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Criteria for Two-Dimensional Circle Planes

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Abstract. We give some easy to use criteria for deciding whether a Möbius plane, Laguerre plane or Minkowski plane, given in some normal form, is 2-dimensional. As an application of our results we prove that a Laguerre plane or Minkowski plane with a given topology on the point set is a (topological) 2-dimensional plane if and only if each derived affine plane at points of at least one parallel class is 2-dimensional with respect to the induced topology.

1. Introduction

Let us recall the basic definitions and facts about (2-dimensional) circle planes.

A *circle plane* consists of a set of points, a set of at least two circles, and a collection of at most two equivalence relations on the point set, called *parallelisms*. We consider circles as subsets of the point set. We say that two points are (non-)parallel if and only if they are (not) in relation for any of the equivalence relations. The identity relation is called the trivial parallelism. For a circle plane the following axioms must be satisfied:

(A1) *Joining*: Three pairwise non-parallel points can be uniquely joined by a circle.

(A2) *Touching*: For two non-parallel points p, q and a circle c through p there exists a uniquely determined circle through q that touches c at p, i.e., intersects c only in the point p, or coincides with c.

(A3) *Parallel projection*: Parallel classes with respect to a non-trivial parallelism and circles intersect in a unique point.

(A4) Parallel classes with respect to different non-trivial parallelisms intersect in a unique point.

(A5) *Richness*: Each circle contains at least three points.

If a circle plane is equipped only with the trivial parallelism, one obtains a *Möbius plane*. Note that axioms A3 and A4 do not apply to Möbius planes and that non-parallel simply means distinct.

If a circle plane is equipped with precisely one non-trivial parallelism $||$, one obtains a *Laguerre plane*. Note that axiom A4 does not apply to Laguerre planes.

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If a circle plane is equipped with two distinct non-trivial parallelisms, written as $||_v$ and $||_h$, one obtains a *Minkowski plane*.

A *topological* circle plane is a circle plane where the set of points and the set of circles carry non-indiscrete topologies such that the geometric operations of joining, touching, parallel projection, intersecting parallel classes with respect to different parallelisms, and intersecting distinct circles are continuous on their domains of definition. The topologies of the point set and the circle set of a topological circle plane determine each other, i.e., for a given topology on the point set of an abstract circle plane there is at most one topology on the circle set for which the circle plane becomes topological, and vice versa.

In this paper we will mainly be concerned with the *2-dimensional circle planes*, i.e., topological circle planes with locally compact 2-dimensional point sets. Each 2-dimensional circle plane can be represented in a standard form; to be precise, there exists a homeomorphism of the point set such that the image of the plane under this homeomorphism looks as follows:

(1) *2-dimensional Möbius planes*: The point set is the 2-sphere, and all circles are simply closed curves on this 2-sphere.

(2) *2-dimensional Laguerre planes*: The point set is the cylinder $S^1 \times \mathbf{R}$, the circles are graphs of continuous functions $S^1 \to \mathbf{R}$, and the parallel classes of $||$ are the verticals in $S^1 \times \mathbf{R}$, i.e., the sets $\{(a, y) | y \in \mathbf{R}\}$, $a \in S^1$.

(3) *2-dimensional Minkowski planes*: The point set is the torus $S^1 \times S^1$, and the circles are graphs of homeomorphisms $S^1 \to S^1$. The parallel classes of $||_v$ are the verticals in $S^1 \times S^1$, i.e., the sets $\{(a, y) | y \in S^1\}$, $a \in S^1$ and the parallel classes of $||_h$ are the horizontals in $S^1 \times S^1$, i.e., the sets $\{(x, a) | x \in S^1\}$, $a \in S^1$.

The point sets of the three types of 2-dimensional circle planes as described above carry natural topologies and are even metrizable. Circles are compact subsets thereof; indeed, they are homeomorphic to the unit circle S^1. When the circle sets are topologized by the Hausdorff metric with respect to a metric that induces the topology of the point set, then the planes are topological in the above sense. Every geometric isomorphism between 2-dimensional circle planes is a homeomorphism when restricted to the point sets and the circle sets, cf. [26, 3.4], [9, 2.5], [21, 5.1]. This implies that an abstract circle plane can be made into a 2-dimensional plane in at most one way.

The classical 2-dimensional Möbius, Laguerre and Minkowski planes are obtained as the geometries of non-trivial plane sections of an elliptic quadric, an elliptic cone with its vertex removed, and a ruled quadric, respectively, in the 3-dimensional projective space over the real numbers. Here the topologies on the point sets and the circle sets are induced from the topologies on the point set and (hyper)plane set of the surrounding 3-dimensional projective space, respectively.

Frequently, the point sets of 2-dimensional circle planes get identified with the sets $\mathbf{R}^2 \cup \{\infty\}$, $(\mathbf{R} \cup \{\infty\}) \times \mathbf{R}$ and $(\mathbf{R} \cup \{\infty\})^2$, respectively, such that the parallel classes correspond to the verticals (and horizontals) in these sets. When dealing with concrete examples of planes given in this form one usually looks at that part of the planes that lives on $\mathbf{R}^2$. In these descriptions the circles of the classical 2-dimensional planes are, essentially, Euclidean lines plus a collection of Euclidean circles, parabolas and hyperbolas, respectively. The aim of this paper is to give some criteria for deciding when, given a circle plane on one of the above sets, it actually gives rise to a 2-dimensional circle plane.

For more details about (2-dimensional) circle planes the reader is referred to [27], [6], [7], [21].

Associated with every point p of a (2-dimensional) circle plane there is an incidence structure whose point set consists of all points not parallel to p and whose lines are all circles passing through p and all parallel classes not passing through p. This incidence structure turns out to be a (2-dimensional) affine plane. We call it the *derived affine plane at* p. A circle that does not pass through p induces an oval in the projective closure of this affine plane. It is for this reason that we will also need some results about (2-dimensional) affine planes and projective planes, and ovals in such planes. An *oval* in a (2-dimensional) projective plane is a set of points that contains no three distinct collinear points. Moreover, for every point in the set there is a uniquely determined line that intersects the oval in just this point. A line is called an exterior line, tangent or secant of the oval if it has 0, 1, 2 points in common with the oval, respectively. A *topological* oval in a 2-dimensional projective plane is an oval that is homeomorphic to the unit circle. For general definitions and facts about 2-dimensional affine (projective) planes and topological ovals in these planes the reader is referred to [19] and [2] (see also [8]).

2. Möbius planes

Let F be a set of continuous functions $\mathbf{R} \to \mathbf{R}$ and K be a set of continuous simply closed curves in $\mathbf{R}^2$. We define an incidence structure $M_{F,K}$ having a point set and a circle set. Incidence is defined by inclusion. The point set of $M_{F,K}$ is $\mathbf{R}^2 \cup \{\infty\}$, and the circles are the elements of K and the graphs of the functions in F to every one of which the point ∞ has been adjoined.

Proposition 1. *Let F be a set of continuous functions $\mathbf{R} \to \mathbf{R}$ and K be a set of continuous simply closed curves in $\mathbf{R}^2$. If $M_{F,K}$ satisfies axioms A1 and A2, then the point set of $M_{F,K}$ can be identified with the 2-sphere such that the resulting incidence structure is a 2-dimensional Möbius plane.*

Proof: Since $M_{F,K}$ satisfies axioms A1, A2, and A5, it is a Möbius plane. Furthermore, the graphs of the functions in F correspond to the lines of the derived affine plane at the point ∞. We identify the point set of $M_{F,K}$ in the natural way with the one-point compactification of $\mathbf{R}^2$. Under this identification the point set of $M_{F,K}$ turns into a 2-sphere and all circles into simply closed curves on this sphere. Now [27, Korollar 7.6], guarantees that $M_{F,K}$ is a 2-dimensional Möbius plane. □

3. Laguerre planes

Let F be a set of functions $\mathbf{R} \cup \{\infty\} \to \mathbf{R}$ such that all functions in F are continuous when restricted to $\mathbf{R}$. We define an incidence structure L_F (that looks very much like a Laguerre plane) consisting of a point set, a circle set and an equivalence relation (parallelism) on the point set. Incidence is defined by inclusion. The point set is $(\mathbf{R} \cup \{\infty\}) \times \mathbf{R}$, the circles are the sets $c_f := \{(x, f(x)) | x \in \mathbf{R} \cup \{\infty\}\}$, $f \in F$, and the equivalence classes (parallel classes) are the verticals in $(\mathbf{R} \cup \{\infty\}) \times \mathbf{R}$, i.e., the sets $\{(a, y) | y \in \mathbf{R}\}$, $a \in \mathbf{R} \cup \{\infty\}$. Furthermore, let F_a, $a \in \mathbf{R}$ be the set of all functions $f \in F$ such that $f(\infty) = a$. We define an incidence structure $L_{a,F}$ consisting of a point set and a line set. Incidence is again defined by inclusion. The point set is $\mathbf{R}^2$. The lines are the vertical (straight lines) in $\mathbf{R}^2$ and the sets $l_f := \{(x, f(x)) | x \in \mathbf{R}\}$, where $f \in F_a$. We note that if L_F is a Laguerre plane, then $L_{a,F}$ is the derived affine plane at the point (∞, a).

We want to prove the following

Proposition 2. *Let F be a set of functions $\mathbf{R} \cup \{\infty\} \to \mathbf{R}$ such that all functions in F are continuous when restricted to $\mathbf{R}$. If L_F satisfies axioms A1 and A2, then the point set of L_F can be identified with the cylinder $S^1 \times \mathbf{R}$ such that the resulting incidence structure is a 2-dimensional Laguerre plane.*

In order to prove this result we need to know some facts about parabolic curves in 2-dimensional affine planes.

Let A be a 2-dimensional affine plane whose line set contains all verticals in $\mathbf{R}^2$. A continuous curve c in A is *parabolic* if c together with the ideal point of the verticals is an oval in the projective closure of A, i.e., c is the graph of a continuous function $\mathbf{R} \to \mathbf{R}$, every non-vertical line intersects c in 0, 1, 2 points, and through every point of c there passes a uniquely determined non-vertical line that intersects c only in this point. A non-vertical line in A is called an exterior line, tangent or secant of c if it has 0, 1, 2 points in common with c, respectively.

We will need the following result

Lemma 1. *Let A be a 2-dimensional affine plane that contains the verticals in $\mathbf{R}^2$ as lines, let p_v be the ideal point of the verticals, and let c be a parabolic curve in A. Moreover, let t be an arbitrary tangent of c and let l be an arbitrary non-vertical line in A.*

Then the set $c \cup \{p_v\}$ is a topological oval in the projective closure of A. Furthermore, the set $\mathbf{R}^2 \setminus c$ has two components, and only one of the components has points in common with t (let $O(c)$ denote this component and let $I(c)$ denote the other component). Finally, all of l except possibly a compact subset of l is contained in $O(c)$, i.e., the "ends" of l are contained in $O(c)$.

Proof: Let W be the ideal line of A and $\bar{A}$ be the projective closure of A, i.e., a 2-dimensional projective plane. Let $g : \mathbf{R} \to \mathbf{R}$ be the continuous function whose graph is c, and let $\bar{c} := c \cup \{p_v\}$. Finally, let $\bar{t}$ and $\bar{l}$ be the lines in $\bar{A}$ that correspond to t and l.

We verify that $\bar{c}$ is a topological oval in $\bar{A}$ (see also [16, Lemma 2.2]). For this it suffices to show that $\bar{c}$ is compact, i.e., homeomorphic to S^1, or equivalently, we have to show that $\lim_{x \to \pm\infty}(x, g(x)) = p_v$ in $\bar{A}$. Let $y \in \mathbf{R}$ and let $\bar{l}_x$, with $x \in \mathbf{R}$ denote the line in $\bar{A}$ that connects the point $(y, g(y))$ with the point $(x, g(x))$ if $x \neq y$, and denote the tangent at the point $(y, g(y))$ if $x = y$. The map $h : \mathbf{R} \setminus \{y\} \to W \setminus \{p_v, \bar{l}_y \wedge W\} : x \mapsto W \wedge \bar{l}_x$ is continuous and, since we are dealing with an oval, bijective. Hence h is a homeomorphism. We conclude that either $\lim_{x \to +\infty} \bar{l}_x = \bar{l}_y$ or $\lim_{x \to +\infty} \bar{l}_x$ is the vertical through the point $(y, g(y))$. Since $y \in \mathbf{R}$ was an arbitrary choice, the first possibility implies that all tangents of $\bar{c}$ pass through the point $\bar{l}_y \wedge W$. W.l.o.g. we may assume that A contains the horizontal (straight lines) in $\mathbf{R}^2$ as lines and that $\bar{l}_y \wedge W$ is the ideal point of these lines. This implies that all lines in A that are neither vertical nor horizontal lines are graphs of strictly increasing or strictly decreasing homeomorphisms. Furthermore, g itself is either strictly increasing or decreasing, since all verticals intersect c exactly once and no horizontal line intersects c in more than one point. If g is strictly increasing (decreasing), then all lines that are graphs of strictly decreasing (increasing) functions are tangents of c. In particular, there are infinitely many tangents through every point of c. This is a contradiction. We conclude that $\lim_{x\to+\infty} \bar{l}_x$ is the vertical through the point $(y, g(y))$. With the same argument we conclude that $\lim_{x\to-\infty} \bar{l}_x$ is also the vertical through the point $(y, g(y))$. Thus the only accumulation point of c on W is p_v. Hence $\bar{c}$ is homeomorphic to the unit circle and is therefore a topological oval in $\bar{A}$.

As an immediate consequence of the definition of an oval, we know that a non-vertical line in A is an exterior line, tangent or secant of c if and only if the corresponding line in $\bar{A}$ is an exterior line, tangent or secant of $\bar{c}$, respectively. By [8, 2.3 b)], $O(c)$ and $I(c)$ correspond to the sets of outer points and inner points of $\bar{c}$, respectively. The same result guarantees that the ends of l are contained in $O(c)$ if l is a tangent or a secant. If l is an exterior line, it has to be contained in $O(c)$ because it intersects $\bar{t}$ at some point. $\square$

Proof of Proposition 2: Since L_F satisfies axioms A1 - A3, and A5, it is a Laguerre plane. Let c be a circle through the origin $(0,0) \in \mathbf{R}^2$ and let f_a, $a \in \mathbf{R}$ denote the uniquely determined (by A2) function in F_a such that the circle c_{f_a} touches c at $(0,0)$. Every single one of the points $(x,y) \in \mathbf{R}^2$, $x \neq 0$ is contained in precisely one of the circles c_{f_a}, $a \in \mathbf{R}$. W.l.o.g. we may therefore assume that $f_a(1) = a$. By making this choice, we introduce an order on the tangent pencil $\{c_{f_a}\}_{a\in\mathbf{R}}$. Now we can provide the point set of L_F with the topology that is generated by the open sets in $\mathbf{R}^2$ and the sets $\{(x, f_a(x)) | a_1 < a < a_2, |x| > |x'|\} \cup \{(\infty, a) | a_1 < a < a_2\}$, $a_1, a_2, x' \in \mathbf{R}, a_1 < a_2$. Equipped with this topology the point set is a topological space homeomorphic to the cylinder $S^1 \times \mathbf{R}$.

We show that the circles in L_F are homeomorphic to the unit circle S^1. For all $a \in \mathbf{R}$ the derived affine plane $L_{a,F}$ at the point (∞, a) is 2-dimensional, since all its lines are homeomorphic to $\mathbf{R}$ (see [19, Theorem 2.12]). Furthermore, axioms A1 and A2 imply that $l_{f_{a'}}$, $a' \neq a$ is a parabolic curve in this plane. With this information at hand, it suffices to check that for all $a, a_1, a_2 \in \mathbf{R}$, $a_1 < a < a_2$, $f \in F_a$, the ends of the line l_f in the affine plane $L_{a,F}$ stay in between the two parabolic curves $l_{f_{a_1}}$ and $l_{f_{a_2}}$. By Lemma 1, this is the case, since l_{f_a} is a tangent of both parabolic curves.

By [7, 3.10], the Laguerre plane L_F is 2-dimensional. $\square$

Knarr and Weigand [14] proved that a ternary field $(\mathbf{R}^n, \tau)$, $n = 1, 2, 4, 8$ with continuous ternary operation τ coordinatizes a $2n$-dimensional affine plane. The following similar result is an immediate consequence of Proposition 2.

Corollary. *Let $g : \mathbf{R}^4 \to \mathbf{R}$ be a continuous function, and let F be the set of all functions*

$$\mathbf{R} \cup \{\infty\} \to \mathbf{R} : x \mapsto \begin{cases} g(a,b,c,x) & \text{for } x \in \mathbf{R} \\ a & \text{for } x = \infty \end{cases} \qquad a, b, c \in \mathbf{R}.$$

If L_F satisfies axioms A1 and A2, then the point set of L_F can be identified with the cylinder $S^1 \times \mathbf{R}$ such that the resulting incidence structure is a 2-dimensional Laguerre plane.

We note that it is possible to associate with any 2-dimensional Laguerre plane L a continuous function $g : \mathbf{R}^4 \to \mathbf{R}$, with corresponding set F of continuous functions $\mathbf{R} \to \mathbf{R}$ as in the lemma, such that $L_F = L$ (see [18, 2.1]).

4. Minkowski planes

Let F be a set of bijective maps $\mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$ such that for each $f \in F$ the restriction $f^* : \mathbf{R} \setminus \{f^{-1}(\infty)\} \to \mathbf{R} \setminus \{f(\infty)\}$ is continuous. Let F_∞ be the set of all functions $f \in F$ such that $f(\infty) = \infty$. For a function $f \in F_\infty$ we find that f^* is a homeomorphism $\mathbf{R} \to \mathbf{R}$. We define an incidence structure A_F consisting of a point set and a line set. The point set is $\mathbf{R}^2$ and the lines are the verticals and horizontals in $\mathbf{R}^2$ and the graphs of the functions f^* where $f \in F_\infty$. We define a second incidence structure M_F (that looks very much like a Minkowski plane) consisting of a point set, a circle set and and two equivalence relations (parallelisms) $||_v$ and $||_h$ defined on the point set. The point set is $(\mathbf{R} \cup \{\infty\})^2$ and the circles are the sets $c_f := \{(x, f(x)) | x \in \mathbf{R} \cup \{\infty\}\}$, $f \in F$. The equivalence classes (parallel classes) of $||_v$ are the verticals in $(\mathbf{R} \cup \{\infty\})^2$, i.e., the sets $\{(a, y) | y \in \mathbf{R} \cup \{\infty\}\}$, $a \in \mathbf{R} \cup \{\infty\}$ and the equivalence classes of $||_h$ are the horizontals in $(\mathbf{R} \cup \{\infty\})^2$, i.e., the sets $\{(x, a) | x \in \mathbf{R} \cup \{\infty\}\}$, $a \in \mathbf{R} \cup \{\infty\}$. We note that if M_F is a Minkowski plane, then A_F is the derived affine plane at the point (∞, ∞).

Proposition 3. *Let F be a set of bijective maps $\mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$ such that for each $f \in F$ the restriction $f^* : \mathbf{R} \setminus \{f^{-1}(\infty)\} \to \mathbf{R} \setminus \{f(\infty)\}$ is continuous. If M_F satisfies axioms A1 and A2, then the point set of M_F can be identified with the torus $S^1 \times S^1$ such that the resulting incidence structure is a 2-dimensional Minkowski plane.*

In order to prove this result we need to know some facts about hyperbolic curves in 2-dimensional affine planes.

Let A be a 2-dimensional affine plane whose line set contains all verticals and horizontals in $\mathbf{R}^2$. A subset h of $\mathbf{R}^2$ homeomorphic to $\mathbf{R} \setminus \{0\}$ is called a *hyperbolic curve* if h together with the ideal points of the verticals and horizontals is an oval in the projective closure of A, i.e., h is the graph of a continuous bijection $\mathbf{R} \setminus \{a\} \to \mathbf{R} - \{b\}$, $a, b \in \mathbf{R}$, every line that is not a vertical or horizontal intersects h in 0, 1, 2 points, and through every point of h there passes a uniquely determined line that intersects h only at this point and that is neither a horizontal nor a vertical.

We will need the following result

Lemma 2. *Let A be a 2-dimensional affine plane that contains the verticals and horizontals in $\mathbf{R}^2$ as lines, and let p_v be the ideal point of the verticals and p_h be the ideal point of the horizontals. Moreover, let h be a hyperbolic curve in A which is the graph of the continuous bijection $g : \mathbf{R} \setminus \{a\} \to \mathbf{R} \setminus \{b\}$, $a, b \in \mathbf{R}$.*

Then the set $h \cup \{p_v, p_h\}$ *is a topological oval in the projective closure of* A. *Furthermore, we find* $\lim_{x\to\pm\infty} g(x) = b$ *(this implies* $\lim_{x\to a^+} g(x) = +\infty$ *or* $-\infty$ *and* $\lim_{x\to a^-} g(x) = -\infty$ *or* $+\infty$*, respectively).*

Proof: We first prove the second part of the lemma. Clearly, $\lim_{x\to+\infty} g(x) \in \{b, +\infty, -\infty\}$. Let us assume that $\lim_{x\to\infty} g(x) = \infty$. Then $\lim_{x\to a^+} g(x) = b$ and g is strictly strictly increasing for $x > a$. Furthermore, $g(x) < b$ for $x < a$. All lines in A that are neither vertical nor horizontal are graphs of strictly increasing or strictly decreasing functions. Let $(x, g(x)) \in h$, $x > a$. Then there are infinitely many lines corresponding to strictly decreasing functions that intersect h only at this point. A contradiction. The assumptions that $\lim_{x\to+\infty} g(x) = -\infty$, or $\lim_{x\to-\infty} g(x) = +\infty, -\infty$ lead to similar contradictions. This proves the second part of the lemma. The first part of the lemma follows immediately. □

Proof of Proposition 3: Since M_F satisfies axioms A1-A5, it is a Minkowski plane. As in the proof of Proposition 2, we conclude that A_F is a 2-dimensional affine plane. Furthermore, axioms A1 and A2 imply that the graphs of all functions f^*, $f \in F \setminus F_\infty$ are hyperbolic curves in A_F. We identify $\mathbf{R} \cup \{\infty\}$ with the unit circle in the natural way. Following this we identify the point set of M_F with the torus $S^1 \times S^1$. By the second part of Lemma 2, it is clear that all circles that do not pass through the point (∞, ∞) are homeomorphic to the unit circle. The circles through this point correspond to the non-vertical and non-horizontal lines in A_F. All these lines intersect any of the horizontals and verticals precisely once, i.e., the only possible accumulation point for these lines on the torus is the point (∞, ∞). This shows that the circles through this point are homeomorphic to the unit circle as well.

Now [21, Satz 4.4] guarantees that M_F is a 2-dimensional Minkowski plane. □

5. Applications

There are many constructions of Möbius planes, Laguerre planes and Minkowski planes that fit into our respective setups, e.g., the constructions in [4], [10], [11], [12], [13], [15], [16], [20], [22], [23], [24], [25]. Some of these planes have only been looked at from an incidence geometric point of view, and the proof that they are actually 2-dimensional planes has never been worked out. Our results fill this gap. Examples of such planes are the planes constructed in [1], [4], [11], [12], [13], [20].

As one further application of our results we prove that 2-dimensional circle planes can be characterized in terms of certain derived affine planes being 2-dimensional.

Proposition 4. *A Möbius plane, Laguerre plane or Minkowski plane with a given topology on the point set is a 2-dimensional plane if and only if each derived affine plane is 2-dimensional with respect to the induced topology.*

Proof: Since lines of a derived affine plane of a 2-dimensional circle plane are homeomorphic to $\mathbf{R}$ and closed subsets of the point set of the affine plane, [19, Theorem 2.12] guarantees that the affine plane is 2-dimensional.

Conversely, assume that each derived affine plane of a 2-dimensional circle plane is a 2-dimensional affine plane with respect to the induced topology. According to [27, Korollar 7.6], [7, 3.10], [21, Satz 4.4], all we have to show is that circles are homeomorphic to S^1 and parallel classes are homeomorphic to $\mathbf{R}$ or S^1 in the case of Laguerre planes and Minkowski planes, respectively. However, each punctured circle is homeomorphic to $\mathbf{R}$ since such a circle becomes a line in the derived affine plane at the point that has been removed. Furthermore, each circle is the union of two punctured circles. In particular, the complement of each open neighborhood of one of the deleted points is a closed bounded subset of $\mathbf{R}$ with respect to the other deleted point. Hence a circle is the union of two compact subsets, and thus is compact itself. Therefore a circle is homeomorphic to the one-point compactification of $\mathbf{R}$, that is, each circle is homeomorphic to S^1.

A similar argument applies to show that parallel classes of Minkowski planes are homeomorphic to S^1. Finally, each parallel class of a Laguerre plane becomes a line in the derived affine plane at a point not on this parallel class. Hence each parallel class must be homeomorphic to $\mathbf{R}$. $\square$

For Laguerre planes and Minkowski planes we can weaken the above condition.

Proposition 5. *A Laguerre plane or Minkowski plane with a given topology on the point set is a 2-dimensional plane if and only if each derived affine plane at points of at least one parallel class is 2-dimensional with respect to the induced topology.*

Proof: Suppose that each derived affine plane at points of one parallel class is 2-dimensional with respect to the induced topology. We fix one point p of such a parallel class.

In the case of a Laguerre plane, circles not passing through p induce parabolic curves in the derived 2-dimensional affine plane at p. Since derived affine planes at points parallel to p have the same point set and are 2-dimensional with respect to the same topology, those parabolic curves are homeomorphic to $\mathbf{R}$. In particular, the describing functions of those circles are continuous on their domains of definition. Thus all assumptions of Proposition 2 are satisfied. Hence we have a 2-dimensional Laguerre plane.

In the case of a Minkowski plane, circles not passing through p induce hyperbolic curves in the derived 2-dimensional affine plane at p. Since derived affine planes at points parallel to p are 2-dimensional with respect to the same topology and because two such derived planes have a point set $\mathbf{R} \setminus \{a\} \times \mathbf{R}$ for some $a \in \mathbf{R}$ with the same topology in common, those curves are homeomorphic to $\mathbf{R}^* = \mathbf{R} \setminus \{0\}$. In particular, the describing functions of those circles are continuous on their domains of definition. Thus all assumptions of Proposition 3 are satisfied. Hence we have a 2-dimensional Minkowski plane. $\square$

Example: Let $f : \mathbf{R} \to \mathbf{R}$ be a function whose graph in the real Desarguesian affine plane $\mathbf{R}^2$ plus the ideal point of the verticals is an oval c_f in the real Desarguesian projective plane. Let us assume that f is not continuous, that is, the corresponding oval c_f of the 2-dimensional real Desarguesian projective plane is not topological. (Such ovals exist by [17].) In the planar description, as given in section 3, of the *ovoidal* Laguerre plane over c_f, that is, the geometry of non-trivial plane sections of a cone over this oval (with its vertex removed), circles are of the form $\{(x, af(x)+bx+c) \mid x \in \mathbf{R}\} \cup \{(\infty, a)\}$ for $a, b, c \in \mathbf{R}$. In the Euclidean topology of $\mathbf{R}^2$ only the derived affine plane at $(\infty, 0)$ is 2-dimensional. The derived affine plane at (∞, a), $a \neq 0$ is not topological with respect to the given topology of $\mathbf{R}^2$, although one can find topologies such that these derived planes become 2-dimensional.

It is not known whether results similar to Propositions 2-5 hold in the case of 4-dimensional circle planes (see [5] for general information about such planes).

One more remark that should not be left unsaid is the following

Proposition 6. *An incidence structure I with point set $\mathbf{R}^2$ that "looks like" the restriction of a 2-dimensional Möbius, Laguerre or Minkowski plane to the complement of a point, a parallel class, two parallel classes, respectively, can be extended to one of the respective 2-dimensional circle planes in at most one way.*

Here "looks like" means that I is given in the form of the incidence structures $M_{F,K}$, L_F, M_F as introduced in section 2, 3, 4, respectively, with the difference that the point set is always $\mathbf{R}^2$, the elements of F are continuous functions $\mathbf{R} \to \mathbf{R}$ in the case of the Möbius and Laguerre planes and, in the case of Minkowski planes, continuous bijections $\mathbf{R} \setminus \{a\} \to \mathbf{R} \setminus \{b\}$, $a, b \in \mathbf{R}$.

Proof: In the case of the Möbius and Minkowski planes Proposition 6 is clearly true.

In the case of Laguerre planes it makes more sense to speak of a line set of I rather than a circle set. We have to show that if I is extendable to a Laguerre plane, we are able to reconstruct the derived affine planes at the points on the parallel class that has to be added by just looking at the lines in I. If two lines in I intersect in exactly one point we will say that they intersect transversally if points of one of the lines are contained in both components of the other line in $\mathbf{R}^2$. Using Lemma 1, we can show that the set A_l of all lines in I that intersect a given line l transversally, plus all those lines that intersect all lines in A_l transversally in exactly one point, plus the verticals have to be contained in any derived affine plane containing l of any 2-dimensional Laguerre plane that extends I. On the other hand, all these lines already form the line set of an affine plane.

Now, every single one of the derived affine planes we just reconstructed corresponds to a point on the parallel class that is to be added. We just have to make sure that the ordering on these points can also be reconstructed. But this follows from the fact that any pencil of lines in I that all touch in a point, i.e., the lines do not intersect transversally, contains exactly one representative line from any of the derived planes under discussion. So, the ordering on the points of the missing parallel class can also be reconstructed by just having a look at one of these pencils. $\square$

For finite Laguerre planes the corresponding result is not true. For example, let $\mathbf{F}$ be a finite field of order $2^n, n \geq 3$. Then the incidence structure with point set $\mathbf{F^2}$ and lines $L(a,b,c) := \{(x, ax^2+bx+c) | x \in \mathbf{F}\}$ $a, b, c \in \mathbf{F}$ can be extended in two different ways by a copy of $\mathbf{F}$ to (non-isomorphic) Laguerre planes. If $(a), a \in \mathbf{F}$ denote the elements of this copy the two essentially different extensions of a line are $L(a,b,c) \cup \{(a)\}$ and $L(a,b,c) \cup \{(b)\}$. If we extend in the first way, the resulting Laguerre plane is the Miquelian Laguerre plane, if we extend in the second way, the resulting Laguerre plane will be an ovoidal non-Miquelian Laguerre plane.

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