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On Conformally Flat Hypersurfaces and Guichard's Nets

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Abstract. A local classification result for conformally flat hypersurfaces in a four dimensional space form is obtained: they are characterized by the existence of special principal curvature coordinates. By means of conformal coordinate charts these coordinates are related to (conformal classes of) Guichard nets in Euclidean three space which reflect some geometric properties of the conformally flat hypersurface.

1. Introduction

A Riemannian manifold (M, g) is called CONFORMALLY FLAT, if there exist conformal coordinate charts around each point, or — equivalently — if to each point $p \in M$ there exists a function u defined on some neighbourhood of this point such that the metric $e^{2u}g$ on that neighbourhood is flat.

For hypersurfaces in space forms we may pose the following

Question: *How does a hypersurface, which is conformally flat with respect to the induced metric, look like; or — more precisely — how can we characterize conformally flat hypersurfaces by means of their extrinsic geometry?*

Surfaces in three space are always conformally flat: this is the well known theorem on the existence of isothermal coordinates. For higher dimensional hypersurfaces this question was first posed by E. Cartan in 1917 [3], who gave a satisfactory answer to the question in case the dimension of the hypersurface is greater then three:

*These notes are part of my doctoral thesis [5].

A hypersurface of dimension $n \geq 4$ in a space form is conformally flat if and only if it is a (branched) channel hypersurface[1]*, i.e. if it envelopes a (perhaps stationary) one parameter family of spheres*[2].

In the remaining case of three dimensional hypersurfaces in four space the answer appears to be more difficult — in dimension three the corresponding equations for the principal curvatures are differential instead of algebraic as in higher dimensions. The only characterization in this case is — as far as I know — a somewhat surrealistic

Theorem of Cartan: *A three dimensional hypersurface with three distinct principal curvatures in a four dimensional space form is conformally flat if and only if the six umbilic distributions (i.e. the distributions of those (complex) planes, where the second fundamental form is a multiple of the first fundamental form) are integrable.*

In these notes I will try to give a more satisfactory answer to the above question, which also leads to a possibility to "look at" three dimensional conformally flat hypersurfaces.

Before starting I will give some

Examples: (i) It is well known (see for example [3]) that three dimensional *(branched) channel hypersurfaces* are conformally flat.

(ii) From the fact that a Riemannian product manifold $(M^2 \times I, g + dt)$ is conformally flat if (and only if) (M^2, g) is of constant Gaussian curvature [7] it follows that the *conformal product hypersurfaces*[3]

- cones over a surface of constant Gaussian curvature in the three sphere $S^3 \subset I\!R^4$,
- cylinders over a surface of constant Gaussian curvature in Euclidean three space $I\!R^3 \subset I\!R^4$ and
- rotational hypersurfaces over a surface of constant Gaussian curvature in hyperbolic three space H^3 thought as the Poincaré half space[4]

are conformally flat. These are conformally flat hypersurfaces with (in general) three distinct principal curvatures.

(iii) But not all three dimensional hypersurfaces are conformally flat: here the *tubes of the Veronese surface* in S^4 are appropriate examples[5]

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[1]In 1985 M. doCarmo, M. Dajczer and F. Mercuri [2] gave a classification of *compact* conformally flat hypersurfaces in Euclidean space of dimension $n+1 \geq 5$: topologically these hypersurfaces may be classified by the number of their handles quite analogous to the classification of compact surfaces.

[2]Channel hypersurfaces may be characterized as quasiumbilic hypersurfaces (i.e. there is one principal curvature of multiplicity at least $n-1$); thus a characterization of conformally flat hypersurfaces in terms of their principal curvatures is obtained.

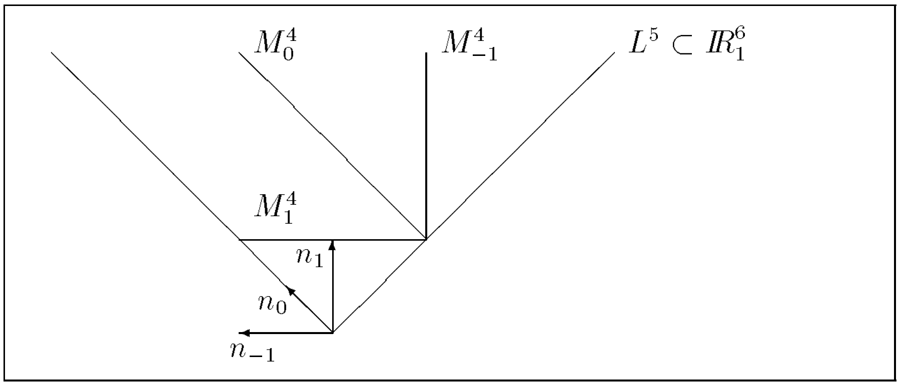
[3]These hypersurfaces will be discussed in section 4 in detail.

[4]The space $S^1 \times H^3$ is conformally equivalent to Euclidean four space $I\!R^4 \setminus \{x \| x_1 = x_2 = 0\}$!

[5]There are two classes of Möbius homogeneous hypersurfaces in four dimensional Möbius space [9]: cones over Clifford tori in S^3, which are conformally flat, and tubes of the Veronese surface, which are not.

2. Submanifolds of the light cone

It is well known that the Euclidean, hyperbolic and elliptic geometries are subgeometries of the Möbius geometry [1, par.19]: identifying the Möbius transformations of four dimensional (Möbius) space with the isometries of Minkowski space $I\!R^6_1$ we get the corresponding groups of motions as those subgroups fixing some vector $n_k \in I\!R^6_1$ — according to the sign of $k = \langle n_k, n_k\rangle$ hyperbolic ($k > 0$), Euclidean ($k = 0$) or elliptic ($k < 0$) groups of motions are obtained.



Moreover, it is not difficult to see that

$$M^4_k := \{y \in L^5 |\, \langle y, n_k\rangle = -1\} \tag{1}$$

is (endowed with the induced metric) a Riemannian manifold of constant sectional curvature k — here $L^5 = \{y \in I\!R^6_1 |\, \langle y, y\rangle = 0\}$ denotes the light cone in $I\!R^6_1$.

Now let $f : M^3 \to M^4_k$ be an immersion with unit normal field n (in our model of M^4_k such a unit normal field is characterized by $\langle n_k, n\rangle = 0$, $\langle f, n\rangle = 0$ and $\langle df, n\rangle \equiv 0$) and let $u \in C^\infty(M)$ be some function on M^3. Then a conformal change

$$\langle df, df\rangle \rightsquigarrow e^{2u}\langle df, df\rangle \tag{2}$$

of the induced metric may be modelled by the passage $f \rightsquigarrow e^u \cdot f$. In terms of Möbius geometry a unit normal field to $f : M^3 \to L^5$ with $\langle f, n\rangle = 0$ may be interpreted as a sphere congruence which is enveloped by f — the passage $n \rightsquigarrow n + a \cdot f$ for some function $a \in C^\infty(M)$ models the passage to another sphere congruence enveloped by f: in our research it will sometimes be convenient to fix a special sphere congruence (resp. a special unit normal field). These remarks give rise to the following

Definition: *Let $f : M^3 \to L^5$ be an immersion such that the induced metric $\langle df, df\rangle$ is positive definit, let $n : M^3 \to S^5_1 = \{y \in I\!R^6_1 |\, \langle y, y\rangle = 1\}$ be a unit normal field of f with $\langle f, n\rangle = 0$ (i.e. a sphere congruence which is enveloped by f) and let $u, a \in C^\infty(M)$ be functions on M^3. Then the passage*

$$f \rightsquigarrow e^u \cdot f, \quad n \rightsquigarrow n + a \cdot f \tag{3}$$

is called a CONFORMAL DEFORMATION *of the* STRIP *given by (f, n).*

Hence we can deform conformally flat immersions $f : M^3 \to M^4_k$ using a "conformal deformation" to a flat immersion $f : M^3 \to L^5$ (with spacelike differential) and vice versa[6].

To do further calculations we will choose some "adapted" framing for a given strip (f, n). After fixing a pseudo orthonormal basis of $\mathbb{R}^6_1$, i.e. a basis $(e_1, \dots, e_6)$ with

$$(\langle e_i, e_j \rangle)_{ij} = \begin{pmatrix} I & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix} =: E_6, \tag{4}$$

a framing may be represented by some matrix valued function

$$F : M \to O_1(6) \cong \{A \in Gl(6) \,|\, A^t E_6 A = E_6\}. \tag{5}$$

Definition: *A framing $F : M \to O_1(6)$ is called an* ADAPTED FRAMING *for the strip (f, n) if for each $p \in M$*

$$\mathrm{span}\{F(p)e_1, F(p)e_2, F(p)e_3\} = d_p f(T_p M), \;\; F(p)e_4 = n(p) \text{ and } F(p)e_5 = f(p). \tag{6}$$

Writing down the *connection form* of such an adapted framing for the strip (f, n) we obtain

$$\Phi := F^{-1} dF =: \begin{pmatrix} \Omega & \boldsymbol{\eta} \\ -E_3 \boldsymbol{\eta}^t & \boldsymbol{\nu} \end{pmatrix} : TM \to \boldsymbol{o}_1(6) \cong \{X \in \boldsymbol{gl}(6) \,|\, (E_6 X)^t + (E_6 X) = 0\} \tag{7}$$

where the components[7] are of the form

$$\begin{aligned} \Omega &= \begin{pmatrix} 0 & -\omega_{12} & -\omega_{13} \\ \omega_{12} & 0 & -\omega_{23} \\ \omega_{13} & \omega_{23} & 0 \end{pmatrix} : \quad TM \to \boldsymbol{o}(3), \\ \boldsymbol{\nu} &= \begin{pmatrix} 0 & 0 & \nu \\ -\nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \quad TM \to \boldsymbol{o}_1(3), \\ \boldsymbol{\eta} &= \begin{pmatrix} -\psi_1 & \omega_1 & \chi_1 \\ -\psi_2 & \omega_2 & \chi_2 \\ -\psi_3 & \omega_3 & \chi_3 \end{pmatrix} : \quad TM \to \boldsymbol{gl}(3). \end{aligned} \tag{8}$$

Herein the form Ω describes the Levi-Civita connection of the induced metric $\langle df, df \rangle = \sum \omega_i^2$, the second fundamental form of f with respect to n is given by $II := \sum \psi_i \omega_i$ and the forms χ_i give us the second fundamental form with respect to the isotropic normal field $\hat{f} = F e_6$.

We define a symmetric product on the space of $\boldsymbol{o}_1(6)$-valued 1-forms by $[\Phi \wedge \Psi](x, y) := [\Phi(x), \Psi(y)] - [\Phi(y), \Psi(x)]$; using this notation we may formulate the following classical

Lemma: *To a $\boldsymbol{o}_1(6)$-valued 1-form Φ there exists a map $F : M \to O_1(6)$ such that $\Phi = F^{-1} dF$ if and only if it satisfies the Maurer-Cartan equation*

$$0 = d\Phi + \tfrac{1}{2}[\Phi \wedge \Phi]. \tag{9}$$

For the connection form of some adapted framing the Maurer-Cartan equation gives us the fundamental equations for our strip (f, n):

[6] Since we do local theory we may assume the function u to be defined on all M^3.

[7] In the context of maps to symmetric spaces this decomposition is very natural: this will be described in some forthcoming paper [6], see also [5].

(i) the Codazzi equation

$$0 = d\boldsymbol{\eta} + \Omega \wedge \boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\nu}; \tag{10}$$

writing this equation in components we obtain

$$0 = d\psi_i - \sum_{j=1}^{3} \omega_{ij} \wedge \psi_j + \omega_i \wedge \nu, \tag{11}$$

$$0 = d\omega_i - \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \tag{12}$$

$$0 = d\chi_i - \sum_{j=1}^{3} \omega_{ij} \wedge \chi_j - \psi_i \wedge \nu; \tag{13}$$

(ii) the Ricci equation

$$d\boldsymbol{\nu} = E_3 \boldsymbol{\eta}^t \wedge \boldsymbol{\eta}; \tag{14}$$

here the component equations are

$$0 = \sum_{j=1}^{3} \psi_j \wedge \omega_j, \tag{15}$$

$$0 = \sum_{j=1}^{3} \chi_j \wedge \omega_j, \tag{16}$$

$$d\nu = \sum_{j=1}^{3} \chi_j \wedge \psi_j; \tag{17}$$

(iii) the Gauß equation

$$\boldsymbol{\varrho} = (\varrho_{ij})_{ij} = \boldsymbol{\eta} \wedge E_3 \boldsymbol{\eta}^t \tag{18}$$

which gives us only one kind of component equations

$$\varrho_{ij} := -d\omega_{ij} + \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} = \psi_i \wedge \psi_j + \omega_i \wedge \chi_j + \chi_i \wedge \omega_j \tag{19}$$

— the form $\boldsymbol{\varrho}$ gives us the curvature tensor of $(M^3, \sum \omega_i^2)$.

First we should remark that the Ricci equation (15) shows that the second fundamental form II of (f, n) is symmetric with respect to the induced metric $\sum \omega_i^2$ — as it should be. So we may arrange the tangential part $\{Fe_1, Fe_2, Fe_3\}$ of the framing such that the corresponding Weingarten tensor field is diagonal:

$$\psi_i = a_i \omega_i, \tag{20}$$

where the functions a_i denote the principal curvatures of the strip (f, n). Such a framing we refer to as an *adapted principal curvature framing.* Defining the vector fields x_1, x_2 and x_3 by the equations

$$df(x_i) := Fe_i \tag{21}$$

we obtain unit principal curvature vector fields on M^3.

Applying a conformal deformation to the strip (f, n) an adapted framing for the deformed strip $(\tilde{f}, \tilde{n}) = (e^u \cdot f, n + a \cdot f)$ is given by

$$\tilde{F} = F \cdot T, \text{ where } T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -u_1 e^{-u} \\ 0 & 1 & 0 & 0 & 0 & -u_2 e^{-u} \\ 0 & 0 & 1 & 0 & 0 & -u_3 e^{-u} \\ 0 & 0 & 0 & 1 & 0 & -a e^{-u} \\ u_1 & u_2 & u_3 & a & e^u & -\frac{1}{2} e^{-u} (\Sigma u_j^2 + a^2) \\ 0 & 0 & 0 & 0 & 0 & e^{-u} \end{pmatrix} \tag{22}$$

with $u_i := du(x_i)$. Note that if F is a principal curvature framing for (f, n) the new framing $\tilde{F}$ will be a principal curvature framing for $(\tilde{f}, \tilde{n})$: conformal deformations preserve the principal curvature directions x_i. Calculating the corresponding connection form $\tilde{\Phi}$ we obtain the following transformation formulae:

$$\begin{aligned} \tilde{\omega}_i &= e^u \omega_i, && (23) \\ \tilde{\omega}_{ij} &= \omega_{ij} + u_i \omega_j - u_j \omega_i, && (24) \\ \tilde{\psi}_i &= \psi_i - a\omega_i, && (25) \\ \tilde{\chi}_i &= e^{-u} (\chi_i + a\psi_i - \tfrac{1}{2} a^2 \omega_i - \tau_i) \\ & \quad \text{with } \tau_i := du_i - \sum_{j=1}^{3} u_j \omega_{ij} - u_i du + \tfrac{1}{2} \sum_{j=1}^{3} u_j^2 \omega_i, && (26) \\ \tilde{\nu} &= e^{-u} (\nu - da + a du - \sum u_j \psi_j); && (27) \end{aligned}$$

for the corresponding curvature forms we find the equation

$$\tilde{\varrho}_{ij} = \varrho_{ij} - (\omega_i \wedge \tau_j + \tau_i \wedge \omega_j). \tag{28}$$

With the help of the *Schouten forms*

$$\sigma_i = \varrho_i - \tfrac{\varrho}{4} \omega_i, \tag{29}$$

where

$$\begin{aligned} \varrho_i &= \textstyle\sum_{j=1}^3 \varrho_{ij}(x_j, .), \\ \varrho &= \textstyle\sum_{i=1}^3 \varrho_i(x_i) \end{aligned} \tag{30}$$

describe the Ricci tensor and the scalar curvature, we may express the curvature forms by

$$\varrho_{ij} = \omega_i \wedge \sigma_j + \sigma_i \wedge \omega_j \tag{31}$$

since on a three dimensional manifold the conformal Weyl tensor always vanishes. Hence our immersion f is conformally flat if and only if there is some function $u \in C^\infty(M)$ such that $\sigma_i = \tau_i$. The integrability condition is given in the famous

Theorem of Weyl-Schouten: *A Riemannian manifold (M^n, g) is conformally flat if and only if the Schouten tensor is Codazzi in the case $n = 3$:*

$$0 = d\sigma_i - \sum \omega_{ij} \wedge \sigma_j; \tag{32}$$

the conformal Weyl tensor vanishes in the case $n > 3$:

$$0 = \varrho_{ij} - (\omega_i \wedge \sigma_j + \sigma_i \wedge \omega_j). \tag{33}$$

Writing down *Cartan's forms*

$$\beta_i := \sigma_i - \chi_i \tag{34}$$

the combination of this integrability condition with the Codazzi equations (13) gives us the following

Lemma of Cartan: *An immersion $f : M^3 \to L^5$ (with spacelike differential) is conformally flat if and only if*

$$0 = d\beta_i - \sum \omega_{ij} \wedge \beta_j + \psi_i \wedge \nu. \tag{35}$$

Herein the forms in the formula come from an arbitrary adapted framing of some strip (f, n).

Reformulating the Gauß equation (19) using Cartan's forms β_i we can calculate these forms:

$$\beta_i = \begin{cases} \frac{1}{2}(a_1a_2 - a_2a_3 + a_3a_1)\omega_1 \\ \frac{1}{2}(a_1a_2 + a_2a_3 - a_3a_1)\omega_2 \\ \frac{1}{2}(-a_1a_2 + a_2a_3 + a_3a_1)\omega_3 \end{cases}, \tag{36}$$

where we use a principal curvature framing[8] (cf.[3, eqs.28"]).

3. Guichard nets

Let us start with Cartan's theorem (p.316): a three dimensional hypersurface with three distinct principal curvatures is conformally flat if and only if the six umbilic distributions are integrable. It is not difficult to see that these distributions are given by

$$0 = \alpha_i \pm \alpha_j \quad (i \neq j), \tag{37}$$

where the forms α_i are defined by

$$\begin{aligned} \alpha_1 &= \sqrt{a_3 - a_1}\sqrt{a_1 - a_2}\,\omega_1, \\ \alpha_2 &= \sqrt{a_1 - a_2}\sqrt{a_2 - a_3}\,\omega_2, \\ \alpha_3 &= \sqrt{a_2 - a_3}\sqrt{a_3 - a_1}\,\omega_3 : \end{aligned} \tag{38}$$

for example, choosing an orthonormal basis[9] $\{x = x_3, y = \frac{\sqrt{a_2-a_3}}{\sqrt{a_2-a_1}}x_1 \mp \frac{\sqrt{a_1-a_3}}{\sqrt{a_1-a_2}}x_2\}$ for the distribution $\alpha_1 \pm \alpha_2 = 0$ (x_i still denoting unit principal curvature vector fields corresponding to the principal curatures a_i) we see that $I\!I(x,x) = I\!I(y,y) = a_3$ and $I\!I(x,y) = 0$.

Moreover the forms α_i are invariant under conformal deformations and so it seems to be convenient to give them a name:

[8]In a coordinate free manner the tensor $B = \sum \beta_i x_i$ is given by

$$B = \mathrm{tr}A \cdot A - A^2 - \tfrac{1}{4}((\mathrm{tr}A)^2 - \mathrm{tr}A^2).$$

Using this formula it is very easy to calculate the Schouten tensor of a hypersurface immersion into a space form M^4_k.

[9]Clearly two of the forms α_i take values in the imaginary axis while the third has real values — so we have to work on the complexified tangent bundle $T^{\mathbb{C}}M$.

Definition: *The forms α_i defined by (38) will be called* CONFORMAL FUNDAMENTAL FORMS *of the strip (f, n).*

From Cartan's theorem one may presume the following

Theorem: *A hypersurface in a four dimensional space form is conformally flat if and only if the forms α_i are integrable (closed).*

Proof: First consider the open set where (f, n) has three distinct principal curvatures. Setting

$$b_i := \begin{cases} \frac{1}{2}(a_1a_2 - a_2a_3 + a_3a_1) \\ \frac{1}{2}(a_1a_2 + a_2a_3 - a_3a_1) \\ \frac{1}{2}(-a_1a_2 + a_2a_3 + a_3a_1) \end{cases}, \tag{39}$$

the Gauß equation (19) reads

$$0 = b_i + b_j - a_ia_j \quad (i \neq j)\ ; \tag{40}$$

combination with the Codazzi equations (12) yields

$$\begin{aligned} \sqrt{2b_i - a_i^2}d\alpha_i &= \sqrt{2b_i - a_i^2}d(\sqrt{2b_i - a_i^2}\omega_i) \\ &= (db_i - a_ida_i) \wedge \omega_i + (2b_i - a_i^2)d\omega_i \\ &= ((db_i - a_i\nu) - a_i(da_i - \nu)) \wedge \omega_i \\ &\quad + \sum((b_i - b_j) - a_i(a_i - a_j) + (b_i + b_j - a_ia_j))\omega_{ij} \wedge \omega_j \\ &= (d\beta_i - \sum \omega_{ij} \wedge \beta_j + \psi_i \wedge \nu) - a_i(d\psi_i - \sum \omega_{ij} \wedge \psi_j + \omega_i \wedge \nu). \end{aligned} \tag{41}$$

Hence by Cartan's lemma (p.321) and the Codazzi equations (11) the assertion follows.

Now let us consider the interior of this set's complement — the set where f parametrizes a (branched) channel hypersurface. Here it is clear that f is conformally flat; we have to show that the conformal fundamental forms are closed. Choosing an appropriate normal field n we may assume $a_2 = a_3 = 0$. But now the normal field n is exactly the one parameter family of sphere enveloped by f, hence we have

$$\begin{aligned} 0 = dn(x_2) = -\nu(x_2) \cdot f, \\ 0 = dn(x_3) = -\nu(x_3) \cdot f, \end{aligned} \tag{42}$$

which together with the Codazzi equations (11) implies

$$d\alpha_1 = i \cdot d\psi_1 = -i \cdot \omega_1 \wedge \nu = 0. \tag{43}$$

The other conformal fundamental forms are clearly identically zero.

By a continuity argument we get the equivalence on all M^3. □

According to this theorem we are given canonical coordinates

$$(g_1, g_2, g_3) : M \to I\!R^3_2 \tag{44}$$

on conformally flat hypersurfaces[10]; since the form

$$(\alpha_1, \alpha_2, \alpha_3) : TM \to \mathbb{R}^3_2 \cong \mathbb{R} \times i\mathbb{R} \times i\mathbb{R} \tag{45}$$

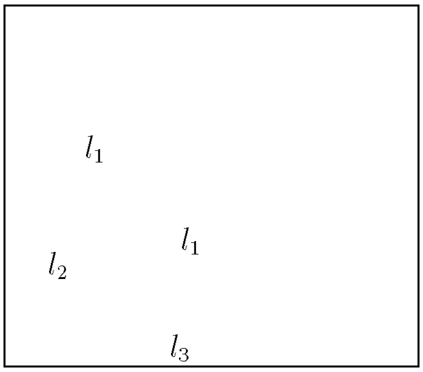
may be interpreted as a real form these coordinates have a (real) geometric interpretation[11]. Obviously these are principal curvature coordinates and hence they built an orthogonal coordinate system, i.e. the coordinate surfaces $g_i = const$ intersect orthogonally. For *Lamé's functions*

$$l_i = \|\tfrac{\partial}{\partial g_i}\| \tag{46}$$

we derive the following relation:

$$\sum_{i=1}^{3} l_i^2 = \tfrac{1}{(a_3-a_1)(a_1-a_2)} + \tfrac{1}{(a_1-a_2)(a_2-a_3)} + \tfrac{1}{(a_2-a_3)(a_3-a_1)} = 0. \tag{47}$$

This relation is conformally invariant; geometrically it means that the coordinate surfaces divide the Riemannian manifold $(M^3, \langle df, df\rangle)$ into infinitesimal rectangular parallelotopes such that two of the six diagonal rectangles are squares.



These remarks give rise to the following

Definition: *A triply orthogonal system of surfaces in a three dimensional Riemannian manifold*

$$g = (g_1, g_2, g_3) : (M, \langle .,. \rangle) \to \mathbb{R}^3_2 \tag{48}$$

will be called a GUICHARD NET[12] *if its Lamé functions l_i satisfy the condition (47).*

Combining this definition with the above theorem we obtain

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[10] Since we do local theory we may assume these coordinates to be defined all over M^3.

[11] Nevertheless it will be convenient to do the calculations with the complex forms.

[12] Triply orthogonal systems of surfaces satisfying this relation were first considered by C. Guichard in 1905 [4]. He refered to these triply orthogonal systems as an analogue of isothermic coordinates on Riemannian surfaces.

Corollary: *On each conformally flat hypersurface of a four dimensional space form there exists a Guichard net consisting of principal curvature surfaces.*

We now will calculate the connection form corresponding to a strip (f,n), where the immersion f is conformally flat, in terms of its Guichard net. Since the unit principal curvature vector fields $x_i = \frac{1}{l_i}\frac{\partial}{\partial g_i}$ are unit normal fields to the coordinate surfaces $g_i = const$ and since the directions x_j $(j \neq i)$ are principal curvature directions of these surfaces (Dupin's theorem, see [8]) the principal curvatures a_{ij} of the surfaces $g_i = const$ in the directions x_j are

$$a_{ij} = -\omega_{ij}(x_j) = \tfrac{1}{\|\nabla g_j\|} d\|\nabla g_j\|(x_i); \tag{49}$$

hence the Levi-Civita connection of $(M^3, \langle df, df\rangle)$ will be given by the forms

$$\omega_{ij} = a_{ji}\omega_i - a_{ij}\omega_j = \tfrac{a_{ji}}{\|\nabla g_i\|}dg_i - \tfrac{a_{ij}}{\|\nabla g_j\|}dg_j \tag{50}$$

— the metric fundamental forms in our coordinates are simply

$$\omega_i = \tfrac{1}{\|\nabla g_i\|}dg_i. \tag{51}$$

Remembering that the conformal fundamental forms were defined using the principal curvatures of the strip (f,n) we obtain

$$a_3 - a_1 = \tfrac{l_2}{l_3 l_1}, a_1 - a_2 = \tfrac{l_3}{l_1 l_2}, a_2 - a_3 = \tfrac{l_1}{l_2 l_3}; \tag{52}$$

choosing the normal field (the enveloped sphere congruence) such that one principal curvature vanishes this gives

$$\psi_1 = -\tfrac{\|\nabla g_3\|}{\|\nabla g_2\|}dg_1, \psi_2 = \tfrac{\|\nabla g_3\|}{\|\nabla g_1\|}dg_2, \psi_3 = 0. \tag{53}$$

It is a bit more difficult to determine the normal connection form ν: using the Codazzi equations (11) and (12) we first derive

$$\begin{array}{rcl} 0 & = & (da_i - \nu)\wedge\omega_i + \sum_{j=1}^3 (a_i - a_j)\omega_{ij}\wedge\omega_j \\ & = & (da_i - \nu - \sum a_{ji}(a_i - a_j)\omega_j)\wedge\omega_i, \end{array} \tag{54}$$

which implies

$$da_i - \nu = k_i\omega_i + \sum a_{ji}(a_i - a_j)\omega_j \tag{55}$$

with suitable functions k_i. Now f is conformally flat and hence the form α_2 for example is integrable; this gives us

$$\begin{array}{rcl} 0 & = & 2\sqrt{(a_1-a_2)(a_2-a_3)}\,d(\sqrt{(a_1-a_2)(a_2-a_3)}\,\omega_2) \\ & = & ((a_1-a_2)(da_2-da_3) + (da_1-da_2)(a_2-a_3))\wedge\omega_2 \\ & & +2(a_1-a_2)(a_2-a_3)d\omega_2 \\ & = & ((a_1-a_2)k_3 + (a_2-a_3)(a_3-a_1)(a_{31}-a_{32}))\omega_2\wedge\omega_3 \\ & & +((a_2-a_3)k_1 + (a_3-a_1)(a_1-a_2)(a_{12}-a_{13}))\omega_1\wedge\omega_2, \end{array} \tag{56}$$

and so we get

$$\begin{array}{rcl} \nu & = & \nu - da_3 \\ & = & -(a_3-a_1)a_{13}\omega_1 + (a_2-a_3)a_{23}\omega_2 + \frac{(a_2-a_3)(a_3-a_1)}{a_1-a_2}(a_{31}-a_{32})\omega_3 \\ & = & -a_{13}\frac{\|\nabla g_3\|}{\|\nabla g_2\|}dg_1 + a_{23}\frac{\|\nabla g_3\|}{\|\nabla g_1\|}dg_2 + (a_{31}-a_{32})\frac{\|\nabla g_3\|^2}{\|\nabla g_1\|\|\nabla g_2\|}dg_3. \end{array} \tag{57}$$

Finally, to determine the forms χ_i we will assume f to be flat. So we have $\chi_i = -\beta_i$ which yields

$$\chi_1 = \tfrac{\|\nabla g_3\|^2}{2\|\nabla g_1\|} dg_1, \chi_2 = \tfrac{\|\nabla g_3\|^2}{2\|\nabla g_2\|} dg_2, \chi_1 = \tfrac{1}{2}\|\nabla g_3\| dg_3. \tag{58}$$

Summerizing these results we get the connection form of a strip (f, n) — f being flat and n being a principal curvature sphere congruence of f — by

$$\begin{array}{lll} \omega_1 = \frac{1}{\|\nabla g_1\|} dg_1 & \psi_1 = -\frac{\|\nabla g_3\|}{\|\nabla g_2\|} dg_1 & \chi_1 = \frac{\|\nabla g_3\|^2}{2\|\nabla g_1\|} dg_1 \\ \omega_2 = \frac{1}{\|\nabla g_2\|} dg_2 & \psi_2 = \frac{\|\nabla g_3\|}{\|\nabla g_1\|} dg_2 & \chi_2 = \frac{\|\nabla g_3\|^2}{2\|\nabla g_2\|} dg_2 \\ \omega_3 = \frac{1}{\|\nabla g_3\|} dg_3 & \psi_3 = 0 & \chi_3 = -\frac{1}{2}\|\nabla g_3\| dg_3 \end{array}$$
$$\omega_{ij} = \tfrac{a_{ji}}{\|\nabla g_i\|} dg_i - \tfrac{a_{ij}}{\|\nabla g_j\|} dg_j$$
$$\nu = -a_{13}\tfrac{\|\nabla g_3\|}{\|\nabla g_2\|} dg_1 + a_{23}\tfrac{\|\nabla g_3\|}{\|\nabla g_1\|} dg_2 + (a_{31} - a_{32})\tfrac{\|\nabla g_3\|^2}{\|\nabla g_1\|\|\nabla g_2\|} dg_3 \ , \tag{59}$$

On the other hand we may use the flatness assumption to derive *Lamé's equations*

$$\begin{array}{rcl} 0 & = & d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj} \\ & = & -\omega_i \wedge da_{ji} - da_{ij} \wedge \omega_j + (a_{ji}^2 + a_{ij}^2 + a_{ki}a_{kj})\omega_i \wedge \omega_j \\ & & +a_{ki}(a_{ji} - a_{jk})\omega_i \wedge \omega_k + a_{kj}(a_{ij} - a_{ik})\omega_k \wedge \omega_j \end{array} \tag{60}$$

for $i \neq j$ respectively — in components — (see [8])

$$\begin{array}{ll} x_1 a_{23} = a_{13}(a_{23} - a_{21}) & x_1 a_{32} = a_{12}(a_{32} - a_{31}) \\ x_2 a_{31} = a_{21}(a_{31} - a_{32}) & x_2 a_{13} = a_{23}(a_{13} - a_{12}) \\ x_3 a_{12} = a_{32}(a_{12} - a_{13}) & x_3 a_{21} = a_{31}(a_{21} - a_{23}) \end{array}$$
$$\begin{array}{c} x_1 a_{12} + x_2 a_{21} = a_{12}^2 + a_{21}^2 + a_{31}a_{32} \\ x_2 a_{23} + x_3 a_{32} = a_{23}^2 + a_{32}^2 + a_{12}a_{13} \\ x_3 a_{31} + x_1 a_{13} = a_{31}^2 + a_{13}^2 + a_{23}a_{21} \ . \end{array} \tag{61}$$

Using $a_{ij} = l_j x_i l_j^{-1}$ and $x_i = \frac{1}{l_i}\frac{\partial}{\partial g_i}$ this system of nine partial differential equations for the principal curvatures of the coordinate surfaces can be reduced to a system of six partial differential equations for Lamé's functions

$$\begin{array}{rcl} \frac{\partial^2 l_1}{\partial g_2 \partial g_3} & = & \frac{1}{l_2}\frac{\partial l_1}{\partial g_2}\frac{\partial l_2}{\partial g_3} + \frac{1}{l_3}\frac{\partial l_1}{\partial g_3}\frac{\partial l_3}{\partial g_2} \\ \frac{\partial^2 l_2}{\partial g_3 \partial g_1} & = & \frac{1}{l_3}\frac{\partial l_2}{\partial g_3}\frac{\partial l_3}{\partial g_1} + \frac{1}{l_1}\frac{\partial l_2}{\partial g_1}\frac{\partial l_1}{\partial g_3} \\ \frac{\partial^2 l_3}{\partial g_1 \partial g_2} & = & \frac{1}{l_1}\frac{\partial l_3}{\partial g_1}\frac{\partial l_1}{\partial g_2} + \frac{1}{l_2}\frac{\partial l_3}{\partial g_2}\frac{\partial l_2}{\partial g_1} \end{array}$$
$$\begin{array}{l} 0 = \frac{\partial}{\partial g_2}\left(\frac{1}{l_2}\frac{\partial l_1}{\partial g_2}\right) + \frac{\partial}{\partial g_1}\left(\frac{1}{l_1}\frac{\partial l_2}{\partial g_1}\right) + \frac{1}{l_3^2}\frac{\partial l_1}{\partial g_3}\frac{\partial l_2}{\partial g_3} \\ 0 = \frac{\partial}{\partial g_3}\left(\frac{1}{l_3}\frac{\partial l_2}{\partial g_3}\right) + \frac{\partial}{\partial g_2}\left(\frac{1}{l_2}\frac{\partial l_3}{\partial g_2}\right) + \frac{1}{l_1^2}\frac{\partial l_2}{\partial g_1}\frac{\partial l_3}{\partial g_1} \\ 0 = \frac{\partial}{\partial g_1}\left(\frac{1}{l_1}\frac{\partial l_3}{\partial g_1}\right) + \frac{\partial}{\partial g_3}\left(\frac{1}{l_3}\frac{\partial l_1}{\partial g_3}\right) + \frac{1}{l_2^2}\frac{\partial l_3}{\partial g_2}\frac{\partial l_1}{\partial g_2}. \end{array} \tag{62}$$

Since we assumed f to be flat we may think of M^3 as a subset of $I\!R^3$ and of f as an isometric immersion[13] — so we are given some Guichard net in $I\!R^3$. This Guichard net is in fact uniquely determined up to a Möbius transformation $m : I\!R^3 \to I\!R^3$.

[13] Remember that we do local theory, so we don't worry about domains

$$\begin{tikzcd}
& M \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "m"] & \\
L^5 & & \mathbb{R}^3_2 \\
& \tilde{M} \arrow[ul, "\tilde{f} = e^u f \circ m^{-1}"] \arrow[ur, "\tilde{g} = g \circ m^{-1}"'] &
\end{tikzcd}$$

Conversely, one may ask whether from a given Guichard net in $\mathbb{R}^3$ we can reconstruct our conformally flat hypersurface (or equivalently our flat immersion into the light cone L^5). This is the question whether integrability of the connection form given by (59) holds when it is defined via an arbitrary Guichard net in $\mathbb{R}^3$. The answer is given in the following

Theorem: *To every conformally flat immersion $f: M^3 \to M^4_k$ with three distinct principal curvatures we may associate a canonically given Guichard net $g: U \subset \mathbb{R}^3 \to \mathbb{R}^3_2$ — this Guichard net is uniquely determined up to a Möbius transformation of $\mathbb{R}^3$;*

conversely, given a Guichard net $g: U \subset \mathbb{R}^3 \to \mathbb{R}^3_2$ we may write down the connection form (59) which then satisfies the Maurer-Cartan equation and hence gives a conformal immersion $f: \tilde{U} \subset U \to M^4_k$ with three distinct principal curvatures — here Möbius equivalent Guichard nets yield conformally equivalent immersions.

Proof: The first part is clear from the above corollary; the second part is a quite horrible calculation which I prefer to leave to the reader[14]: besides Lamé's equations and Guichard's condition one should use $d^2(\frac{1}{l_3}) = 0$. $\square$

4. Examples

To demonstrate how the geometry of conformally flat hypersurfaces may be encoded in the geometry of the associated Guichard nets I will examine the *conformal product hypersurfaces* mentioned in the introduction. Therefore, let $p: U \subset \mathbb{R}^2 \to M^3_k$ denote a principal curvature parametrization of a surface of constant Lipschitz- Killing curvature $c = a_1 \cdot a_2$ — and hence of constant Gaussian curvature $K = c + k$. To assure that the conformal product hypersurface will have three distinct principal curvatures we will assume that $c \neq 0$ and $a_1 \neq a_2$.

In the case $k > 0$

$$f: U \times (0, \infty) \to L^5, f(x, y, r) := \tfrac{1}{r}(rp(x, y), \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\|rp(x, y)\|^2) \tag{63}$$

will parametrize a cone over the K-surface $p(U)$ — here we think of $M^3_k \cong S^3(\frac{1}{\sqrt{k}}) \subset \mathbb{R}^4$ and of $\mathbb{R}^4$ as $\operatorname{span}\{e_1, \dots, e_4\} \subset \mathbb{R}^6_1$.

$$n: U \times (0, \infty) \to S^5_1, n(x, y, r) := (n_p(x, y), 0, 0), \tag{64}$$

[14] It may also be found in my thesis [5].

$n_p : M^2 \to S^3 \subset I\!R^4$ denoting the unit normal field of p, will give us a suitable unit normal field of f.

In the case $k = 0$

$$f : U \times I\!R \to L^5, f(x,y,t) := (t, p(x,y), \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}(t^2 + \|p(x,y)\|^2)) \tag{65}$$

gives us a parametrization of the cylinder over the K-surface $p(U) \subset I\!R^3 \subset I\!R^4$; an appropriate unit normal field is

$$n : U \times I\!R \to S^5_1, n(x,y,t) := (0, n_p(x,y), 0, -\sqrt{2}\langle n_p, p\rangle(x,y)). \tag{66}$$

Here we should remark that no conformal deformation is included and so the sphere congruence given by n is the tangent plane congruence, i.e. all spheres contain the point at infinity $Fe_6 = (0, \ldots, 0, \sqrt{2}) \in I\!R^6_1$.

In the case $k < 0$

$$(x,y,s) \mapsto (p_1(x,y)\cos s, p_1(x,y)\sin s, p_2(x,y), p_3(x,y)) \tag{67}$$

will describe a rotational hypersurface in $I\!R^4$ over the K-surface in Poincaré's half space $H^3 = \{(p_1,p_2,p_3) \in I\!R^3 |\, p_1 > 0\}$. Mapped to the light cone L^5 this gives — after a dilatation by p_1^{-1} —

$$(x,y,s) \mapsto (\cos s, \sin s, \tfrac{p_2}{p_1}(x,y), \tfrac{p_3}{p_1}(x,y), \tfrac{1}{\sqrt{2}p_1}(x,y), -\tfrac{1}{\sqrt{2}p_1}\sum_{j=1}^{3} p_j^2(x,y)). \tag{68}$$

Here we recover our initial K-surface from the last four components when we think of H^3 as a submanifold of $I\!R^4_1 \cong \mathrm{span}\{e_3, \ldots, e_6\}$. So it will be convenient to start directly with this point of view:

$$f : U \times I\!R \to L^5, f(x,y,s) := (\tfrac{1}{\sqrt{-k}}\cos s, \tfrac{1}{\sqrt{-k}}\sin s, p(x,y)) \tag{69}$$

parametrizes a rotational hypersurface over the K-surface $p(U)$ given in $H^3(\frac{1}{\sqrt{-k}}) \subset I\!R^4_1$.

$$n : U \times I\!R \to S^5_1, n(x,y,s) := (0, 0, n_p(x,y)) \tag{70}$$

defines a suitable unit normal field.

Calculating the connection forms of adapted principal curvature framings of all the strips given by (f, n) we see that in every case

$$\omega_{23} = \omega_{31} = 0 \tag{71}$$

— for a cone over a K-surface in a sphere with $c < 0$ we have for example

$$\begin{array}{lll} \omega_1 = \cos u \cdot dx & \psi_1 = -\sqrt{-c}\sin u \cdot dx & \chi_1 = \frac{k}{2}\cos u \cdot dx \\ \omega_2 = \sin u \cdot dy & \psi_2 = \sqrt{-c}\cos u \cdot dy & \chi_2 = \frac{k}{2}\sin u \cdot dy \\ \omega_3 = \frac{1}{r\sqrt{k}}dr & \psi_3 = 0 & \chi_3 = -\frac{\sqrt{k}}{2r}dr \\ \multicolumn{3}{c}{\omega_{12} = u_y dx + u_x dy\ ,\ \ \omega_{23} = \omega_{31} = 0} \\ \multicolumn{3}{c}{\nu = 0,} \end{array} \tag{72}$$

the function u describing the angle between the asymptotic lines on the K-surface.

So from (59) we see that on $(U \times I, \langle df, df\rangle)$ the curvatures

$$a_{32} = a_{31} = 0 \text{ (and } a_{23} = a_{13} = 0); \tag{73}$$

it follows that the surfaces $g_3 = const$ are totally geodesic — especially they are umbilic surfaces. Since the induced metric $\langle df, df\rangle$ on $U \times I$ is conformally flat we may choose a conformal coordinate system $h : (U \times I, \langle df, df\rangle) \to \mathbb{R}^3$ which will map the surfaces $g_3 = const$ to a family of pieces of spheres[15]

Conversely, starting from a Guichard net $g : \mathbb{R}^3 \to \mathbb{R}^3_2$ with a family $g_3 = const$ of umbilic surfaces we will see that the "reconstruction" of the corresponding conformally flat hypersurfaces yield exactly those given above. First we should remark that an easy calculation[16] shows that the orthogonal trajectories of the surface family $g_3 = const$ consists of circles. Now three cases may occur: picking two spheres from the surfaces[17] $g_3 = const$ they may

(i) not meet — here we may apply some Möbius transformation which maps the two spheres to concentric spheres; the orthogonal trajectories will be straight lines which meet in a point, the family $g_3 = const$ will consist of concentric spheres;

(ii) meet in a point — reflection at a sphere with this point as center transforms these two spheres to parallel planes; the orthogonal trajectories are then straight lines orthogonal to these planes, the family $g_3 = const$ will consist of parallel planes; or

(iii) meet in a circle — here we may choose the center of the reflection sphere on this circle; we obtain a family of planes $g_3 = const$ intersecting in a line and its orthogonal intersecting circles.

From these informations we are able to determine the connection form (59) starting with adapted coordinates on $\mathbb{R}^3$. During this procedure only one integration constant λ arises — this corresponds to the fact that different K-surfaces may determine (from the conformal point of view) the same conformal product hypersurface: this is obvious in the case of a cone over a K-surface in a sphere of sectional curvature k. For the family $g_3 = const$ describing a family of parallel planes and g_3 taking real values we obtain for example the connection form

$$\begin{array}{lll} \omega_1 = -\frac{i}{\lambda}\cos u \cdot dg_1 & \psi_1 = i \sin u \cdot dg_1 & \chi_1 = -\frac{i\lambda}{2}\cos u \cdot dg_1 \\ \omega_2 = -\frac{i}{\lambda}\sin u \cdot dg_2 & \psi_2 = -i\cos u \cdot dg_2 & \chi_2 = -\frac{i\lambda}{2}\sin u \cdot dg_2 \\ \omega_3 = \frac{1}{\lambda} \cdot dg_3 & \psi_3 = 0 & \chi_3 = -\frac{\lambda}{2} \cdot dg_3 \\ \multicolumn{3}{c}{\omega_{12} = \frac{\partial}{\partial g_2}u \cdot dg_1 + \frac{\partial}{\partial g_1}u \cdot dg_2, \quad \omega_{23} = \omega_{31} = 0} \\ \multicolumn{3}{c}{\nu = 0.} \end{array} \tag{74}$$

Setting

$$\begin{array}{l} g_1 = i\lambda x, \; g_2 = i\lambda y, \; c = -\lambda^2; \\ g_3 = \ln r \text{ and } k = \lambda^2 \end{array} \tag{75}$$

this yields exactly the connection form (72) of a cone over a K-surface in $S^3(\frac{1}{\lambda})$. We should remark that the three cases of Guichard nets — concentric spheres, parallel planes and planes

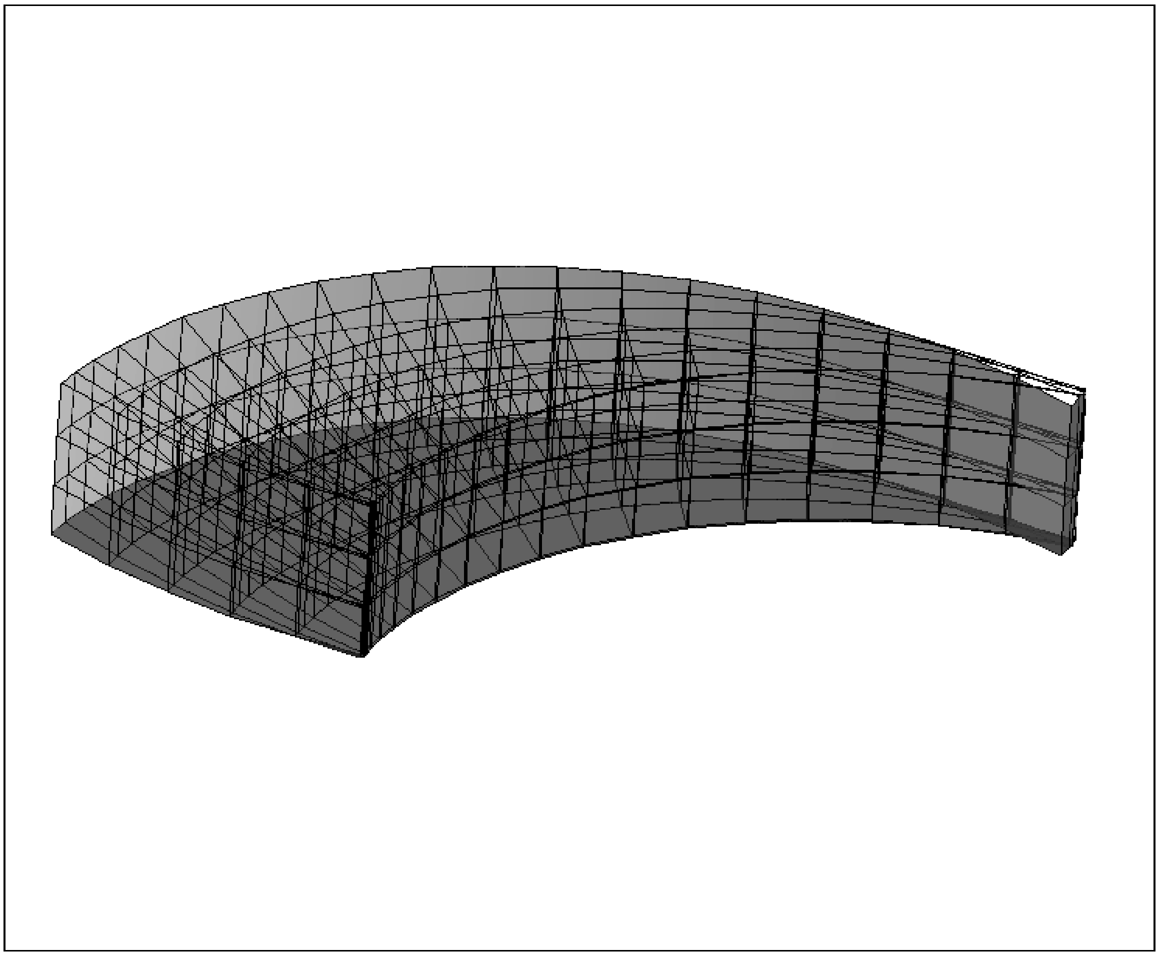
[15]Applying some appropriate conformal deformation $(f, n) \rightsquigarrow (e^u f, n)$ which transforms the induced metric to a flat one we may obtain such a conformal coordinate system by integrating the forms ω_i.

[16]Calculate $d^2(l_3^{-1})$.

[17]Here we pick *spheres* so we perhaps have to complete (geometrically) the surfaces $g_3 = const$.

intersecting in a line — correspond to the Gaussian curvature $K = c + k$ of the K-surface satisfying $K > 0$, $K = 0$ and $K < 0$.

In the following figure we see the Guichard net corresponding to a cone over an (intrinsic) flat surface in a sphere[18]:

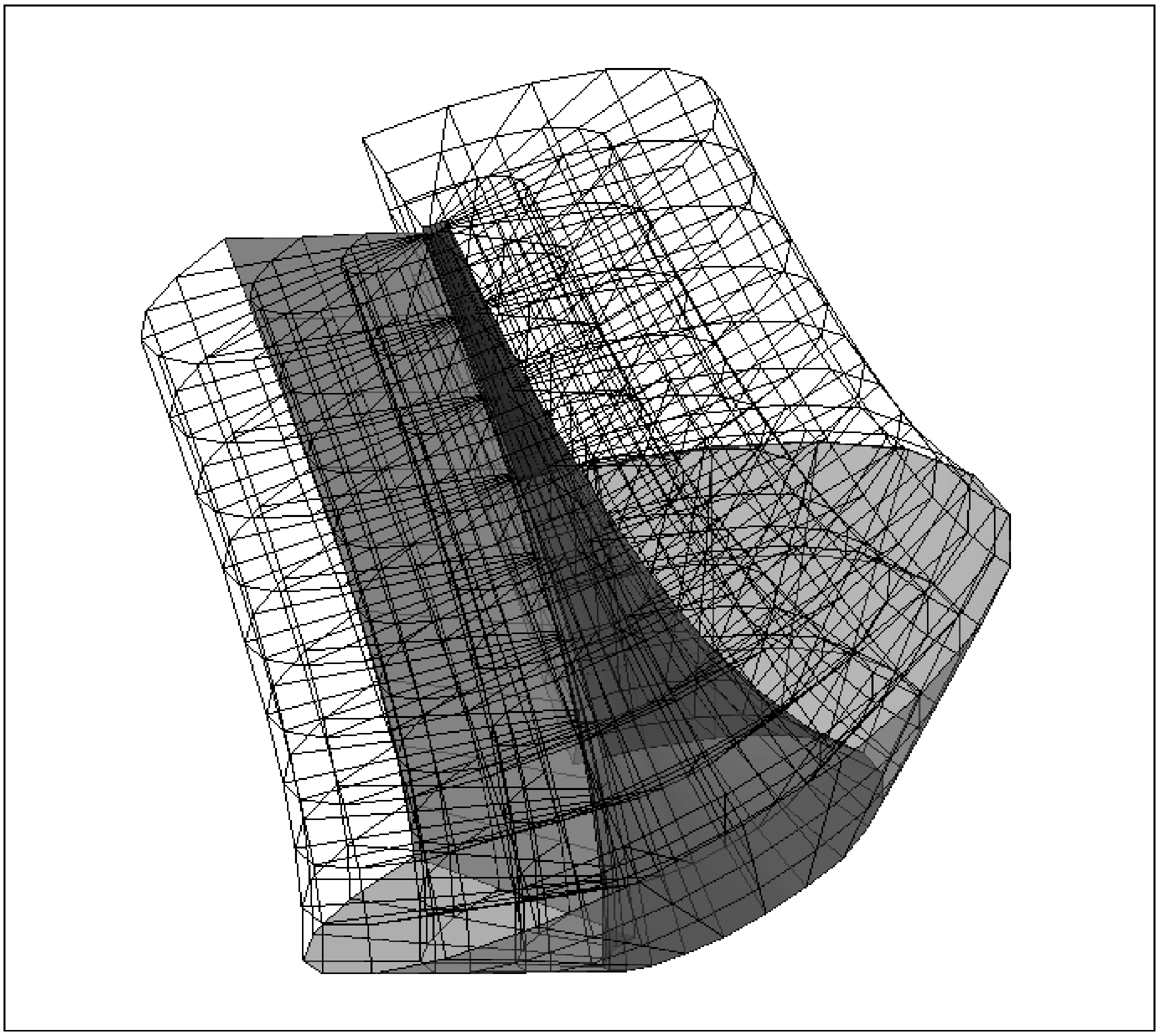


[18]Given two curves $c_1, c_2 : \mathbb{R} \to \mathbb{R}^2$ parametrized by arc length the function

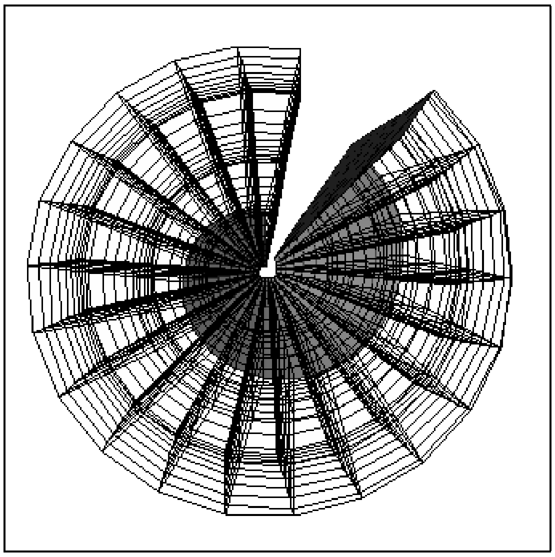
$$h : \mathbb{R}^3 \to \mathbb{R}^3, h(x, y, z) := (c_1(x + y) + c_2(x - y), 2z)$$

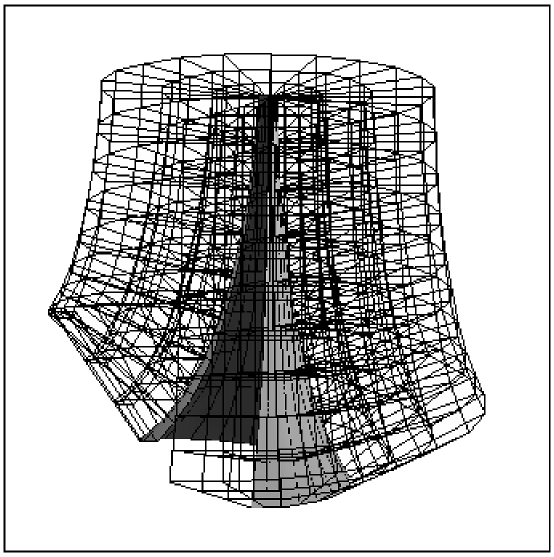
parametrizes a Guichard net: $||h_x||^2 + ||h_y||^2 = ||h_z||^2$. Since all the surfaces $g_3 \cong z = const$ are parallel planes and since $g_1 \cong ix$ and $g_2 \cong iy$ give the imaginary coordinate surfaces this construction yields Guichard nets corresponding to cones over K-surfaces in the sphere with Lipschitz-Killing curvature $c < 0$. This is the way the figure is constructed.

The following figure shows a Guichard net which does not correspond to one of the discussed conformal product hypersurfaces: none of the coordinate surfaces are umbilic — this should be plausible by looking at the three pictures below: if one family would consist of planes intersecting in a line the other two families would consist of rotational surfaces. This shows that there are conformally flat hypersurfaces of four dimensional space forms other than (branched) channel hypersurfaces and conformal product hypersurfaces[19].



[19]The shown Guichard net is built from parallel surfaces of a Dini's helix.





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