domain $\mathcal{F}$ consisting of labelled simplices, each of them representing Γ-orbit in $\mathcal{C}$. Moreover, the σ_i-operations ($i \in I$) define how to glue these simplices together along their faces, and how to pair the faces of $\mathcal{F}$, respectively. The face pairing isometries generate the group Γ. Again, the symmetric matrix function $C \mapsto M(C) = (m_{ij}(C))$ $i, j \in I := \{0, 1, 2, 3\}$ on $\mathcal{C}$ determines the combinatorial structure of $\mathcal{T}$. By Γ-factorization we again obtain the matrix function in formula (2.3) and the D-symbol $\mathcal{D}(\mathcal{M})$ for $(\mathcal{T}, \Gamma)$.

If $\mathcal{T}_{cu}$ is the familiar cube tiling of the Euclidean space $\mathbf{E}^3$ then the matrix function is constant on $\mathcal{C}_{cu}$:

$$(m_{ij}) = \begin{pmatrix} 1 & 4 & 2 & 2 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}. \tag{3.1}$$

Here $m_{01} = 4$ describes the square faces, since 8 simplices meet at any 23-edge in the barycentric subdivision $\mathcal{C}_{cu}$ cyclically along their 0- and 1-faces. A 23-edge connects a (2-dimensional) face centre with a (3-dimensional) solid centre. Analogously, $m_{23} = 4$ implies that 4 cubes of $\mathcal{T}_{cu}$, i.e. 8 simplices of $\mathcal{C}_{cu}$ meet at any 01-edge cyclically along the 2- and 3-faces. To describe the 3 faces and edges at any vertex of the cube, we write $m_{21} = 3$ in (3.1). Indeed, 6 simplices of $\mathcal{C}_{cu}$ meet at any 03-edge cyclically along the 1- and 2-faces.

Any group Γ, acting on the cube tiling $\mathcal{T}_{cu}$, induces a D-symbol $\mathcal{D}(\mathcal{M})$ with the constant matrix (3.1). The D-symbol also describes a fundamental domain $\mathcal{F}$ for Γ, consisting of $|\mathcal{D}|$ simplices as indicated above.

The automorphism group of the combinatorial cube tiling $\mathcal{T}$ can be considered as the crystallographic space group $\Gamma = \mathbf{Pm\bar{3}m}$ of the metric cube tiling ($\mathcal{T}_{cu}$, $\Gamma = \mathbf{Pm\bar{3}m}$). The fundamental domain $\mathcal{F} = AEKQ$ is the simplex illustrated in Fig.2.a. Thus the D-diagram is a node, together with the four σ_i-loops ($i \in I$), as Table 1.No.1 shows it. All

Figure 2

the identifications on $\mathcal{F}$ are plane reflections m_0, m_1, m_2, m_3 in the corresponding simplex faces. The matrix (3.1) implies the defining relations for Γ in Table 1, as usual for so-called Coxeter groups generated by plane reflections.

In general, the partial D-symbol components obtained from $\mathcal{D}$ by cancelling any σ_k-operation, characterizes the lower-dimensional simplicial neighbourhoods of the k-midpoint orbit(s) in $\mathcal{C}$ and their stabilizer(s) Γ^k in Γ.

E.g. by cancelling σ_3-operation in Table 1.No.1, we get $\mathcal{D}^3(\mathcal{M}) = \mathcal{D}_{012}(\mathcal{M})$ with the corresponding 3×3 submatrix of (3.1). This describes a spherical tiling around any 3-vertex of $\mathcal{C}_{cu}$ with stabilizer $\Gamma^3(Q) = \mathbf{m\bar{3}m}$ as a crystallographic point group, the symmetry group of a fixed cube. Analogously, we get the stabilizer $\Gamma^2(K)$ of the 2-midpoint and every point in its Γ-orbit (and for the corresponding cube face orbit), resp. the stabilizer $\Gamma^0(A)$ of the 0-vertex orbit in Fig.2.a.

We are searching for all the symmetry breakings $(\mathcal{T}, \Gamma)$ of the Euclidean cube tiling $\mathcal{T}_{cu} = \mathcal{T}$ where, in addition, the isometry group Γ acts on $\mathcal{T}$ with one transitivity class of $\mathcal{T}$-faces.

To do this, we classify all D-symbols $\mathcal{D}(\mathcal{M})$ (up to D-isomorphism) *with constant matrix function* $\mathcal{M} : \mathcal{D} \to \mathbf{N}_{I\times I}$, $D \mapsto (m_{ij}(D))$ *by (3.1) such that the partial symbol* $\mathcal{D}^2(\mathcal{M}) = \mathcal{D}_{013}(\mathcal{M})$, *obtained from* $\mathcal{D}(\mathcal{M})$ *by cancelling the* σ_2*-operation, has a connected D-diagram. Moreover, all the stabilizers* $\Gamma^l < \Gamma$, *according to the components of the partial symbol* $\mathcal{D}^l(\mathcal{M}) = \mathcal{D}_{ijk}(\mathcal{M})$ ($i, j, k, l \in I$ are different), *are finite* (spherical) *point groups.*

In other words we will look for all the above D-symbols and the corresponding fundamental domains $\mathcal{F}$. Such a domain $\mathcal{F}$ will contain a piece of a cube face as Fig.2 shows, i.e. $\mathcal{F}$ consists of $|\mathcal{D}| = 1, 2, 4, 8, 16$ simplices supporting the cube face from one side or both sides. The face pairing on $\mathcal{F}$ will generate a crystallographic space group Γ in $\mathbf{E}^3$.

As we will see, there are 43 cases, enumerated in Tables 1–3, referring to Fig.2.

3.1. Table 1

Consider the cases of Fig.2.a–h, only the upper domains with 3-midpoint (cube centre) Q. Then all the cube centres are in one Γ-orbit and the face stabilizer $\Gamma^2(K)$ changes the sides of the face. Any such D-symbol $\mathcal{D}(\mathcal{M})$ has exactly one $\mathcal{D}^{23} = \mathcal{D}_{01}$ component (if we cancel the σ_2- and σ_3-operations and keep σ_0 and σ_1 of $\mathcal{D}(\mathcal{M})$).

For instance, let us consider the case No.11 where in Fig.2.e we have chosen the domain $\mathcal{F} = ABFGQ$. The D-diagram dictates the face pairing generators r_1, r, r', r_3 as described there (see also Sect.4.2). The components of $\mathcal{D}_{01} = \mathcal{D}^{23}$, $\mathcal{D}_{02} = \mathcal{D}^{13}$, $\mathcal{D}_{03} = \mathcal{D}^{12}$, $\mathcal{D}_{12} = \mathcal{D}^{03}$, $\mathcal{D}_{13} = \mathcal{D}^{02}$, $\mathcal{D}_{23} = \mathcal{D}^{01}$, respectively, provide the defining relations:

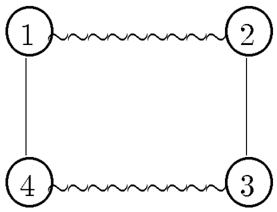
$$\begin{aligned} 1 &= r_1^2 = r_1 r r' = r_3^2 = (r_1 r_3)^2 = r^3 = {r'}^3 = \\ &= \text{trivial} = (r_3 r r_3 r')^2 \end{aligned} \tag{3.1.1}$$

which also describe the corresponding stabilizers

$$\begin{array}{ll} \Gamma^{23}(KQ) = \mathbf{2}, & \Gamma^{13}(FQ, GQ, EQ) = \mathbf{1}, \\ \Gamma^{12}(EK) = \mathbf{2}, & \Gamma^{12}(FK, GK) = \mathbf{2}, \\ \Gamma^{03}(AQ) = \mathbf{3}, & \Gamma^{03}(BQ) = \mathbf{3}, \\ \Gamma^{02}(AK, BK) = \mathbf{1}, & \Gamma^{01}(BF, AG, AE, BE) = \mathbf{2} \end{array}$$

of the simplex edge classes of $\mathcal{C}_{cu}$, respectively.

How to get the last relation, say, from $\mathcal{D}_{23} = \mathcal{D}^{01}$ whose diagram is



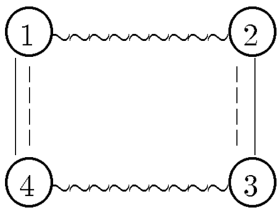
with $m_{23} = 4$.

This dictates the cyclic order of the Γ-images of the representing simplices from $\mathcal{F}$ around BF in $\mathcal{C}_{cu}$ as follows (see Fig.2.e):

$$\begin{aligned} & D_3 = BFKQ; \quad D_2^{r'} = (BEKQ)^{r'}; \quad D_1^{r_3 r'} = (AEKQ)^{r_3 r'}; \\ (3.1.2) \qquad & D_4^{r r_3 r'} = (AGKQ)^{r r_3 r'}; \quad D_3^{r_3 r r_3 r'} = (BFKQ)^{r_3 r r_3 r'}; \\ & \text{and so on up to } D_3^{(r_3 r r_3 r')(r_3 r r_3 r')} = D_3, \end{aligned}$$

since we have 8 simplices because of $m_{23} = 4$. So we have just obtained the relation $(r_3 r r_3 r')^2 = 1$. The general procedure for dimension d, where we go around a $(d-2)$-dimensional simplex "edge" in $\mathcal{C}$ to obtain a defining relation among the identification generators of Γ, are described in [15] and is called *Poincaré algorithm,* since it was first applied by him for $d = 2, 3$ (of course in other formulation [19], see also [12] and our Sect.4.2).

Table 1.No.11 contains also the corresponding stabilizers Γ^k $(k \in I)$ for k-midpoint orbits in $\mathcal{C}_{cu}$. For instance $\Gamma^0(A, B) = \mathbf{23}$ (as a point group) has the partial symbol $\mathcal{D}^0 = \mathcal{D}_{123}$ by cancelling σ_0-operation.



Thus we can produce a local fundamental domain $\mathcal{F}^0$ consisting of 4 spherical simplices glued, say, around vertex A. From $\mathcal{D}^0$ and from (3.1) we derive two 3-rotations and one 2-rotation as in case of point group **23**. As we see, this tiling has exactly one Γ-orbit of 3-, 2-, 1-, 0-midpoints each. So we have obtained a symmetry breaking No.(8) of the cube tiling in a restricted sense among the 17 ones.

Table 1

1 (1)	221. $Pm\bar{3}m$	Fig.2.a (C) $\mathcal{F} = AEKQ$	$m_0 : EKQ,\ \ m_1 : AKQ,\ \ m_2 : AEQ,\ \ m_3 :\ AEK$
			$(m_i m_j)^{m_{ij}} = 1$ by (3.1); $\ i, j \in I := \{0, 1, 2, 3\}$
			$\Gamma^3(Q) = m\bar{3}m, \quad \Gamma^2(K) = 4/mmm,$ $\Gamma^1(E) = 4/mmm, \quad \Gamma^0(A) = m\bar{3}m$

2 (2)	200. $Pm\bar{3}$	Fig.2.c (M) $\mathcal{F} = AEKFQ$	$m_0 : EKQ,\ m_0' : FKQ,\ m_3 : AEKF,\ \ r : AEQ \longrightarrow AFQ$
			$m_0^2,\ \ {m_0'}^2,\ \ m_3^2,\ \ r^3,\ \ (m_0 m_0')^2,\ \ (m_0 m_3)^2,\ \ (m_0' m_3)^2,$ $m_0 r m_0' r^{-1},\ \ (m_3 r m_3 r^{-1})^2$
			$\Gamma^3(Q) = m\bar{3}, \quad \Gamma^2(K) = mmm,$ $\Gamma^1(E, F) = mmm, \quad \Gamma^0(A) = m\bar{3}$

3 (3)	226. $Fm\bar{3}c$	Fig.2.c (N) $\mathcal{F} = AEKFQ$	$m_0 : EKQ, \quad m_0' : FKQ, \quad r_3 : AEKF \longrightarrow AFKE,$ $r : AEQ \longrightarrow AFQ$
			$m_0^2,\ \ {m_0'}^2,\ \ r_3^2,\ \ r^3;\ \ (m_0 m_0')^2,\ \ m_0 r_3 m_0' r_3,\ \ m_0 r m_0' r^{-1},$ $(r_3 r)^4$
			$\Gamma^3(Q) = m\bar{3}, \quad \Gamma^2(K) = \bar{4}2m,$ $\Gamma^1(E, F) = 4/m, \quad \Gamma^0(A) = 432$

Table 1 (continued)

4	225. $Fm\bar{3}m$	Fig.2.b (M) $\mathcal{F} = ABKQ$	$m_1 : BKQ, \quad m_1' : AKQ, \quad m_2 : ABQ, \quad m_3 : ABK$
			$m_1^2, \; m_1'^2, \; m_2^2, \; m_3^2; \; (m_1 m_1')^2, \; (m_1 m_2)^3, \; (m_1 m_3)^2, \; (m_1' m_2)^3, \; (m_1' m_3)^2, \; (m_2 m_3)^4$
			$\Gamma^3(Q) = \bar{4}3m, \quad \Gamma^2(K) = mmm, \quad \Gamma^1(E) = 4mm, \quad \Gamma^0(A) = m\bar{3}m, \quad \Gamma^0(B) = m\bar{3}m$
5 (4)	215. $P\bar{4}3m$	Fig.2.b (N) $\mathcal{F} = ABKQ$	$m_1 : BKQ, \; m_1' : AKQ, \; m_2 : ABQ, \; r : EAK \longrightarrow EBK$
			$m_1^2, \; m_1'^2, \; m_2^2, \; r^2, \; (m_1 m_1')^2, \; (m_1 m_2)^3, \; m_1 r m_1' r, \; (m_1' m_2)^3, \; (m_2 r m_2 r)^2$
			$\Gamma^3(Q) = \bar{4}3m, \quad \Gamma^2(K) = \bar{4}2m, \quad \Gamma^1(E) = \bar{4}2m, \quad \Gamma^0(A, B) = \bar{4}3m$
6 (5)	226. $Fm\bar{3}c$	Fig.2.b (M) $\mathcal{F} = ABKQ$	$r_1 : KAQ \longrightarrow KBQ, \; r_2 : EAQ \longrightarrow EBQ, \; m : ABK$
			$r_1^4, \; r_2^2, \; m^2; \; (r_1 r_2)^3, \; m r_1 m r_1^{-1}, \; (m r_2 m r_2)^2$
			$\Gamma^3(Q) = 432, \quad \Gamma^2(K) = 4/m, \quad \Gamma^1(E) = \bar{4}2m, \quad \Gamma^0(A, B) = m\bar{3}$
		Fig.2.c (M) $\mathcal{F} = AEKFQ$	$r_1 : KFQ \longrightarrow KEQ, \; r : AEQ \longrightarrow AFQ, \; m : AEKF$
			$r_1^4, \; r^3, \; m^2; \; (r_1 r)^2, \; m r_1 m r_1^{-1}, \; (m r m r^{-1})^2$
			$\Gamma^3(Q) = 432, \quad \Gamma^2(K) = 4/m, \quad \Gamma^1(E, F) = \bar{4}2m, \quad \Gamma^0(A) = m\bar{3}$

Table 1 (continued)

7 (6)	207. $P432$	Fig.2.b (N) $\mathcal{F} = ABKQ$	$r_1\colon KAQ \longrightarrow KBQ, \quad r_2\colon EAQ \longrightarrow EBQ,$ $r_3\colon EAK \longrightarrow EBK$
			$r_1^4, \quad r_2^2, \quad r_3^2; \quad (r_1r_2)^3, \quad (r_1r_3)^2, \quad (r_2r_3)^4$
			$\Gamma^3(Q) = 432, \quad \Gamma^2(K) = 422,$ $\Gamma^1(E) = 422, \quad \Gamma^0(A,B) = 432$
		Fig.2.c (N) $\mathcal{F} = AEKF$	$r_1\colon KFQ \longrightarrow KEQ, \quad r\colon AEQ \longrightarrow AFQ,$ $r_3'\colon AFK \longrightarrow AEK$
			$r_1^4, \quad r^3, \quad {r_3'}^2; \quad (r_1r)^2, \quad (r_1r_3')^2, \quad (rr_3')^4$
			$\Gamma^3(Q) = 432, \quad \Gamma^2(K) = 422,$ $\Gamma^1(E,F) = 422, \quad \Gamma^0(A) = 432$

8	229. $Im\bar{3}m$	Fig.2.d (M) $\mathcal{F} = ABCQ$	$m_1\colon ACQ,\ m_2\colon ABQ\ ,\ m_3\colon ABC, \ \ r\colon FBQ \longrightarrow FCQ$
			$m_1^2, \quad m_2^2, \quad m_3^2, \quad r^2; \quad (m_1m_2)^3, \quad (m_1m_3)^2,$ $m_1rm_2r, \quad (m_2m_3)^4, \quad (m_3rm_3r)^2$
			$\Gamma^3(Q) = \bar{3}m,\ \Gamma^2(K) = mm2,\ \ \Gamma^1(E) = 4/mmm,$ $\Gamma^1(F) = \bar{4}2m,\ \ \Gamma^0(A) = m\bar{3}m,\ \Gamma^0(B,C) = 4/mmm$

9 (7)	166. $R\bar{3}m$	Fig.2.g(d) (C) $\mathcal{F} = ABCQ$ (N)	$m_1\colon ACQ, \quad m_2\colon ABQ, \quad r\colon FBQ \longrightarrow FCQ,$ $r_3\colon KAB \longrightarrow KCB$
			$m_1^2, \quad m_2^2, \quad r^2, \quad r_3^2; \quad (m_1m_2)^3, \quad m_1rm_2r,$ $m_1r_3m_1r_3, \quad m_2r_3r_2r_3m_2r_3r_2r_3$
			$\Gamma^3(Q) = \bar{3}m, \quad \Gamma^2(K) = 2/m, \quad \Gamma^1(E,F) = 2/m,$ $\Gamma^0(A,B,C) = \bar{3}m$

Table 1 (continued)

10	202. $Fm\bar{3}$	Fig.2.d(e) (M) $\mathcal{F} = ABCQ$	$r_1: KAQ \longrightarrow KCQ, \quad r: BCQ \longrightarrow BAQ, \quad m: ABC$
			$r_1^2, \quad r^3, \quad m^2; \quad (r_1 r)^3, \quad m r_1 m r_1, \quad (m r m r^{-1})^2$
			$\Gamma^3(Q) = 23, \quad \Gamma^2(K) = 2/m, \quad \Gamma^1(E,F) = mm2,$ $\Gamma^0(A,C) = m\bar{3}, \quad \Gamma^0(B) = m\bar{3}$

11 (8)	195. $P23$	Fig.2.e(d) (N) $\mathcal{F} = ABFGQ$	$r_1: KFQ \longrightarrow KGQ, \quad r: AGQ \longrightarrow AEQ,$ $r': BEQ \longrightarrow BFQ, \quad r_3: KFBE \longrightarrow KGAE$
			$r_1^2, \quad r^3, \quad {r'}^3, \quad r_3^2\ ; \quad r_1 r r', \quad (r_1 r_3)^2, \quad (r_3 r r_3 r')^2$
			$\Gamma^3(Q) = 23, \quad \Gamma^2(K) = 222,$ $\Gamma^1(E,F,G) = 222, \quad \Gamma^0(A,B) = 23$

12 (9)	219. $F\bar{4}3c$	Fig.2.d(e) (N) $\mathcal{F} = ABCQ$	$z: KCB \longrightarrow KBA, \quad r_1: KAQ \longrightarrow KCQ,$ $r: BCQ \longrightarrow BAQ$
			$r_1^2, \quad r^3; \quad z z r_1, \quad (z r^{-1} z r^{-1})^2, \quad (r_1 r)^3$
			$\Gamma^3(Q) = 23, \quad \Gamma^2(K) = \bar{4}, \quad \Gamma^1(E,F) = \bar{4},$ $\Gamma^0(A,B,C) = 23$

Table 1 (continued)

13	209. $F432$	Fig.2.d(e) (N) $\mathcal{F} = ABCQ$	$r_1 : KAQ \longrightarrow KCQ, \quad r : BCQ \longrightarrow BAQ,$ $r_3 : BAK \longrightarrow BCK$
			$r_1^2\ ,\quad r^3,\quad r_3^2\ ;\quad (r_1r)^3,\quad (r_1r_3)^2,\quad (rr_3)^4$
			$\Gamma^3(Q) = 23, \quad \Gamma^2(K) = 222, \quad \Gamma^1(E,F) = 4,$ $\Gamma^0(A,C) = 432, \quad \Gamma^0(B) = 432$

14	223. $Pm\bar{3}n$	Fig.2.f (M) $\mathcal{F} = ABCDQ$	$r : ADQ \longrightarrow ABQ, \quad r_1 : FBQ \longrightarrow FCQ,$ $r_2 : GCQ \longrightarrow GDQ, \quad m : ABCD \longrightarrow ABCD$
			$r^3,\quad r_1^2\ ,\quad r_2^2\ ,\quad m^2\ ;\quad rr_1r_2\ ,\quad (mrmr^{-1})^2,$ $(mr_1mr_1)^2,\quad (mr_2mr_2)^2$
			$\Gamma^3(Q) = 32, \quad \Gamma^2(K) = m, \quad \Gamma^1(E,H) = mm2,$ $\Gamma^1(F) = \bar{4}2m, \quad \Gamma^1(G) = \bar{4}2m, \quad \Gamma^0(A) = m\bar{3},$ $\Gamma^0(B,C,D) = mmm$

15	211. $I432$	Fig.2.f (N) $\mathcal{F} = ABCDQ$	$r : ADQ \longrightarrow ABQ, \quad r_1 : FBQ \longrightarrow FCQ,$ $r_2 : GCQ \longrightarrow GDQ, \quad r_3 : ABC \longrightarrow ADC$
			$r^3,\quad r_1^2\ ,\quad r_2^2\ ,\quad r_3^2\ ;\quad rr_1r_2\ ,\quad (rr_3)^4,\quad (r_3r_1r_3r_2)^2$
$\sigma_0 :\ (1,2)(3,4)(5,6)(7,8)$ $\sigma_1 :\ (1,8)(2,3)(4,5)(6,7)$ $\sigma_2 :\ (1,8)(2,7)(3,4)(5,6)$ $\sigma_3 :\ (1,8)(2,7)(3,6)(4,5)$			$\Gamma^3(Q) = 32, \quad \Gamma^2(K) = 2, \quad \Gamma^1(E,H) = 4,$ $\Gamma^1(F,G) = 222, \quad \Gamma^0(A) = 432, \quad \Gamma^0(B,C,D) = 422$

16 (10)	167. $R\bar{3}c$	Fig.2.h(f) (N) $\mathcal{F} = ABCDQ$	$r : ADQ \longrightarrow ABQ, \quad r_1 : FBQ \longrightarrow FCQ,$ $r_2 : GCQ \longrightarrow GDQ, \quad z : ABCD(K) \longrightarrow CDBA(K)$
			$r^3,\quad r_1^2\ ,\quad r_2^2\ ,\quad z^2;\quad rr_1r_2\ ,\quad zr_2zr^{-1}zr_1zr$
$\sigma_0\ ,\ \sigma_1\ ,\ \sigma_2$ as before $\sigma_3 :\ (1,5)(2,6)(3,7)(4,8)$			$\Gamma^3(Q) = 32, \quad \Gamma^2(K) = \bar{1}, \quad \Gamma^1(E,F,G,H) = 2,$ $\Gamma^0(A,B,C,D) = \bar{3}$

Table 1 (continued)

17 (11)	155. $R32$	Fig.2.h(f) (N) $\mathcal{F}=ABCDQ$	$r\colon ADQ \longrightarrow ABQ,\quad r_1\colon FBQ \longrightarrow FCQ,$ $r_2\colon GCQ \longrightarrow GDQ,\quad r_3\colon BCD \longrightarrow BAD$
			$r^3,\quad r_1^2\,,\quad r_2^2\,,\quad r_3^2\,;\quad rr_1r_2,\quad rr_3r_1r_3r^{-1}r_3r_2r_3$
$\sigma_0\,,\ \sigma_1\,,\ \sigma_2$ as before $\sigma_3\colon\ (1,4)(2,3)(5,8)(6,7)$			$\Gamma^3(Q)=32,\quad \Gamma^2(K)=2,\quad \Gamma^1(E,F,G,H)=2,$ $\Gamma^0(A,B,C,D)=32$
18	204. $Im\bar{3}$	Fig.2.f (M) $\mathcal{F}=ABCDQ$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,\quad m\colon ABCD$
			$r^3,\quad m^2;\quad rzz,\quad (mrmr^{-1})^2,\quad (mzmz^{-1})^2$
$\sigma_0\,,\ \sigma_1$ as before $\sigma_2\colon\ (1,8)(2,7)(3,5)(4,6)$ $\sigma_3\colon\ (1)(2)(3)(4)(5)(6)(7)(8)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^2(K)=m,\quad \Gamma^1(E,H)=mm2,$ $\Gamma^1(F,G)=mm2,\ \ \Gamma^0(A)=m\bar{3},\quad \Gamma^0(B,C,D)=mmm$
19 (12)	206. $Ia\bar{3}$	Fig.2.f (N) $\mathcal{F}=ABCDQ$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,$ $r_3\colon EADG \longrightarrow EBCG$
			$r^3,\quad r_3^2\,;\quad rzz,\quad r_3r^{-1}r_3zr_3z^{-1}r_3r$
$\sigma_0\,,\ \sigma_1\,,\ \sigma_2$ as before $\sigma_3\colon\ (1,2)(3,8)(4,7)(5,6)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^2(K)=2,\quad \Gamma^1(E,F,G,H)=2,$ $\Gamma^0(A,B,C,D)=\bar{3}$
20	222. $Pn\bar{3}n$	Fig.2.f (N) $\mathcal{F}=ABCDQ$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,$ $r_3\colon ABC \longrightarrow ADC$
			$r^3,\quad r_3^2\,;\quad rzz,\quad (rr_3)^4,\quad (zr_3zr_3)^2$
$\sigma_0\,,\ \sigma_1\,,\ \sigma_2$ as before $\sigma_3\colon\ (1,8)(2,7)(3,6)(4,5)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^2(K)=2,\quad \Gamma^1(E,H)=4,$ $\Gamma^1(F,G)=\bar{4},\quad \Gamma^0(A)=432,\quad \Gamma^0(B,C,D)=422$
21 (13)	148. $R\bar{3}$	Fig.2.h(f) (N) $\mathcal{F}=ABCDQ$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,$ $z_3\colon ABCD(K) \longrightarrow CDAB(K)$
			$r^3,\quad z_3^2\,;\quad rzz,\quad z_3z^{-1}z_3rz_3z^{-1}z_3r$
$\sigma_3\colon\ (1,5)(2,6)(3,7)(4,8)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^2(K)=\bar{1},\quad \Gamma^1(E,F,G,H)=\bar{1},$ $\Gamma^0(A,B,C,D)=\bar{3}$
22 (14)	167. $R\bar{3}c$	Fig.2.h(f) (N) $\mathcal{F}=ABCDQ$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,$ $r_3\colon BAD \longrightarrow BCD$
			$r^3,\quad r_3^2\,;\quad rzz,\quad r_3zr_3rr_3zr_3r$
$\sigma_3\colon\ (1,4)(2,3)(5,8)(6,7)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^2(K)=2,\quad \Gamma^1(E,F,G,H)=\bar{1},$ $\Gamma^0(A,B,C,D)=32$

If we cut off the simplex $D_4 = AGKQ$ along AKQ, and glue it $GKQ \to FKQ$ by the 2-rotation r_1, then we get a new fundamental domain $ABCQ$ in Fig.2.d with the same D-symbol but another presentation for $\Gamma = \mathbf{P23}$ as follows:

$$r_1 : KCQ \longrightarrow KAQ, \quad r' : BAQ \longrightarrow BCQ,$$
$$r_3 : KAE \longrightarrow KBE, \quad r_3' : KBF \longrightarrow KCF$$

are the generators, and

$$(3.1.3) \qquad \begin{aligned} 1 &= r_1^2 = \text{trivial} = r_3^2 = {r_3'}^2 = (r_1 r') = \\ &= {r'}^3 = r_3 r_3' r_1 = r_3 r' r_3' {r'}^{-1} \end{aligned}$$

are the defining relations. Furthermore, we have the stabilizers

$$\Gamma^3(Q) = \mathbf{23}, \quad \Gamma^2(K) = \mathbf{222}, \quad \Gamma^1(E,F) = \mathbf{222}, \quad \Gamma^0(A,B,C) = \mathbf{23}$$

for the 3-, 2-, 1-, 0-midpoint orbits under $\Gamma = \mathbf{P23}$.

As we see, *different fundamental domains may represent the same D-symbol and tiling* as we indicated in Sect.2. These phenomena appear in Table 1.No.6–7, and they are indicated at No.10–13 as well.

3.2. Table 2

Consider the cases of Fig.2.i–j. Again, all the cube centres will be in one Γ-orbit, although the face stabilizer $\Gamma^2(K)$ does not change the sides of the face. Thus, any D-symbol $\mathcal{D}(\mathcal{M})$ has two $\mathcal{D}^{23} = \mathcal{D}_{01}$ components, but only one $\mathcal{D}^3 = \mathcal{D}_{012}$ component, describing a symmetry group $\Gamma^3(Q)$ of the cube by a local spherical fundamental domain $\mathcal{F}^3$. There are 2 cube face orbits (K, K'; by $\mathcal{D}_{01}$) in $\Gamma^3(Q)$ (by $\mathcal{D}_{012}$), as Fig.2.i or 2.j illustrate. They have to be connected by σ_3-operation of $\mathcal{D}$, i.e. by identifying the 2 cube faces under Γ.

For example, consider No.25 with Fig.2.j where the D-symbol $\mathcal{D}(\mathcal{M})$ contains 16 elements as simplices

$$D_1 = ALKQ, \quad D_2 = XLKQ, \quad \ldots, \quad D_8 = AEKQ,$$
$$D_9 = AEK'Q, \quad D_{10} = BEK'Q, \quad \ldots, \quad D_{16} = AHK'Q.$$

We prescribe the following operations in $\mathcal{D}$ $(D_x \leftrightarrow x)$:

$$(3.2.1) \qquad \begin{aligned} &\sigma_0 : (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16) && \cdots\cdots \\ &\sigma_1 : (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15) && - - - \\ &\sigma_2 : (8,9)(7,10)(1,4)(2,3)(5,16)(6,15)(11,14)(12,13) && \text{———} \\ &\sigma_3 : (1,15)(2,16)(3,9)(4,10)(5,11)(6,12)(7,13)(8,14) && \sim\sim\sim \end{aligned}$$

Table 2

23 (15)	160. $R3m$	Fig.2.i(g) (N) $\mathcal{F} = ABCX$	$m_1: BCX, \quad m_2: AXC, \quad s: XAB \longrightarrow ABC$
			$m_1^2\,, \quad m_2^2\,; \quad (m_1m_2)^3, \quad m_1sm_2s^{-1}, \quad m_1s^{-2}m_2s^2$
$\sigma_0:\ (1,2)(3,4)(5,6)(7,8)$ $\sigma_1:\ (1)(2,3)(4)(5)(6,7)(8)$ $\sigma_2:\ (1)(2)(3,5)(4,6)(7)(8)$ $\sigma_3:\ (1,5)(2,6)(3,7)(4,8)$			$\Gamma^3(Q) = 3m, \quad \Gamma^2(K,K') = m, \quad \Gamma^1(E,F,G) = m,$ $\Gamma^0(A,B,C,X) = 3m$

24	217. $I\bar{4}3m$	Fig.2.i (N) $\mathcal{F} = ABCX$	$m_1: BCX, \quad m_2: AXC, \quad z: EAXB \longrightarrow EBCA$
			$m_1^2\,, m_2^2\,, (zz)^2; (m_1m_2)^3, m_1zm_2z^{-1}, (m_1z^{-1}m_2z)^2$
$\sigma_0\,,\ \sigma_1\,,\ \sigma_2$ as before $\sigma_3:\ (1,8)(2,7)(3,6)(4,5)$			$\Gamma^3(Q) = 3m, \quad \Gamma^2(K,K') = m, \quad \Gamma^1(E) = \bar{4},$ $\Gamma^1(F,G) = mm2, \quad \Gamma^0(A,B) = \bar{4}2m, \quad \Gamma^0(C,X) = \bar{4}3m$

25 (16)	161. $R3c$	Fig.2.j(h) (N) $AXYBCDQ{=}\mathcal{F}$	$r: XADC(Q) \longrightarrow XYBC(Q), \quad c: XYBA \longrightarrow ABCD$
			$r^3, \quad ccrc^{-1}r^{-1}c^{-1}r$
Formula (3.2.1)			$\Gamma^3(Q) = 3,\ \Gamma^2(K,K') = 1,\ \ \Gamma^1(E,F,G,H,L,M,N) = 1,$ $\Gamma^0(A,X,Y,B,C,D) = 3$

26	205. $Pa\bar{3}$	Fig.2.j (N) $AXYBCDQ{=}\mathcal{F}$	$r: XADC(Q) \longrightarrow XYBC(Q), \quad z: AXYB \longrightarrow ABCD$
			$r^3, \quad zzrzrz^{-1}r^{-1}$
$\sigma_3:\ (1,9)(2,10)(3,11)(4,12)$ $(5,13)(6,14)(7,15)(8,16)$			$\Gamma^3(Q) = 3,\ \Gamma^2(K,K') = 1,\ \ \Gamma^1(E,F,G,H,L,M,N) = 1,$ $\Gamma^0(A,Y,C) = \bar{3}, \quad \Gamma^0(X,B,D) = \bar{3}$

27	218. $P\bar{4}3n$	Fig.2.j (N) $AXYBCDQ{=}\mathcal{F}$	$r: XADC(Q) \longrightarrow XYBC(Q),$ $z: (E)AXYB \longrightarrow (E)BCDA$
			$r^3, \quad (zz)^2; \quad (rzrz)^2, \quad (rzrz^{-1})^2$
$\sigma_3:\ (1,11)(2,12)(3,13)(4,14)$ $(5,15)(6,16)(7,9)(8,10)$			$\Gamma^3(Q) = 3, \quad \Gamma^2(K,K') = 1, \quad \Gamma^1(E) = \bar{4},$ $\Gamma^1(H,N) = \bar{4}, \quad \Gamma^1(L,M,G,F) = 2,$ $\Gamma^0(X,C) = 23,\ \Gamma^0(A,B,D,Y) = 222$

28	197. $I23$	Fig.2.j (N) $AXYBCDQ{=}\mathcal{F}$	$r: XADC(Q) \longrightarrow XYBC(Q), \quad s: XYBA \longrightarrow CBAD$
			$r^3, \quad (rsr^{-1}s^{-1})^2, \quad (ssr)^2$
$\sigma_3:\ (1,14)(2,13)(3,12)(4,11)$ $(5,10)(6,9)(7,16)(8,15)$			$\Gamma^3(Q) = 3, \quad \Gamma^2(K,K') = 1, \quad \Gamma^1(L,M,F,G) = 2,$ $\Gamma^1(N,E,H) = 2, \quad \Gamma^0(X,C) = 23,$ $\Gamma^0(A,Y,B,D) = 222$

Table 2 (continued)

29 (17)	146. $R3$	Fig.2.j(h) (N) $AXYBCDQ{=}\mathcal{F}$	$r: XADC(Q) \longrightarrow XYBC(Q), \quad s: XABY \longrightarrow ABCD$
			$r^3, \quad rsrsrs^{-1}s^{-1}$
$\sigma_3:$ $(1,10)(2,9)(3,16)(4,15)$ $(5,14)(6,13)(7,12)(8,11)$			$\Gamma^3(Q)=3$, $\Gamma^2(K,K')=1$, $\Gamma^1(L,M,N,E,F,G,H)=1$, $\Gamma^0(A,X,Y,B,C,D)=3$

Drawing the D-diagram is left to the reader. These induce the identifications on the fundamental domain $\mathcal{F}$. The curved face $XADC(Q)$ is paired with $XYBC(Q)$ by rotation r about the diagonal XQC. The second generator $c : XYBA \to ABCD$ is a glide reflection with glide plane $LHFN$ and glide vector $\overrightarrow{LH}$. We have the partial symbol $\mathcal{D}^{03} = \mathcal{D}_{12}$ with 4 components but only one non-trivial relation $1 = r^3$, and the partial symbol $\mathcal{D}^{01} = \mathcal{D}_{23}$ with 2 components for the relation $1 = c^2rc^{-1}r^{-1}c^{-1}r$, as Table 2.No.25 shows, again by Poincaré algorithm as in (3.1.1–2). In the same manner we get every k-midpoint stabilizer Γ^k of $\mathcal{D}^k$, obtained from $\mathcal{D}(\mathcal{M})$ by cancelling the σ_k-operation, respectively $k \in I$. No.(16) refers to the list of symmetry breakings in the restricted sense.

Now the space group 161.$\mathbf{R\bar{3}c} = \Gamma$ indicates how to deform the cube tiling to a rhombohedron tiling (in analogy to Fig.2.h) by expanding it in direction XC. Such a phenomenon also occurs in other cases. The sign (N) means that non-convex realization of $(\mathcal{T}, \Gamma)$ is possible without marking, but with deformation of faces $XYBA$ and $ABCD = (XYBA)^c$.

We remark that the fundamental domain $\mathcal{F}$ in Fig.2.j provides the minimal presentation for the space group $\mathbf{R\bar{3}c}$ with 2 generators and 2 defining relations in the sense of [13]. Cutting $\mathcal{F}$ at ABQ and gluing $ABCDQ$ to $XYBAQ$ by glide reflection $c^{-1} : ABCD \mapsto XYBA$, we get another fundamental domain $\mathcal{F}'$ with other (redundant) presentation but the same D-symbol. This $\mathcal{F}'$ (as in Fig.2.f or h) would be a *Dirichlet–Voronoi cell corresponding to the* $\mathbf{R\bar{3}c}$*-orbit of* K.

3.3. Table 3

The upper and lower domains together show our fundamental domain $\mathcal{F}$ in Fig.2.a–h. The face stabilizer $\Gamma^2(K)$ does not change the sides of the face and we have two orbits for cube centres Q and Q_2. Thus, any such D-symbol $\mathcal{D}(\mathcal{M})$ has two $\mathcal{D}^3$-components describing the solid stabilizers $\Gamma^3(Q)$ and $\Gamma^3(Q_2)$, respectively. These facts mean that we are describing cube tilings in 2 colours, i.e. the spatial chess board, with isometry group Γ acting transitively on the faces.

The automorphism group Aut $\mathcal{T}$ of this chess board $\mathcal{T}$ is just the space group $\Gamma = \mathbf{Fm\bar{3}m}$ under No.225, as described in Table 3.No.30, $(1^\star)$. Γ is a Coxeter group, generated by plane reflections in the side faces of $\mathcal{F} = AEQQ_2$. In $\mathcal{F}$ we can choose a piece AEK of the cube face with great freedom, and we also get a non-convex realization of $(\mathcal{T},\ \Gamma = \mathbf{Fm\bar{3}m})$.

The other tilings $(\mathcal{T}, \Gamma)$ in Table 3 are all the face transitive symmetry breakings of chess board $(\mathcal{T},\ \Gamma = \mathbf{Fm\bar{3}m})$ as indicated in the Introduction (see also our Sect.4.3).

Table 3

30 (1*)	225. $Fm\bar{3}m$	Fig.2.a (N) $\mathcal{F} = AEQQ_2$	$m_0: EQQ_2$, $m_1: AQQ_2$, $m_2: AEQ$, $m_2': AEQ_2$
			m_0^2, m_1^2, m_2^2, ${m_2'}^2$; $(m_0m_1)^4$, $(m_0m_2)^2$, $(m_0m_2')^2$, $(m_1m_2)^3$, $(m_2m_2')^2$, $(m_1m_2')^3$
			$\Gamma^3(Q) = m\bar{3}m$, $\Gamma^3(Q_2) = m\bar{3}m$, $\Gamma^2(K) = 4mm$, $\Gamma^1(E) = mmm$, $\Gamma^0(A) = \bar{4}3m$
31 (2*)	202. $Fm\bar{3}$	Fig.2.c (N) $AEKFQQ_2{=}\mathcal{F}$	$m_0: EQQ_2$, $m_0': FQQ_2$, $r: AEQ \longrightarrow AFQ$, $r': AFQ_2 \longrightarrow AEQ_2$
			m_0^2, ${m_0'}^2$, r^3, ${r'}^3$; $(m_0m_0')^2$, $m_0rm_0'r^{-1}$, $m_0'r'm_0{r'}^{-1}$, $(rr')^2$
			$\Gamma^3(Q) = m\bar{3}$, $\Gamma^3(Q_2) = m\bar{3}$, $\Gamma^2(K) = mm2$, $\Gamma^1(E,F) = 2/m$, $\Gamma^0(A) = 23$
32	216. $F\bar{4}3m$	Fig.2.b (N) $ABQQ_2 = \mathcal{F}$	$m_1: BQQ_2$, $m_1': AQQ_2$, $m_2: ABQ$, $m_2': ABQ_2$
			m_1^2 , ${m_1'}^2$, m_2^2 , ${m_2'}^2$; $(m_1m_1')^2$, $(m_1m_2)^3$, $(m_1m_2')^3$, $(m_1'm_2)^3$, $(m_1'm_2')^3$, $(m_2m_2')^2$
			$\Gamma^3(Q) = \bar{4}3m$, $\Gamma^3(Q_2) = \bar{4}3m$, $\Gamma^2(K) = mm2$, $\Gamma^1(E) = mm2$, $\Gamma^0(A) = \bar{4}3m$, $\Gamma^0(B) = \bar{4}3m$

Table 3 (continued)

<table>
<tr><td rowspan="2">33
(3*)</td><td rowspan="2">209.
$F432$</td><td rowspan="2">Fig.2.b (N)
$ABQQ_2 = \mathcal{F}$</td><td>$r_1\colon QAQ_2 \longrightarrow QBQ_2$, $r_2\colon EAQ \longrightarrow EBQ$,
$r_2'\colon EAQ_2 \longrightarrow EBQ_2$</td></tr>
<tr><td>r_1^4, r_2^2, ${r_2'}^2$; $(r_1r_2)^3$, $(r_1r_2')^3$, $(r_2r_2')^2$</td></tr>
<tr><td colspan="3">$\sigma_0\colon$ $(1,2)(3,4)$
$\sigma_1 = \sigma_2 = \sigma_0$
$\sigma_3\colon$ $(1,4)(2,3)$</td><td>$\Gamma^3(Q) = 432$, $\Gamma^3(Q_2) = 432$, $\Gamma^2(K) = 4$,
$\Gamma^1(E) = 222$, $\Gamma^0(A,B) = 23$</td></tr>
</table>

<table>
<tr><td rowspan="2">34</td><td rowspan="2">224.
$Pn\bar{3}m$</td><td rowspan="2">Fig.2.d (N)
$ABCQQ_2 = \mathcal{F}$</td><td>$m_1\colon AQ_2CQ$, $m_2\colon ABQ$, $m_2'\colon ABQ_2$,
$r_2\colon FBQ \longrightarrow FCQ$, $r_2'\colon FCQ_2 \longrightarrow FBQ_2$</td></tr>
<tr><td>m_1^2, m_2^2, ${m_2'}^2$, r_2^2, ${r_2'}^2$; $(m_1m_2)^3$, $(m_1m_2')^3$,
$(m_2m_2')^2$, $m_1r_2m_2r_2$, $m_1r_2'm_2'r_2'$, $(r_2r_2')^2$</td></tr>
<tr><td colspan="3">$\sigma_0\colon$ $(1,2)(3,4)(5,6)(7,8)$
$\sigma_1\colon$ $(1)(2,3)(4)(5)(6,7)(8)$
$\sigma_2\colon$ $(1)(2)(3,4)(5)(6)(7,8)$
$\sigma_3\colon$ $(1,5)(2,6)(3,7)(4,8)$</td><td>$\Gamma^3(Q) = \bar{3}m$, $\Gamma^3(Q_2) = \bar{3}m$, $\Gamma^2(K) = m$, $\Gamma^1(E) = mm2$,
$\Gamma^1(F) = 222$, $\Gamma^0(A) = \bar{4}3m$, $\Gamma^0(B,C) = \bar{4}2m$</td></tr>
</table>

<table>
<tr><td rowspan="2">35</td><td rowspan="2">166.
$R\bar{3}m$</td><td rowspan="2">Fig.2.g(d) (N)
$ABCQQ_2 = \mathcal{F}$</td><td>$m_1\colon AQ_2CQ$, $m_2\colon QAB$, $r_2\colon FBQ \longrightarrow FCQ$,
$m_2'\colon BCQ$, $r_2'\colon EAQ_2 \longrightarrow EBQ$</td></tr>
<tr><td>m_1^2, m_2^2, r_2^2, ${m_2'}^2$, ${r_2'}^2$; $(m_1m_2)^3$, $m_1r_2m_2r_2$,
$(m_1m_2')^3$, $m_1r_2'm_2'r_2'$, $m_2r_2'm_2r_2'$, $r_2m_2'r_2m_2'$</td></tr>
<tr><td colspan="3">σ_0, σ_1, σ_3 as before
$\sigma_2\colon$ $(1)(2)(3,4)(5,6)(7)(8)$</td><td>$\Gamma^3(Q) = \bar{3}m$, $\Gamma^3(Q_2) = \bar{3}m$, $\Gamma^2(K) = m$,
$\Gamma^1(E) = 2/m$, $\Gamma^1(F) = 2/m$, $\Gamma^0(A,B,C) = 3m$</td></tr>
</table>

<table>
<tr><td rowspan="2">36</td><td rowspan="2">196.
$F23$</td><td rowspan="2">Fig.2.d(e) (N)
$ABCQQ_2 = \mathcal{F}$</td><td>$r_1\colon QAQ_2 \longrightarrow QCQ_2$, $r\colon BCQ \longrightarrow BAQ$,
$r'\colon BAQ_2 \longrightarrow BCQ_2$</td></tr>
<tr><td>r_1^2, r^3, ${r'}^3$, $(r_1r)^3$, $(r_1r')^3$, $(rr')^2$</td></tr>
<tr><td colspan="3">$\sigma_0\colon$ $(1,2)(3,4)(5,6)(7,8)$
$\sigma_1\colon$ $(1,4)(2,3)(5,8)(6,7)$
$\sigma_2\colon$ $(1,4)(2,3)(5,8)(6,7)$
$\sigma_3\colon$ $(1,5)(2,6)(3,7)(4,8)$</td><td>$\Gamma^3(Q) = 23$, $\Gamma^3(Q_2) = 23$, $\Gamma^2(K) = 2$,
$\Gamma^1(E,F) = 2$, $\Gamma^0(A,C) = 23$ $\Gamma^0(B) = 23$</td></tr>
</table>

Table 3 (continued)

37	208. $P4_232$	Fig.2.f (N) $ABCDQQ_2{=}\mathcal{F}$	$r: ADQ \longrightarrow ABQ,\quad r_1: FBQ \longrightarrow FCQ,$ $r_2: GCQ \longrightarrow GDQ,\quad r': ADQ_2 \longrightarrow ABQ_2,$ $r'_1: FBQ_2 \longrightarrow FCQ_2,\quad r'_2: GCQ_2 \longrightarrow GDQ_2,$
			$r^3,\quad r_1^2\,,\quad r_2^2\,,\quad {r'}^3,\quad {r'_1}^2,\quad {r'_2}^2;\quad rr_1r_2\,,\quad r'r'_1r'_2\,,$ $(r^{-1}r')^2,\quad (r_1r'_1)^2,\quad (r_2r'_2)^2$
Formula (3.3.1)			$\Gamma^3(Q) = 32,\quad \Gamma^3(Q_2) = 32,\quad \Gamma^2(K) = 1,$ $\Gamma^1(E,H) = 2,\quad \Gamma^1(F) = 222,\quad \Gamma^1(G) = 222,$ $\Gamma^0(A) = 23,\quad \Gamma^0(B,C,D) = 222$

38	155. $R32$	Fig.2.h(f) (N) $ABCDQQ_2{=}\mathcal{F}$	$r: ADQ \longrightarrow ABQ,\quad r_1: FBQ \longrightarrow FCQ,$ $r_2: GCQ \longrightarrow GDQ,\quad r': CDQ_2 \longrightarrow CBQ_2,$ $r'_1: EBQ_2 \longrightarrow EAQ_2,\quad r'_2: HAQ_2 \longrightarrow HDQ_2$
			$r^3,\quad r_1^2\,,\quad r_2^2\,,\quad {r'}^3,\quad {r'_1}^2,\quad {r'_2}^2;\quad rr_1r_2\,,\quad r'r'_1r'_2,$ $rr'_1r^{-1}r'_2\,,\quad r'r_1{r'}^{-1}r_2$
$\sigma_0\,,\ \sigma_1\,,\ \sigma_3$ as before $\sigma_2:\ (1,8)(2,7)(3,4)(5,6)$ $(12,13)(11,14)(9,10)(15,16)$			$\Gamma^3(Q) = 32,\quad \Gamma^3(Q_2) = 32,\quad \Gamma^2(K) = 1,$ $\Gamma^1(E,H) = 2,\quad \Gamma^1(F,G) = 2,\quad \Gamma^0(A,B,C,D) = 3$

39	228. $Fd\bar{3}c$	Fig.2.f (N) $ABCDQQ_2{=}\mathcal{F}$	$r: ADQ \longrightarrow ABQ,\quad r_1: FBQ \longrightarrow FCQ,$ $r_2: GCQ \longrightarrow GDQ,\quad r': ADQ_2 \longrightarrow ABQ_2,$ $z: Q_2BC \longrightarrow Q_2CD$
			$r^3,\ r_1^2\,,\ r_2^2\,,\ {r'}^3;\ rr_1r_2\,,\ r'zz,\ (r^{-1}r')^2,\ \ r_1zr_2z^{-1}$
$\sigma_0\,,\ \sigma_1\,,\ \sigma_3$ as before $\sigma_2:\ (1,8)(2,7)(3,4)(5,6)$ $(9,16)(10,15)(11,13)(12,14)$			$\Gamma^3(Q) = 32,\quad \Gamma^3(Q_2) = \bar{3},\quad \Gamma^2(K) = 1,$ $\Gamma^1(E,H) = 2,\quad \Gamma^1(F,G) = 2,$ $\Gamma^0(A) = 23,\quad \Gamma^0(B,C,D) = \bar{4}$

40	167. $R\bar{3}c$	Fig.2.h(f) (N) $ABCDQQ_2{=}\mathcal{F}$	$r: ADQ \longrightarrow ABQ,\quad r_1: FBQ \longrightarrow FCQ,$ $r_2: GCQ \longrightarrow GDQ,\quad r': CDQ_2 \longrightarrow CBQ_2,$ $z: Q_2BA \longrightarrow Q_2AD$
			$r^3,\quad r_1^2\,,\quad r_2^2\,,\quad {r'}^3;\quad rr_1r_2\,,\quad r'zz,\quad rzrz,\quad r'r_1{r'}^{-1}r_2$
$\sigma_0\,,\ \sigma_1\,,\ \sigma_3$ as before $\sigma_2:\ (1,8)(2,7)(3,4)(5,6)$ $(12,13)(11,14)(9,15)(10,16)$			$\Gamma^3(Q) = 32,\quad \Gamma^3(Q_2) = \bar{3},\quad \Gamma^2(K) = 1,$ $\Gamma^1(E,H) = \bar{1},\quad \Gamma^1(F,G) = 2,$ $\Gamma^0(A,B,C,D) = 3$

Table 3 (continued)

41	201. $Pn\bar{3}$	Fig.2.f (N) $ABCDQQ_2{=}\mathcal{F}$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,$ $r'\colon ADQ_2 \longrightarrow ABQ_2,\quad z'\colon Q_2BC \longrightarrow Q_2CD$
			$r^3,\quad {r'}^3;\quad (r^{-1}r')^2,\quad rzz,\quad r'z'z',\quad (z^{-1}z')^2$
σ_0 , σ_1 , σ_3 as before $\sigma_2\colon$ $(1,8)(2,7)(3,5)(4,6)$ $(9,16)(10,15)(11,13)(12,14)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^3(Q_2)=\bar{3},\quad \Gamma^2(K)=1,$ $\Gamma^1(E,H)=2,\quad \Gamma^1(F,G)=2,$ $\Gamma^0(A)=23,\quad \Gamma^0(B,C,D)=222$

42 (4*)	205. $Pa\bar{3}$	Fig.2.f (C,N) $ABCDQQ_2{=}\mathcal{F}$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,$ $r'\colon DAQ_2 \longrightarrow DCQ_2,\quad z'\colon Q_2CB \longrightarrow Q_2BA$
			$r^3,\quad {r'}^3;\quad rzz,\quad r'z'z',\quad rz'^{-1}zr'^{-1}$
σ_0 , σ_1 , σ_3 as before $\sigma_2\colon$ $(1,8)(2,7)(3,5)(4,6)$ $(14,15)(13,16)(9,11)(10,12)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^3(Q_2)=\bar{3},\quad \Gamma^2(K)=1,$ $\Gamma^1(E,F,G,H)=1,\quad \Gamma^0(A,B,C,D)=3$

43	148. $R\bar{3}$	Fig.2.h(f) (N) $ABCDQQ_2{=}\mathcal{F}$	$r\colon ADQ \longrightarrow ABQ,\quad z\colon QBC \longrightarrow QCD,$ $r'\colon CBQ_2 \longrightarrow CDQ_2,\quad z'\colon Q_2DA \longrightarrow Q_2AB$
			$r^3,\quad {r'}^3;\quad rzz,\quad r'z'z',\quad rz'^{-1}rz'^{-1},\quad r'z^{-1}r'z^{-1}$
σ_0 , σ_1 , σ_3 as before $\sigma_2\colon$ $(1,8)(2,7)(3,5)(4,6)$ $(12,13)(11,14)(9,15)(10,16)$			$\Gamma^3(Q)=\bar{3},\quad \Gamma^3(Q_2)=\bar{3},\quad \Gamma^2(K)=1,$ $\Gamma^1(E,H)=\bar{1},\quad \Gamma^1(F,G)=\bar{1},$ $\Gamma^0(A,B,C,D)=3$

For example, consider Table 3.No.37 with Fig.2.f. The D-symbol $\mathcal{D}(\mathcal{M})$ contains 16 elements as simplices

$$D_1 = AEKQ,\quad D_2 = BEKQ,\quad \ldots,\quad D_8 = AHKQ,$$
$$D_9 = AEKQ_2,\quad D_{10} = BEKQ_2,\quad \ldots,\quad D_{16} = AHKQ_2,$$

with the matrix function $\mathcal{M}$ constant (3.1), where the following operations are introduced in $\mathcal{D}$ $(D_x \leftrightarrow x)$:

$$\begin{array}{lll} & \sigma_0 : (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16) & \cdots\cdots \\ & \sigma_1 : (1,8)(2,3)(4,5)(6,7)(9,16)(10,11)(12,13)(14,15) & - - - \\ (3.3.1) & \sigma_2 : (1,8)(2,7)(3,4)(5,6)(9,16)(10,15)(11,12)(13,14) & —— \\ & \sigma_3 : (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) & \sim\sim\sim \end{array}$$

We do not draw the D-diagram here. The σ_i-operations, $i \in I$, induce the face pairing generators of $\mathcal{F}$ given in Table 3.No.37. The face and its midpoint K have trivial stabilizer. The partial symbols $\mathcal{D}_{02}$, $\mathcal{D}_{12}$, $\mathcal{D}_{23}$ provide the defining relations for the group $\Gamma = \mathbf{P4_232}$

as space group No.208. The stabilizers Γ^k, given by the partial symbols $\mathcal{D}^k$, obtained from $\mathcal{D}$ by cancelling σ_k-operations ($k \in I$), helped us in choosing this space group from International Tables [10].

We have obtained 4 symmetry breakings of the spatial chess board in the restricted sense, as enumerated $(1^\star)$–$(4^\star)$ in Table 3.

We remark that I. Prok [20] classified all the fundamental cube tilings in $\mathbf{E}^3$, i.e. with exactly one trivial solid stabilizer $\Gamma^3(Q) = 1$. *By computer he found 298 equivariant classes* $(\mathcal{T}, \Gamma)$ *distributed into 130 non-isomorphic space groups.* Our first 29 tilings in Table 1–2 do not belong to these classes, because our solid stabilizers are not trivial.

4. The General Strategy for Classifying d-Dimensional Tilings with Given Transitivity Properties by D-Dymbols

We consider combinatorially equivariant classes $(\mathcal{T}, \Gamma)$ of d-tilings in certain simply connected d-spaces $\mathcal{S}^d$ to be described by the following requirements stated in Assumptions 1–3. One can think of their realizations in Euclidean plane $\mathbf{E}^2$ or space $\mathbf{E}^3$ (see again Sect.2 and 3).

4.1. The Description of a Tiling $(\mathcal{T}, \Gamma)$ by a D-Symbol

Assumption 1 A tiling $\mathcal{T}$ of a simply connected d-space $\mathcal{S}^d$ is characterized by its barycentric subdivision $\mathcal{C}$ of d-simplices with I-labelled i-vertices and opposite i-facets ($(d-1)$-hyperfaces) for any $i \in I := \{0, 1, \dots, d-1, d\}$. A starting simplex $C_1 \in \mathcal{C}$ has a *coordinate mapping* $x : C_1 \to \Delta_I$ onto the *standard real coordinate simplex*

$$\Delta_I := \left\{ x = (x^0, \dots, x^d) \in \mathbf{R}^I \; : \; x^i \geq 0 \text{ for every } i \in I, \quad \sum_{i=0}^{d} x^i = 1 \right\} \tag{4.1.1}$$

The i-facet of C_1 is defined by the i-facet of Δ_I

$$\Delta_I^i := \{x \in \Delta_I \; : \; x^i = 0\}, \quad i \in I. \tag{4.1.2}$$

Then the topological space $\mathcal{S}^d$ is defined as the union $\Delta_I \times \mathcal{C}$ of as many copies of Δ_I as there are simplices $C \in \mathcal{C}$ with i-facet identifications

$$(\Delta_I^i, C) = (\Delta_I^i, \sigma_i C) \quad i \in I. \tag{4.1.3}$$

So the i-adjacent simplex $\sigma_i C$ of any $C \in \mathcal{C}$, i.e. the involutive σ_i-operation on $\mathcal{C}$ is also introduced for any $i \in I$. This σ_i is first a permutation of $\mathcal{C}$, which orders to any simplex its own i-adjacent one. Furthermore, σ_i can be considered a "local hyperface reflection" in the i-facet of an actually fixed simplex C as well that fixes all the points of the common facet (4.1.3), and

$$\sigma_i(x, C) := (x, \sigma_i C) \quad \text{for any } x \in \Delta_I \tag{4.1.4}$$

holds for the points of simplices C and $\sigma_i C$.

Then the (free) Coxeter group

$$\Sigma_I := (\sigma_0, \sigma_1, \ldots, \sigma_d \quad - \quad 1 = \sigma_i^2 \text{ for any } i \in I) \tag{4.1.5}$$

acts on $\mathcal{C}$ from the left by formula

$$(\sigma_i \sigma) C := \sigma_i(\sigma C)$$

for any $C \in \mathcal{C}$, $\sigma \in \Sigma_I$ and $i \in I$. That means, if

$$\sigma = \sigma_{i_r} \cdots \sigma_{i_2} \sigma_{i_1} \in \Sigma_I$$

is a product of generators $\{\sigma_j \; : \; j \in I\}$, then the path through simplices

$$C, \quad \sigma_{i_1} C, \quad \sigma_{i_2}\sigma_{i_1} C, \quad \ldots, \quad \sigma_{i_r} \cdots \sigma_{i_2}\sigma_{i_1} C, \quad \sigma_i \sigma_{i_r} \cdots \sigma_{i_2}\sigma_{i_1} C \tag{4.1.6}$$

arrives at $\sigma_i \sigma C$ entering the i-facet of σC. These also show how to define the action of Σ_I as local mappings of a fixed simplex $C \in \mathcal{C}$ onto the corresponding simplices of $\mathcal{C}$. □

Assumption 2 To describe the tiling $\mathcal{T}$ and so its space $\mathcal{S}^d$, the matrix function

$$M \; : \; \mathcal{C} \longrightarrow \mathbf{N}_{I \times I}, \quad C \longmapsto M(C) := (m_{ij}(C)) \tag{4.1.7}$$

is introduced (with entries from the set $\mathbf{N}$ of natural numbers) by the following properties (4.1.8–10): Any entry $m_{ij}(C)$ is minimal for the equality

$$C = (\sigma_j \sigma_i)^{m_{ij}(C)} C \text{ with product } \overbrace{(\sigma_j \sigma_i) \cdots (\sigma_j \sigma_i)(\sigma_j \sigma_i)}^{m_{ij}(C)-\text{times}}; \tag{4.1.8}$$

$$m_{ii}(C) = 1, \qquad m_{ij}(C) = m_{ji}(C) = m_{ji}(\sigma_i C); \tag{4.1.9}$$

$$m_{ij}(C) = 2 \;\text{ if }\; |i-j| > 1; \qquad m_{ij}(C) > 2 \;\text{ if }\; |i-j| = 1 \tag{4.1.10}$$

hold for any $i, j \in I$ and for any $C \in \mathcal{C}$.

We require $\mathcal{C}$ and $\mathcal{S}^d$ to be simply connected. That means Σ_I acts transitively on $\mathcal{C}$, and that moreover, any relation

$$\sigma C = C, \quad \sigma \in \Sigma_I \tag{4.1.11}$$

is a consequence of the "loop relations"

$$\tau^{-1} (\sigma_j \sigma_i)^{m_{ij}(\tau C)} \tau C = C \tag{4.1.12}$$

for any $C \in \mathcal{C}$, $\tau \in \Sigma_I$, $i, j \in I$ (in the sense of (4.1.6)).

Thus, the generalized Coxeter group $\Sigma_I(\mathcal{C}, M) =: \Sigma_I(\mathcal{T})$ acting freely on $\mathcal{C}$, is defined at (4.1.5), the matrix function (4.1.7–10) and by a fixed simplex C_1. Different C_1's lead to conjugated Coxeter groups (not distinguished in the following [4]). □

Assumption 3 *In addition, we require a group* Γ *of simplicial* (piecewise linear) *homeomorphisms which acts on* $\mathcal{T} = (\mathcal{T}, \Gamma)$ and freely on simplices of $\mathcal{C}$ with right action as

(4.1.13) $$C^{\gamma_1\gamma_2} := (C^{\gamma_1})^{\gamma_2}$$

denotes for any $C \in \mathcal{C}$ and $\gamma_1, \gamma_2 \in \Gamma$. *The transformations in* Γ *shall preserve all incidences of* 0-faces=vertices, 1-faces=edges, ..., $(d-1)$-faces=facets, d-faces=solids of $(\mathcal{T}, \Gamma)$. *That means, we require the* Γ*-invariance of* Σ_I, as

(4.1.14) $$(\sigma C)^{\gamma} = \sigma(C^{\gamma})$$

shows for any $\sigma \in \Sigma_I$ and $\gamma \in \Gamma$. In particular, the γ-image of the i-adjacent of C is the same as the i-adjacent of C^{γ}. *The group* Γ *factorizes* $\mathcal{C}$ *into* Γ*-orbits (*Γ*-images)*

(4.1.15) $$C^{\Gamma} := \{C^{\gamma} \in \mathcal{C} \,:\, \gamma \in \Gamma\} =: D,$$

required to constitute a finite set $\mathcal{D}$ called D-set of $(\mathcal{T}, \Gamma)$. Moreover, there exists an induced action of Σ_I on $\mathcal{D}$ as left σ_i-operations for $i \in I$ defined by

(4.1.16) $$\sigma_i D := \sigma_i C^{\Gamma} := \left\{\sigma_i(C^{\gamma}) = (\sigma_i C)^{\gamma} \,:\, \gamma \in \Gamma\right\} \text{ for } C^{\Gamma} = D$$

and the matrix function $\mathcal{M} = (m_{ij})$ by

(4.1.17) $$\mathcal{M} \,:\, \mathcal{D} \longrightarrow \mathbf{N}_{I\times I}, \quad D \longmapsto \mathcal{M}(D) = M(C) \text{ with any } C \in D.$$

This function has the properties (4.1.8–10) with C's replaced by D's, but without requiring the minimality in (4.1.8).

The system $\mathcal{D}(\mathcal{M})$, *as a finite set with the* I*-adjacencies i.e.* Σ_I*-operations and the matrix function* $\mathcal{M}$*, is called a D-symbol (Delone–Delaney–Dress symbol).* □

As we have seen, all the 14 Euclidean square tilings in $\mathbf{E}^2 =: \mathcal{S}^2$ and all the 43 cube tilings in $\mathbf{E}^3 =: \mathcal{S}^3$ and their D-symbols satisfy these requirements Ass.1–3 with appropriate crystallographic isometry groups Γ of $\mathbf{E}^2$ resp. $\mathbf{E}^3$. The properties and algorithmic procedures, described in Sect.2–3 by examples, can be formulated in the following Propositions 1–3 of principial importance.

4.2. Fundamental Domain and Presentation of the Group Γ for a Tiling $(\mathcal{T}, \Gamma)$

Proposition 1 *Any representative set, containing one simplex* C_x *from each orbit* $D_x \in \mathcal{D}$ *constitutes a fundamental set*

$$\mathcal{F} := \bigcup_{x=1}^{|\mathcal{D}|} C_x$$

which is compact in $\mathcal{S}^d$ since $|\mathcal{D}|$ is finite. Any such $\mathcal{F}$, by gluing its simplices and pairing its free (not glued) facets as the σ_i-operations dictate, represents the D-symbol $\mathcal{D}(\mathcal{M})$. Any fixed $\mathcal{F}$ (may be not connected) presents a finite system of generators for the group Γ. Each of them pairs some σ_i-corresponded free simplex facets of $\mathcal{F}$ in $\mathcal{C}$ $(i \in I)$.

If $\tau_1 C_1 \in D_1$ and $\tau_2 C_1 \in D_2$ and $\sigma_i D_1 = D_2$, with $D_1, D_2 \in \mathcal{D}$, $\tau_1, \tau_2 \in \Sigma_I$, $i \in I$, then we have two cases:

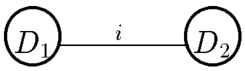
(C.1)

i.e. $D \ni \tau_1 C_1$, $D \ni \tau_2 C_1 = \sigma_i \tau_1 C_1$ then $\tau_1 C_1 \mapsto \tau_2 C_1 = \sigma_i \tau_1 C_1 =: (\tau_1 C_1)^{\mathrm{m}}$ defines a simplicial reflection m for C_1: $\tau_1^{-1}\sigma_i \tau_1 C_1 =: C_1^{\mathrm{m}}$. If $C = \tau C_1$ with $\tau \in \Sigma_I(M, C_1)$ then the m-image of C is defined by

$$C^{\mathrm{m}} := (\tau C_1)^{\mathrm{m}} := (\tau \tau_1^{-1} \sigma_i \tau_1 \tau^{-1}) C = \tau(C_1^{\mathrm{m}}). \tag{4.2.1}$$

In this sense, after having fixed $C_1 \in \mathcal{C}$, we have defined the facet reflection $\mathrm{m} \in \Gamma$ with relation $\mathrm{m}^2 = 1$ expressed in $\Sigma_I(M, C_1)$ by Ass.1–3.

(C.2)



Then $\tau_1 C_1 \mapsto \tau_2 C_1 = \sigma_i \tau_1 C_1 =: (\tau_1 C_1)^{\gamma}$ defines $\gamma \in \Gamma$ with the required extension to $\mathcal{C}$: For any $C = \tau C_1$ with $\tau \in \Sigma_I(M, C_1)$ let its γ-image

$$C^{\gamma} := (\tau C_1)^{\gamma} := \tau(C_1^{\gamma}) = (\tau \tau_2^{-1} \sigma_i \tau_1 \tau^{-1}) C \tag{4.2.2}$$

be defined. Thus the mapping $\gamma \in \Gamma$ and its inverse γ^{-1} by

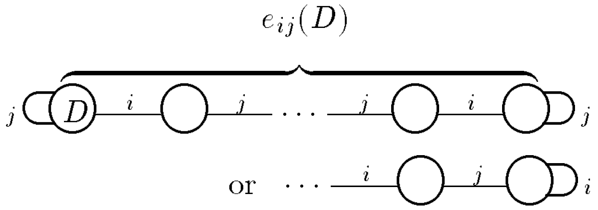
$$C^{\gamma^{-1}} := (\tau C_1)^{\gamma^{-1}} := \tau(C_0^{\gamma^{-1}}) = (\tau \tau_1^{-1} \sigma_1 \tau_2 \tau^{-1}) C$$

are expressed in $\Sigma_I(M, C_1)$. In both cases we obtain generators for Γ. □

Proposition 2 *Any free $(d-2)$-face class of $\mathcal{F}$ under Γ-equivalence provides a defining relation for the previous generators of Γ by the Poincaré algorithm.*

Here "free" means that any element in the $(d-2)$-face Γ-class lies on the $(d-1)$-surface of $\mathcal{F}$, i.e. it is not surrounded by simplices of $\mathcal{C}$, belonging to $\mathcal{F}$. If i- and j-facets of an $\mathcal{F}$-simplex $C \in D \in \mathcal{D}$ join this $(d-2)$-face class, and $m_{ij}(D) := m_{ij}(C)$ is the matrix entry, then we have two cases again:

(C.A)

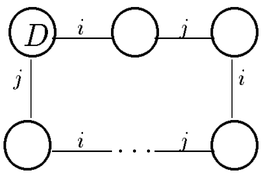


That is, we have a $\langle\sigma_i,\sigma_j\rangle$-chain $\mathcal{D}_{ij}$ in $\mathcal{D}$ with chain length $e_{ij}(D)$. This describes a local fundamental domain $\mathcal{F}_{ij}(D)$ for the stabilizer $\Gamma_{ij}(D)$ of this $(d-2)$-face Γ-class, generated by two facet reflections from Γ. Since $2\cdot m_{ij}(D)$ simplices from $\mathcal{C}$ meet around this $(d-2)$-face of $C\in D$, the natural number

$$v_{ij}(D) := \frac{2\cdot m_{ij}(D)}{2\cdot e_{ij}(D)} \tag{4.2.3}$$

is just the order of the "rotation subgroup" of the stabilizer $\Gamma_{ij}(D)$.

(C.B)



That means, we have in $\mathcal{D}$ a $\langle\sigma_i,\sigma_j\rangle$-cycle $\mathcal{D}_{ij}$ of length $e_{ij}(D)$ which is even. This describes $\mathcal{F}_{ij}(D)$ for the stabilizer $\Gamma_{ij}(D)$ of the $(d-2)$-face Γ-class, which is a "rotation group" of order

$$v_{ij}(D) := \frac{2\cdot m_{ij}(D)}{e_{ij}(D)}. \tag{4.2.4}$$

(A non trivial "rotation" about a $(d-2)$-face means that exactly this $(d-2)$-face is pointwise fixed.)

Similarly to Pr.1, the generators of $\Gamma_{ij}(D)$ can be expressed by the generators of Γ in both cases, and we get the relation

$$R_{ij}^{v_{ij}} = 1. \tag{4.2.5}$$

Considering all the free $(d-2)$-face Γ-classes of $\mathcal{F}$, we obtain a complete set of defining relations for the generators of Γ in accordance with Ass.2 (4.1.11–12). That means, any relation of Γ is a consequence of these $(d-2)$-face relations (4.2.5) in the form $\gamma^{-1}R_{ij}^{v_{ij}}\gamma = 1$, for certain $\gamma\in\Gamma$. □

Proposition 3 Any k-face of $(\mathcal{T},\Gamma)$ is characterized by a common k-vertex of the joint simplices in the barycentric subdivision $\mathcal{C}$, $k\in I$. The action of Γ on $\mathcal{T}$, and so on $\mathcal{C}$, induces these k-face orbits, or k-midpoint orbits, and their stabilizers Γ^k by the components of the partial D-symbol $\mathcal{D}^k$. This $\mathcal{D}^k$ is derived from $\mathcal{D}(\mathcal{M})$ by cancelling the σ_k-operation and keeping the others.

Any component $\mathcal{D}_c^k$ describes the local tiling round the corresponding k-vertex of $\mathcal{C}$, its barycentric subdivision $\mathcal{C}_c^k$ and the stabilizer Γ_c^k, furthermore the $(d-1)$-dimensional local tiling $(\mathcal{T}_c^k,\ \Gamma_c^k)$. We can choose a local fundamental domain $\mathcal{F}_c^k$, glued from those representing $\mathcal{C}_c^k$-simplices of $\mathcal{D}_c^k$, which lie also in the fundamental domain $\mathcal{F}$ of Γ. Then we get the generators of Γ_c^k expressed by generators of Γ. We can derive defining relations for Γ_c^k as consequences of the presentation of Γ by $\mathcal{F}$, analogously as in Pr.1–2. □

Proofs are not difficult and omitted here, see [3], [4], [5], [15]. In Sect.2–3 we have used these propositions in particular cases.

4.3. Equivariant Classifications and Symmetry Breakings of Tilings

Definition 1 *Two tilings* $(\mathcal{T}_1, \Gamma_1)$ *and* $(\mathcal{T}_2, \Gamma_2)$ *are combinatorially* (topologically) *equivariant and belong to the same equivariance class, iff there is a combinatorial bijection* $\varphi : \mathcal{T}_1 \to \mathcal{T}_2,\ T \mapsto T^\varphi$ (a topological mapping of the corresponding topological realizations), *preserving all the incidences in* $\mathcal{T}_1$ *resp.* $\mathcal{T}_2$ (i.e. the flag structures and barycentric subdivisions of $\mathcal{T}_1$ resp. $\mathcal{T}_2$) *such that* $\Gamma_2 = \varphi^{-1}\Gamma_1\varphi$*, i.e,* φ *is equivariant isomorphism* (homeomorphism).

The latter requirement means: if $\gamma_1 : \mathcal{T}_1 \to \mathcal{T}_1,\ T_1 \mapsto T_1^{\gamma_1}$ is any element of Γ_1 then

$$\gamma_2 := \varphi^{-1}\gamma_1\varphi : \mathcal{T}_2 \longrightarrow \mathcal{T}_2, \quad T^\varphi \longmapsto (T^{\gamma_1})^\varphi$$

also belongs to Γ_2. Moreover, we obtain all the elements of Γ_2 in this manner. □

By Def.1 we can classify various tilings, and we can ask for their realizations, i.e. their equivariantly isomorphic (homeomeorphic) images, e.g. in a space of constant curvature with a group Γ of isometries. The concept of D-symbol $\mathcal{D}(\mathcal{M})$, defined to a tiling $(\mathcal{T}, \Gamma)$, and their D-isomorphisms are of great importance, since we have the characteristic

Theorem 1 [4, §3] *The tilings* $(\mathcal{T}_1, \Gamma_1)$ *and* $(\mathcal{T}_2, \Gamma_2)$ *are combinatorially equivariant, iff their D-symbols* $\mathcal{D}_1(\mathcal{M}_1)$ *and* $\mathcal{D}_2(\mathcal{M}_2)$ *are D-isomorphic.* □

Definition 2 A *D-isomorphism* is a bijection $\varphi : \mathcal{D}_1 \to \mathcal{D}_2,\ D \mapsto D^\varphi$ with $(\sigma_i D)^\varphi = \sigma_i(D^\varphi)$ and $\mathcal{M}_1(D) = \mathcal{M}_2(D^\varphi)$ for any $D \in \mathcal{D}_1$ and $i \in I$. □

Definition 3 *A tiling* $(\mathcal{T}, \Gamma)$ *can be obtained by symmetry breaking from the tiling* $(\mathcal{T}^\star, \Gamma^\star)$, iff there is an isomorphism

$$\varphi : \mathcal{T} \longrightarrow \mathcal{T}^\star, \text{ such that } \varphi^{-1}\Gamma\varphi \subseteq \Gamma^\star.$$

In terms of D-symbols $\mathcal{D}(\mathcal{M})$ and $\mathcal{D}^\star(\mathcal{M}^\star)$, this means that there exists a surjective mapping $\psi : \mathcal{D} \to \mathcal{D}^\star,\ D \mapsto D^\star$ with $|\mathcal{D}| \geq |\mathcal{D}^\star|$, $(\sigma_i D)^\psi = \sigma_i(D^\psi)$ and $\mathcal{M}(D) = \mathcal{M}^\star(D^\psi)$ for any $D \in \mathcal{D}$ and $i \in I$. *Any such* ψ *is called a D-morphism,* it will be called proper iff $|\mathcal{D}| > |\mathcal{D}^\star|$, otherwise it will be called D-isomorphism. □

We remark that the above $\varphi : \mathcal{T} \to \mathcal{T}^\star$ *is determined by any simplex* $C \in \mathcal{C}(\mathcal{T})$ *and its* φ*-image* $C^\varphi \in \mathcal{C}(\mathcal{T}^\star)$ *uniquely if such* φ *exists.* Analogously $\psi : \mathcal{D} \to \mathcal{D}^\star$ above is uniquely determined by any D and D^ψ if such a D-morphism ψ exists.

These facts are very useful in searching either for symmetry breakings of a given tilings $(\mathcal{T}, \Gamma)$ or for the so-called *maximal tiling* $(\mathcal{T},\ \Gamma = \operatorname{Aut}\mathcal{T})$. Here $\operatorname{Aut}\mathcal{T}$ denotes the set of all self-isomorphisms of $\mathcal{T}$ in a combinatorial sense. But the existence of a topological, etc., metric realization of such a maximal tiling is not clear yet, in general.

4.4. Tilings for a Given D-Symbol and Their Realizations

We shall consider the most important inverse problem.

Problem 1 *Let a finite D-symbol $\mathcal{D}(\mathcal{M})$ be given with its Σ_I-operations and matrix function $\mathcal{M}$ described in (4.1.17) and (4.1.8–10) as in Ass.2 with C's replaced by D's but without requiring minimality in (4.1.8). Does there exist a tiling $(\mathcal{T}, \Gamma)$ in certain simply connected space $\mathcal{S}^d$ so that the D-symbol of $(\mathcal{T}, \Gamma)$ is D-isomorphic to the prescribed D-symbol $\mathcal{D}(\mathcal{M})$?*

If such a tiling exists, it would be unique up to combinatorial (topological) equivariance.

In general, the answer is no, if $\mathcal{D}(\mathcal{M})$ represents a "bad orbifold" in the sense of Thurston (see [8], [9] and [21]). As a simple example, we mention the following. Let $\mathcal{D} = \{1, 2, 3, 4\}$ be a D-set of 4 elements (now we write $D_x \leftrightarrow x$ for brevity) and we define

$$\sigma_0 : (1,2)(3)(4); \quad \sigma_1 : (1)(2,3)(4); \quad \sigma_2 : (1)(2)(3,4)$$

with the matrix function

$$(4.4.1) \quad m_{ij}(D_1) = m_{ij}(D_2) = m_{ij}(D_3) = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}; \qquad m_{ij}(D_4) = \begin{pmatrix} 1 & 6 & 2 \\ 6 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

This is a 2-dimensional D-symbol, $I = \{0, 1, 2\}$. (Draw a picture!) The procedure in Subsection 4.2 provides a (hypothetic) fundamental domain $\mathcal{F}$ by gluing the simplices: D_1 and D_2 along their 0-facets as σ_0 dictates, D_2 and D_3 along their 1-facets by σ_1, D_3 and D_4 along their 2-facets by σ_2. The free facets of $\mathcal{F}$

$$f_0(D_3), \;\; f_0(D_4), \;\; f_1(D_1), \;\; f_1(D_4), \;\; f_2(D_1), \;\; f_2(D_2)$$

would serve 6 facet-reflections denoted by

$$\mu_{03}, \;\; \mu_{04}, \;\; \mu_{11}, \;\; \mu_{14}, \;\; \mu_{21}, \;\; \mu_{22},$$

respectively. Fixing the simplex D_3, say, we have the generators for a (hypothetic) transformation group Γ as follows

$$\begin{aligned}
&\mu_{03} : D_3 \longmapsto D_3^{\mu_{03}} := \sigma_0 D_3; && \mu_{04} : D_3 \longmapsto D_3^{\mu_{04}} := \sigma_2\sigma_0\sigma_2 D_3; \\
&\mu_{11} : D_3 \longmapsto D_3^{\mu_{11}} := (\sigma_1\sigma_0)\sigma_1(\sigma_0\sigma_1) D_3; && \mu_{14} : D_3 \longmapsto D_3^{\mu_{14}} := \sigma_2\sigma_1\sigma_2 D_3; \\
&\mu_{21} : D_3 \longmapsto D_3^{\mu_{21}} := (\sigma_1\sigma_0)\sigma_2(\sigma_0\sigma_1) D_3. && \mu_{22} : D_3 \longmapsto D_3^{\mu_{22}} := \sigma_1\sigma_2\sigma_1 D_3.
\end{aligned}$$

Now $\mu_{03} = \mu_{04}$ follows from $\sigma_0 D_3 = \sigma_2\sigma_0\sigma_2 D_3$ i.e. $D_3 = \sigma_0\sigma_2\sigma_0\sigma_2 D_3$, since $m_{02}(D_3) = 2$. Similarly $\mu_{03} = \mu_{11}$ holds by

$$\sigma_0 D_3 = \sigma_1\sigma_0\sigma_1\sigma_0\sigma_1 D_3 \text{ because of } m_{01}(D_3) = m_{01}(D_2) = m_{01}(D_1) = 3.$$

Thus we have a reflection

$$\mathrm{m}_1 := \mu_{03} = \mu_{04} = \mu_{11}$$

in a facet of $\mathcal{F}$. In the same manner we have another reflection

$$\mathrm{m}_2 := \mu_{14} = \mu_{22} = \mu_{21}$$

in the other facet of $\mathcal{F}$. This is the consequence of

$$m_{12}(D_3) = m_{12}(D_4) = m_{12}(D_2) = 3 \text{ and } m_{02}(D_2) = m_{02}(D_1) = 2.$$

So, our group Γ should be generated by two reflections m_1 and m_2. The action of Γ, however, is not compatible now, because by (4.2.2) the product

$$\begin{aligned}\mathrm{m}_1\mathrm{m}_2 = \mu_{11}\mu_{21} \,:\, D_3 \longmapsto (D_3^{\mu_{11}})^{\mu_{21}} &:= \big[(\sigma_1\sigma_0)\sigma_1(\sigma_0\sigma_1)D_3\big]^{\mu_{21}} = \\ = (\sigma_1\sigma_0)\sigma_1(\sigma_0\sigma_1)(\sigma_1\sigma_0)\sigma_2(\sigma_0\sigma_1)D_3 &= (\sigma_1\sigma_0)\sigma_1\sigma_2(\sigma_0\sigma_1)D_3\end{aligned}$$

would have the order $m_{12}(D_1) = 3$, from one side, and the order $m_{01}(D_4) = 6$ from the other side by

$$\begin{aligned}\mathrm{m}_1\mathrm{m}_2 = \mu_{04}\mu_{14} \,:\, D_3 \longmapsto (D_3^{\mu_{04}})^{\mu_{14}} &:= (\sigma_2\sigma_0\sigma_2D_3)^{\mu_{14}} \\ = \sigma_2\sigma_0\sigma_2\sigma_2\sigma_1\sigma_2D_3 &= \sigma_2(\sigma_0\sigma_1)\sigma_2D_3.\end{aligned}$$

Hence, this example does not give rise to a realizable tiling.

If we choose $m_{01}(D_4) = 3$ in (4.4.1), then the fundamental domain $\mathcal{F}$ and its images under the dihedral group

$$\Gamma = \big(\mathrm{m}_1,\, \mathrm{m}_2 \quad - \quad 1 = \mathrm{m}_1^2 = \mathrm{m}_2^2 = (\mathrm{m}_1\mathrm{m}_2)^3\big)$$

provide a tetrahedron tiling on the sphere $\mathbf{S}^2$. The sphere would be the universal covering manifold of our modified orbifold. The original D-symbol describes a tetrahedron tiling on the "tear-drop" $(+, 0; [2]; \{\,\})$ which is a bad orbifold with the sphere as underlying manifold. This tear-drop is the universal cover of our bad orbifold $(+, 0; [\,]; \{(3, 6)\})$ represented by the original D-symbol, i.e. it is orientable with genus 0, no rotation centre and with dihedral corners of order 3 and 6 on its one boundary component [21]. In the paper [15] the algorithmic extension of Poincaré's polyhedron theorem (Th.8.1) has been formulated for the existence of a tiling with prescribed fundamental polyhedron and for the realization in a space $\mathcal{S}^d$ of constant curvature.

A hopeful strategy of the proof, which is due to A.D. Aleksandrov, can be applied to attack our Problem 1 as well. We can follow the opposite way as in Sect.4.1–2.

First we glue a connected fundamental domain $\mathcal{F}$ from $|\mathcal{D}|$ representing simplices. Choose a starting simplex C_1 corresponding to a vertex D_1 in the D-graph $\mathcal{D}$, and glue to C_1 the other simplices C_x as for D_x the Σ_I-operations dictate ($I = \{0, 1, \dots, d\}$). Though $\mathcal{F}$ is not unique, Σ_I also pairs the free simplex facets of $\mathcal{F}$, starting to generate a group Γ, moreover a combinatorial tiling $(\mathcal{F}, \Gamma)$ and the barycentric simplex tiling $\mathcal{C}(\Sigma_I, M, C_1)$. The matrix function $\mathcal{M}$ describes the branching numbers of $\mathcal{C}$ and also the defining relations for the generators of Γ in the same time, as we indicated at Pr.1–2 and at Ass.1–3 in the other direction.

Our conjecture is, the topological construction is possible (then unique up to equivariance) iff it is compatible round $(d-2)$-faces of $\mathcal{F}$. This seems to be satisfactory to exclude bad orbifolds [21]. As our example (4.4.1) illustrated, this condition is necessary. Of course, we could define the $(d-1)$-facets of $\mathcal{F}$ and also the other dimensional faces in the combinatorial terminology of $\mathcal{C}(\Sigma_I, M, C_1)$. The partial D-symbol $\mathcal{D}^k(\mathcal{M})$ obtained from $\mathcal{D}(\mathcal{M})$ by cancelling the σ_k-operation ($k \in I$), consists of as many components, as many Γ-equivalence classes of the k-labelled vertices of $\mathcal{C}$ we have. Any component of $\mathcal{D}^k(\mathcal{M})$ describes the stabilizer Γ^k of the corresponding k-vertex class and the local tiling $(\mathcal{T}^k, \Gamma^k)$ by means of a $(d-1)$-dimensional fundamental domain $\mathcal{F}^k$. So we may have an inductive procedure to define the tiling $(\mathcal{F}, \Gamma)$ so $\mathcal{C}(\Sigma_I, M, C_1)$ and $(\mathcal{T}, \Gamma)$ further and further from the starting simplex C_1. If the fundamental domain $\mathcal{F}$, with its face structure, has a nice topology, then we can guarantee the space $\mathcal{S}^d = \Delta_I \times \mathcal{C}$ at (4.1.2–4) to be simply connected by Ass.2 (4.1.11–12). We can go by induction on the dimension and on the simplicial distance from C_1.

Of course, this was only a sketch of a hypothetic procedure for constructing the topological realization of our tiling. Together with the colleagues in Bielefeld we are working on this problem [5].

In our case, where the Euclidean space $\mathbf{E}^3$ and its combinatorial cube tiling $\mathcal{T}$ is given, the procedure is straightforward. E.g. the D-symbol under formulas (3.3.1) and (3.1) in Sect.3 implied the fundamental domain $\mathcal{F} := ABCDQQ_2$ (Fig.2.f), the facet pairing generators, given in Table 3.No.37, the defining relations and the vertex stabilizers. The only problem was to read off the group $\Gamma = \mathbf{P4_232}$ from the International Tables [10]. In [16] and [18] we have described some tilings in non-Euclidean metric spaces of constant curvature, moreover tilings in other homogenous Riemannian spaces as in $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$, $\widetilde{\mathbf{SL_2R}}$, **Nil** [19]. Our intention is to give criteria for realizations of D-symbols by polyhedral tilings in $\mathbf{E}^3$ where Γ is a crystallographic space group. This seems to have applications also in non-mathematical fields as in the theory of solid state matter.

4.5. Algorithm for Classifying Euclidean Space Tilings $(\mathcal{T}, \Gamma)$ in $\mathbf{E}^3$ with Given Transitivity Properties

We describe principial steps which will be performed in various computer programs. We follow the above strategy in Sect.4.4. Our joint paper [5] realized this algorithm first for facet-transitive 3-dimensional tilings in $\mathbf{E}^3$.

Step 1 The required transitivity properties determine the substructures and so the structures of the possible D-diagrams $\mathcal{D}$ with Σ_I-operations, $I = \{0, 1, 2, 3\}$. The number $|\mathcal{D}|$ has to be finite as a consequence of the following steps. □

Step 2 For any D-diagram $\mathcal{D}$ we consider the $\langle\sigma_i, \sigma_j\rangle$-orbits i.e. $\mathcal{D}_{ij}$-components (in Pr.1), their types (C.A) or (C.B) and their corresponding parameters e_{ij}. For each $\mathcal{D}_{ij}$ component we choose a rotation order $v_{ij}(D) = 1, 2, 3, \ldots$ and so we choose by (4.2.3–4) a corresponding matrix element $m_{ij}(D)$ for each $D \in \mathcal{D}_{ij}$ according to (4.1.17) and (4.1.8–10). The possible v_{ij}'s will be bounded by the Step 3 or finally by Step 4. So we have completed the D-diagram to a D-symbol $\mathcal{D}(\mathcal{M})$. □

Step 3 *The partial symbols $\mathcal{D}^k$*, obtained by cancelling σ_k-operations from $\mathcal{D}$, $k \in I$, *shall have components determining finite stabilizers Γ^k for any k-midpoint orbit.* So each component $\mathcal{D}_c^k$ describes a spherical tiling $(\mathcal{T}_c{}^k, \Gamma_c^k)$. Then the Euler theorem yields positive curvature for any connected $\mathcal{D}^k$-component $\mathcal{D}_c^k$, i.e.

$$\sum_{D}^{\mathcal{D}_c^k} \left[\left(\sum_{i<j}^{I \setminus \{k\}} \frac{1}{m_{ij}(D)} \right) - 1 \right] > 0. \tag{4.5.1}$$

□

This 2-dimensional "curvature formula" is deciding for 2-dimensional tilings (see [4] and [9] for the existence criteria of 2-tilings). E.g. if $k = 0$, then any component $\mathcal{D}_c^0(\mathcal{M}_c^0)$ of $\mathcal{D}(\mathcal{M})$, with the corresponding submatrix function $\mathcal{M}_c^0$ satisfying (4.5.1), shall describe a spherical tiling $\mathcal{T}_c^0$ around a vertex (0-face) and a symmetry group Γ_c^0 which will be restricted further in Step 4.

Step 4 *The group Γ determined by $\mathcal{D}(\mathcal{M})$ has to be a crystallographic space group from among the 219 isomorphism classes.* Thus the rotation orders in Step 2 correspond to the Barlow condition $v_{ij}(D) = 1, 2, 3, 4, 6$. Moreover, the substructures of stabilizers Γ^k, $k \in I$, help choosing the candidate Γ on the base of Wyckoff positions from [10]. □

Step 5 The space group candidate Γ corresponds to $\mathcal{D}(\mathcal{M})$ indeed, if a combinatorially given fundamental domain $\mathcal{F}$ by $\mathcal{D}(\mathcal{M})$ can be realized also metrically (up to equivariant bijection) as a metric fundamental domain in $\mathbf{E}^3$. *The generalized Poincaré theorem* (in the sense of Sect.4.4 and [15]) says that all the partial D-symbol components $\mathcal{D}_{ij}$ and the corresponding stabilizers Γ_{ij} have to be realized metrically in $\mathbf{E}^3$ by the space group Γ (see e.g. [15] for some details). These criteria guarantee the existence of the equivariant bijection in question. As all the space groups are well-known, such a copy algorithm can be constructed by the coordinate presentation of the stabilizers in Γ. □

The critical points of our algorithm are Step 1 and Step 5. Indeed, in searching all the solid transitive tilings in $\mathbf{E}^3$ we cannot guarantee the finite set of possible D-diagrams. This fact is known since Heesch's and Zamorzayev's examples. But assuming e.g. face transitivity, we can prove that $|\mathcal{D}| \leq 20$ (as in [5]).

In non-Euclidean spaces, as in the spherical space $\mathbf{S}^3$ and the Bolyai–Lobachevskian hyperbolic space $\mathbf{H}^3$, the rotational orders maybe not bounded in Step 2.

A given stabilizer collection for $\mathcal{D}(\mathcal{M})$ in Step 3 may lead to different Γ's of $\mathbf{E}^3$ (as the 10 fixed point free space groups show), moreover, stabilizers in different spaces may coincide.

We are working on the algorithmic problems mentioned in Sect.4.4 together with A. Dress, D. Huson, O. Delgado Friedrichs, I. Prok, J. Szirmai, Z. Lučić, M. Stojanović and other colleagues.

We hope that analogous algorithms have applications in various fields of investigation.

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