Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 35 (1994), No. 2, 193-204.

Remarks to the Dimension of an Atypical Irreducible Representation of the Special Lie Superalgebra sl(1,n)

Hartmut Schlosser
Fachbereich Mathematik/Informatik, Universität Greifswald
Friedrich-Ludwig-Jahnstraße 15a, D – 17489 Greifswald
schloss@math-inf.uni-greifswald.d400.de

1. Introduction.

In [6] and [7] Palev gave explicit expressions for the action of the generators of the Lie superalgebra $sl(1,n)$ at an (irreducible) representation. His description of the representation space W of an atypical irreducible representation in terms of highest weights of $gl(n)$ representations give a possibility to compute the dimension of W numerically.

In this paper we will proof an explicit formula for the dimension of an atypical representation of $sl(1,n)$ using numbers H_l which connect with the Bernoulli numbers B_l.

The results of this paper are announced in [8].

2. Facts from the representation theory of sl(1,n)

We follow the terminology of Palev ([6],[7]). Here we briefly recall the points of his papers which are important for the following. We will use both the language of representations and the language of modules.

The even part of $sl(1,n)$ is isomorphic to $gl(n)$. Let be

$$[m]_n = [m_1, \ldots, m_n]$$

the highest weight of an irreducible representation of $gl(n)$. The m_i are the coordinates of a highest weight considered as functional with respect to the basis of the dual H^* which is dual to the standard basis of the Cartan algebra H. So we will refer to $[m]_n$ as highest weight and the corresponding representation we will denote with $[m]_n$ also (analogous for $sl(1,n)$).

The m_i are in general complex numbers such that $m_i - m_j \in Z_+$ for all i, j with $i < j$, $i, j = 1, \ldots, n$. Furthermore, $[m]_n$ corresponds to a $gl(n)$ fidirmod (=finite-dimensional irreducible module) $V_0([m]_n)$. To $V_0([m]_n)$ we can find an induced $sl(1,n)$

module $\bar{W}([m]_{n+1})$ where $[m]_{n+1} = [m]_n$; the index at the right bracket we use to differ between the representations of $gl(n)$ and $sl(1,n)$ resp.

We denote as Θ-tuple a n-tuple $\Theta = (\Theta_1, \ldots, \Theta_n)$ with $\Theta_j \in \{0,1\}$ for $j = 1, \ldots, n$. If we consider $\bar{W}([m]_{n+1})$ as a $gl(n)$ module then $\bar{W}([m]_{n+1})$ decomposes into a direct sum of $gl(n)$ fidirmods $W([m(\Theta)]_n)$ with highest weights

$$[m(\Theta)]_n = [m_1(\Theta), \ldots, m_n(\Theta)]_n, m_i(\Theta) = m_i + \Theta_i - \sum_{j=1}^{n} \Theta_j. \tag{1}$$

The sum is taken over all Θ-tuple such that $m_i(\Theta) - m_j(\Theta) \in Z_+$ for all i, j with $i < j$ (admissible Θ-tuple). The description of the $[m(\Theta)]_n$ by the Θ-tuple $(\Theta_1, \ldots, \Theta_n)$ is one-by-one.

In the socalled typical case the $sl(1,n)$ module $\bar{W}([m]_{n+1})$ is irreducible. The atypical case is characterized by the condition $m_k = k-1$ for a certain $k \in \{1, \ldots, n\}$. If $m_k = k-1$, $m_j = j-1$ and $k < j$ then $m_k - m_j = k - j < 0$ which contradicts to $m_k - m_j \in Z_+$. So, in a highest weight it exists almost one k with the condition of atypicity.

If $m_k = k-1$ then $\bar{W}([m]_{n+1})$ is an indecomposable $sl(1,n)$ module which is not irreducible. $\bar{W}([m]_{n+1})$ contains a maximal invariant subspace $\bar{I}$ which we can describe as a direct sum of $gl(n)$ fidirmods:

$$\bar{I} = \oplus W([m_1 + \Theta_1 - \sum_{j=1}^{n} \Theta_j, \ldots, m_n + \Theta_n - \sum_{j=1}^{n} \Theta_j])$$

with

$$\Theta_k = 1;\ \Theta_j \in \{0,1\},\ j \neq k,\ j = 1, \ldots, n. \tag{2}$$

If the Θ-tuple is not admissible we drop the corresponding summand.

The factor module $\bar{W}([m]_{n+1})/\bar{I}$ is isomorphic to an analogous direct sum where in (2) the condition $\Theta_k = 1$ is replaced by $\Theta_k = 0$. The factor module is irreducible. This factor module and the corresponding representation is said to be atypical (or nontypical) and is characterized by the highest weight $[m]_{n+1}$.

We denote the irreducible (atypical) representation given by $\bar{W}([m]_{n+1})/\bar{I}$ with $[m]_{n+1}^{irr}$ in difference to $[m]_{n+1}$ denoting the indecomposable representation given by $\bar{W}([m]_{n+1})$.

Furthermore, we set for all possible $\kappa = 0, 1, \ldots$

$$\begin{aligned}[m-(\kappa+1)]_{n+1}^{k-\kappa,\ldots,k} =& [m_1 - (\kappa+1), \ldots, m_{k-\kappa-1} - (\kappa+1), m_{k-\kappa} - \kappa, \ldots, \\ & m_k - \kappa, m_{k+1} - (\kappa+1), \ldots, m_n - (\kappa+1)]_{n+1}.\end{aligned}$$

Similar, we set for all possible $\lambda = 0, 1, \ldots$

$$\begin{aligned}[m+\lambda+1]_{n+1}^{k,\ldots,k+\lambda} =& [m_1 + \lambda + 1, \ldots, m_{k-1} + \lambda + 1, m_k + \lambda, \ldots, \\ & m_{k+\lambda} + \lambda, m_{k+\lambda+1} + \lambda + 1, \ldots, m_n + \lambda + 1]_{n+1}.\end{aligned}$$

$\bar{I}$ carries an atypical representation, too. This representation is irreducible (cf. [3], [4]). The highest weight of this representation is given in general by $[m-1]_{n+1}^{k}$. This corresponds to the Θ-tuple with $\Theta_k = 1$, $\Theta_i = 0$ elsewhere.

But it is possible that this Θ-tuple is not admissible . Then exist a $\kappa > 0, \kappa < k$ such that $m_{k-\kappa} = \ldots = m_k = k-1$ and $m_{k-\kappa-1} > m_{k-\kappa}$ if $k - \kappa > 1$. Here we have the highest weight $[m-(\kappa+1)]_{n+1}^{k-\kappa,\ldots,k}$.

3. The polynomial solution of the functional equation $g(x_1,\ldots,x_n) + g(x_1-1,\ldots,x_n-1) = 2^{n+1}x_1\cdots x_n$

The computation of the dimension of an atypical representation will be reduced in the following to the solution of an equation of the form

$$g(x_1,\ldots,x_n) + g(x_1-1,\ldots,x_n-1) = 2^{n+1}x_1\cdots x_n. \tag{3}$$

For the solution of this difference equation we use some ideas from the theory of difference equations in one variables and some own ideas. First we develop a suitable formalism.

We define for a function $f(x_1,\ldots,x_n)$ in n variables with a suitable domain for all $i,j \in Z_+$:

$$f^{(i)} = f(x_1-i,\ldots,x_n-i);\ (f^{(i)})^{(j)} = f^{(i+j)} \tag{4}$$

and

$$\Delta^0 f(x_1,\ldots,x_n) = f(x_1,\ldots,x_n),\ \Delta^i f(x_1,\ldots,x_n) = \Delta^{i-1}f^{(0)} - \Delta^{i-1}f^{(1)}. \tag{5}$$

It is easy to see by induction that

$$\Delta^i f(x_1,\ldots,x_n) = \sum_{j=0}^{i}(-1)^j\binom{i}{j}f^{(j)}. \tag{6}$$

For $n=1$ and $f(x)=x^i$ in [2] is computed at $x=0$ the value

$$\Delta^l f(0) = \Delta^l 0^i = \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}j^l. \tag{7}$$

It is valid:

$$\Delta^l 0^i = 0,\ i > l. \tag{8}$$

Lemma 1: *For $n>0$ let be $f(x_1,\ldots,x_n) = 2^n x_1\cdots x_n$. Then $\Delta^i f(x_1,\ldots,x_n) = 0$ for all $i>n$.*

Proof: Let $\sigma_l(x_1,\ldots,x_n)$ denote the l-th elementar symmetric function in n variables:

$$\sigma_l(x_1,\ldots,x_n) = x_1\cdots x_l + x_1\cdots x_{l-1}x_{l+1} + \cdots + x_{n-l+1}\cdots x_n.$$

It is easy to see that

$$f^{(j)} = 2^n(x_1-j)\cdots(x_n-j) = 2^n\sum_{l=0}^{n}(-1)^l j^l \sigma_{n-l}(x_1,\ldots,x_n). \tag{9}$$

Now we have by (6) and (9)

$$\begin{aligned}\Delta^i f(x_1,\ldots,x_n) &= \sum_{j=0}^{i}(-1)^j\binom{i}{j}2^n\sum_{l=0}^{n}(-1)^l j^l\sigma_{n-l}(x_1,\ldots,x_n)\\ &= 2^n\sum_{l=0}^{n}(\sum_{j=0}^{i}(-1)^j\binom{i}{j}j^l)(-1)^l\sigma_{n-l}(x_1,\ldots,x_n). \qquad (10)\end{aligned}$$

From (8) follows that the expression $(-1)^i\Delta^l 0^i$ in the brackets is equal to zero for all $i > l$. Because $l \leq n$ the proof is finished.

Remark: It is possible to prove the Lemma directly by induction.

Lemma 2: *Let be again $f(x_1,\ldots,x_n) = 2^n x_1\cdots x_n$. Then*

$$g(x_1,\ldots,x_n) = \sum_{i=0}^{n}\frac{1}{2^i}\Delta^i f(x_1,\ldots,x_n)$$

is a solution of (3).

For the proof we use Lemma 1 and the following arguments:

a) Because $\binom{i}{j} = 0$ for $j > i$ we can replace $\sum_{j=0}^{i}\binom{i}{j}\ldots$ by $\sum_{j=0}^{n}\binom{i}{j}\ldots$, if $n > i$.

b) For $n > 0, 0 < j \leq n+1$ is valid: $\sum_{i=0}^{n}\frac{1}{2^i}(\binom{i}{j} - \binom{i}{j-1}) = -\frac{1}{2^n}\binom{n+1}{j}$ (proof by induction).

Now we prove Lemma 2:

$$\begin{aligned}&\sum_{i=0}^{n}\frac{1}{2^i}\Delta^i f(x_1,\ldots,x_n) + \sum_{i=0}^{n}\frac{1}{2^i}\Delta^i f^{(1)}(x_1,\ldots,x_n)\\ &= \sum_{i=0}^{n}\frac{1}{2^i}(\sum_{j=0}^{i}(-1)^j\binom{i}{j}f^{(j)}) + \sum_{i=0}^{n}\frac{1}{2^i}(\sum_{j=0}^{i}(-1)^j\binom{i}{j}f^{(j+1)})\\ &= \sum_{i=0}^{n}\frac{1}{2^i}f^{(0)} + \sum_{j=1}^{n}\sum_{i=0}^{n}\frac{1}{2^i}(-1)^j\binom{i}{j}f^{(j)} + \sum_{j=1}^{n+1}\sum_{i=0}^{n}\frac{1}{2^i}(-1)^{j-1}\binom{i}{j-1}f^{(j)}\\ &= \sum_{i=0}^{n}\frac{1}{2^i}f^{(0)} + \sum_{j=1}^{n+1}(-\frac{1}{2^n})(-1)^j\binom{n+1}{j}f^{(j)}\\ &= \sum_{i=0}^{n}\frac{1}{2^i}f^{(0)} - \frac{1}{2^n}(\Delta^{n+1}f(x_1,\ldots,x_n) - f^{(0)})\\ &= (\sum_{i=0}^{n}\frac{1}{2^i} + \frac{1}{2^n})f^{(0)} = 2f^{(0)} = 2^{n+1}x_1\cdots x_n.\end{aligned}$$

Lemma 2 and (10) give the way to solve the difference equation. But for an explicit formula we have to look to sums

$$H(n,l) = \sum_{i=0}^{n} \frac{1}{2^i}(-1)^l \sum_{j=0}^{i}(-1)^j \binom{i}{j} j^l = \sum_{i=0}^{n} \frac{1}{2^i}(-1)^{i+l}\Delta^l 0^i \tag{11}$$

for $l = 0, \ldots, n$ which are the coefficients by $2^n \sigma_{n-l}$. (8) shows that $H(n,l) = H(l,l)$ if $l < n$. So we must compute the coefficients $H_l = H(l,l)$ only for $l = 0, 1, \ldots$

Computations with a computer give the following values:

l	H_l	l	H_l
0	1	6	0
1	1/2	7	−17/8
2	0	8	0
3	−1/4	9	31/2
4	0	10	0
5	1/2	11	−691/4

It seems that $H_l = 0$ if l is even; the H_l couldt be similar to the Bernoulli numbers B_l. Furthermore, for the given numbers in the table we have

$$H_l = \frac{2(2^{l+1}-1)}{l+1} B_{l+1} \tag{12}$$

So, we come to the idea that a generating function belongs to the H_l.

Lemma 3:

$$2\frac{e^t}{e^t+1} = \sum_{l=0}^{\infty} H_l \frac{t^l}{l!}$$

Proof: We adopt the following equation from [2]:

$$(e^t - 1)^j = \sum_{l=1}^{\infty} \Delta^l 0^j \frac{t^l}{l!}$$

for $j = 1, 2, \ldots$ and small t. Then we get for such small t:

$$\begin{aligned}
\frac{e^t-1}{e^t+1} &= \frac{e^t-1}{2}\frac{1}{1-\frac{1-e^t}{2}} = \sum_{j=0}^{\infty}(-1)^j (\frac{e^t-1}{2})^{j+1} \\
&= \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{2^j}\sum_{l=1}^{\infty}\Delta^l 0^j \frac{t^l}{l!} \\
&= \sum_{l=1}^{\infty}(\sum_{j=1}^{\infty}(-1)^{j-l}\frac{1}{2^j}\Delta^l 0^j)(-1)^{l+1}\frac{t^l}{l!} = \sum_{l=1}^{\infty}(-1)^{l+1} H_l \frac{t^l}{l!}.
\end{aligned}$$

Because

$$\frac{e^t-1}{e^t+1} = -\frac{e^{-t}-1}{e^{-t}+1}$$

(odd function) all H_l with even index vanishes; further, if l is odd we can drop the factor $(-1)^{l+1}$; moreover, with $H_0 = 1$ we have

$$\frac{e^t-1}{e^t+1} + 1 = 2\frac{e^t}{e^t+1} = \sum_{l=0}^{\infty} H_l \frac{t^l}{l!}.$$

Lemma 3 is proved.

Now we can show the relation (12) between H_l and the Bernoulli numbers B_l. Because it is valid: $B_1 = -1/2$ and for small t

$$\frac{t}{e^t-1} = \sum_{l=0}^{\infty} B_l \frac{t^l}{l!} \tag{13}$$

we get

$$\begin{aligned}
2\frac{e^t}{e^t+1} &= 2\frac{e^t(e^t-1)}{(e^t+1)(e^t-1)} = \frac{2}{t}\left(t - \frac{t}{e^t-1} + \frac{2t}{e^{2t}-1}\right) \\
&= 2 - \frac{2}{t}\sum_{l=0}^{\infty} B_l \frac{t^l}{l!} + \frac{2}{t}\sum_{l=0}^{\infty} B_l \frac{(2t)^l}{l!} \\
&= 2 - \frac{2}{t}\left(\sum_{l=2}^{\infty}(B_l - 2^l B_l)\frac{t^l}{l!}\right) - 2B_1 + 4B_1 \\
&= 1 + \sum_{l=1}^{\infty} 2(2^{l+1}-1)B_{l+1}\frac{t^l}{(l+1)!} = \sum_{l=0}^{\infty} H_l \frac{t^l}{l!}
\end{aligned} \tag{14}$$

and

$$H_0 = 1, H_l = \frac{2(2^{l+1}-1)}{l+1} B_{l+1} \tag{15}$$

for $l = 1, 2, \ldots$

It is possible to get the H_l by computation directly. If we differentiate the function

$$h(x) = 2\frac{e^x}{e^x+1}$$

then $H_l = h^{(l)}(0)$. Unfortunately, the differentiation does not give a general expression for the function $h^{(l)}(x)$. But it is easy to see that

$$h^{(l)}(x) = 2p_l(e^x)\frac{e^x}{(e^x+1)^{l+1}}$$

where $p_l \in Z[y]$. Hence the values $2^l H_l = 2^l h^{(l)}(0)$ are integers for all $l = 0, 1, \ldots$.

Similar to the Bernoulli numbers the H_l satisfy some recursive relations which can be used to calculate the numbers H_l. For instances, we obtain:

a)From

$$2e^t = (e^t + 1)\sum_{l=0}^{\infty} H_l \frac{t^l}{l!}$$

and development in power series we get by comparison of coefficients $2H_0 = 2, 2H_1 + H_0 = 2, 2H_2 + 2H_1 + H_0 = 2$ etc. and in general

$$\sum_{l=0}^{k} \binom{k}{l} H_l = 2 - H_k.$$

b)From

$$2 = (e^{-t} + 1)\sum_{l=0}^{\infty} H_l \frac{t^l}{l!}$$

and development in power series we get

$$H_0 = 1, \sum_{l=0}^{k} (-1)^{k-l+1} \binom{k}{l} H_l = H_k.$$

Furthermore, it is easy to express the H_l by aid of the important values $B_l(1/2)$ and $B_l(1)$ of the Bernoulli polynoms $B_l(x)$.

Proposition:

$$g(x_1, \ldots, x_n) = 2^n \sum_{l=0}^{n} H_l \sigma_{n-l}(x_1, \ldots, x_n)$$

is the unique polynomial solution of (3).

Proof: By the foregoing we have to prove the uniqueness only. For this we look to the equation

$$g(x_1, \ldots, x_n) + g(x_1 - 1, \ldots, x_n - 1) = 0 \tag{16}$$

and show that the only polynomial solution is the nullpolynom. From this fact it is easy to get the uniqueness of the solution of (16) with an arbitrary polynomial on the right side.

Let be $cx_1^{t_1} \cdots x_n^{t_n}$ a summand in g with maximal degree $t_1 + \cdots + t_n$ and

$$g(x_1, \ldots, x_n) = cx_1^{t_1} \cdots x_n^{t_n} + d(x_1, \ldots, x_n).$$

On the left side of (16) we get

$$\begin{aligned} cx_1^{t_1} \cdots x_n^{t_n} + c(x_1 - 1)^{t_1} \cdots (x_n - 1)^{t_n} + d(x_1, \ldots, x_n) + d(x_1 - 1, \ldots, x_n - 1) \\ = 2cx_1^{t_1} \cdots x_n^{t_n} + h(x_1, \ldots, x_n) \end{aligned}$$

where h contains only summands of the degree less or equal to $t_1 + \cdots + t_n$ and the summands with this maximal degree are linear independent from the first summand. Then c must be zero and g has no summand of maximal degree.

Remark: If we assume that the solution $g(x_1, \ldots, x_n)$ of the equation (3) has the form from the proposition then by a (not short) calculation we get the recursive relation b) for the H_l directly. For this it is necessary to use the equation (proof by induction)

$$\sigma_l(x_1 - 1, \ldots, x_n - 1) = \sum_{j=0}^{l} (-1)^{l-j} \binom{n-j}{l-j} \sigma_j(x_1, \ldots, x_n).$$

For later applications we note:

Corollary: *To every $k \in \{1, \ldots, n\}$ and for every natural number $\kappa > 0$, $\kappa < k$ the function $g(x_1, \ldots, x_n)$ satisfy the following equation:*

$$\begin{aligned}
&g(x_1, \ldots, x_{k-\kappa-1}, x_{k-\kappa} = \kappa, \ldots, x_{k-2} = 2, x_{k-1} = 1, x_k, \ldots, x_n) + \\
&(-1)^\kappa g(x_1 - (\kappa+1), \ldots, x_{k-\kappa} = -1, \ldots, x_{k-2} = -\kappa + 1, x_{k-1} = -\kappa, \\
&x_k - (\kappa+1), \ldots, x_n - (\kappa+1)) \\
&= 2^{n+1} \kappa! \prod_{i=1,\ldots,k-\kappa-1;k,\ldots,n} x_i
\end{aligned}$$

Proof: From the proposition we know that

$$g(x_1, \ldots, x_n) = 2^n \sum_{l=0}^{n-1} H_l \sigma_{n-l}(x_1, \ldots, x_n)$$

is the unique polynomial solution of

$$g(x_1, \ldots, x_n) + g(x_1 - 1, \ldots, x_n - 1) = 2^{n+1} x_1 \cdots x_n.$$

If we replace x_i by $x_i - 1$ and x_{k-1} by 1 we get

$$g(x_1 - 1, \ldots, x_{k-1} = 1, \ldots, x_n - 1) + g(x_1 - 2, \ldots, x_{k-1} - 2 = -1, \ldots, x_n - 2) =$$

$$2^{n+1}(x_1 - 1) \cdots (x_n - 1) = 0.$$

and

$$g(x_1, \ldots, x_{k-1} = 1, \ldots, x_n) - g(x_1 - 2, \ldots, x_{k-1} - 2 = -1, \ldots, x_n - 2) = 2^{n+1} x_1 \cdots x_n.$$

Analogous, by the assumption $x_{k-\kappa} = \kappa, \ldots, x_{k-2} = 2, x_{k-1} = 1$ we can produce a chain of equations

$$g(x_1 - s, \ldots, x_n - s) + g(x_1 - (s+1), \ldots, x_n - (s+1)) = 0,$$

$s = 1, \ldots, \kappa$, such that for $x_{k-\kappa} = \kappa, \ldots, x_{k-1} = 1$ we have

$$g(x_1, \ldots, x_n) + (-1)^\kappa g(x_1 - (\kappa+1), \ldots, x_n - (\kappa+1)) = 2^{n+1}\kappa! \prod_{i=1,\ldots,k-\kappa-1;k,\ldots,n} x_i.$$

Remark: In general, $g(x_1, \ldots, x_n)$ is not determined in an unique way by the equation of the corollary.

4. The dimension of an irreducible representation of sl(1,n)

We use the notations of 2.

The dimension D of $\bar{W}([m]_{n+1})$ is determined by the $gl(n)$ fidirmod V_0 with the same highest weight $[m]_n$. The dimension of V_0 is given by the expression (cf. [5])

$$dim V_0 = \prod_{i<j,\, i,j=1,\ldots,n} \frac{m_i - m_j + j - i}{j - i} = \prod_{i<j,\, i,j=1,\ldots,n} \frac{r_{ij}}{j-i} \tag{17}$$

where $r_{ij} = m_i - m_j + j - i$ (for the following we extend this definition of r_{ij} on the case $i > j$ also) and

$$D = 2^n dim V_0.$$

So we can consider the dimension of a typical $sl(1,n)$ fidirmod as known.

On the other side by the sum decomposition given by (1) it is possible to describe D in the following way: Let for a given Θ-tuple $(\Theta_1, \ldots, \Theta_n)$ be $r_{st}(\Theta) = m_s - m_t + t - s + \Theta_s - \Theta_t$. Then

$$D = \sum_\Theta \prod_{s<t,\, s,t=1,\ldots,n} \frac{r_{st}(\Theta)}{t-s} \tag{18}$$

where the summation goes over all admissible $\Theta-$tuple. If $\bar{W}([m]_{n+1})$ is indecomposable and $m_k = k-1$ then the dimensions $D_1 = dim\bar{I}$ and $D_2 = dim\bar{W}([m]_{n+1})/\bar{I}$ are given by the same formula (18) with the difference that the summation goes over all admissible Θ-tuple with $\Theta_k = 1$ and $\Theta_k = 0$ resp. (cf. (2)). This gives a possibility for the numerical calculation of the dimension of an atypical $sl(1,n)$ fidirmod. In the following we will prove an explicit formula for such dimensions.

We have to differ between two cases. In the first one we assume that $k = 1$ or $k > 1$ and $m_{k-1} > m_k = k-1$. We will refer to this case as the general one. In the second case (the exceptional case) we assume that exist a $\kappa > 0$ such that $m_{k-\kappa} = \ldots = m_k = k-1$ and if $k - \kappa > 1$ we have $m_{k-\kappa-1} > m_{k-\kappa}$.

We look to the maximal invariant subspace $\bar{I}$ of $\bar{W}([m]_{n+1})$. $\bar{I}$ itself is an atypical $sl(1,n)$ fidirmod with the highest weight

$$[m']_{n+1} = [m_1 - 1, \dots, m_k, \dots, m_n - 1] = [m-1]_{n+1}^{k;irr}$$

and the same k as in $\bar{W}([m]_{n+1})/\bar{I}$ in the general case. In the exceptional case the highest weight is given by

$$[m']_{n+1}^{irr} = [m_1 - \kappa - 1, \dots, m_{k-\kappa-1} - \kappa - 1, m_{k-\kappa} - \kappa, \dots,$$

$$m_k - \kappa, m_{k+1} - \kappa - 1, \dots, m_n - \kappa - 1] = [m - (\kappa+1)]_{n+1}^{k-\kappa,\dots,k;irr}. \tag{19}$$

Firstly, we consider the general case. We look to the exceptional case after the theorem.

D_1 and D_2 we can compute by the same formula replacing m_i by m_i' and r_{st} by r'_{st}, where $r'_{sk} = r_{sk} - 1$ for $s < k$, $r'_{kt} = r_{kt} + 1$ for $t > k$ and $r'_{st} = r_{st}$ in all other cases. On the other side from the analogous formula to (18) we see that D_2 is a polynomial $f(r_{12}, \dots, r_{n-1,n})$. Because $D_2 + D_1 = D$ we have to look to a polynomial f which satisfied the functional equation

$$f(r_{12}, \dots, r_{n-1,n}) + f(r'_{12}, \dots, r'_{n-1,n}) = 2^n \prod_{s<t,\, s,t=1,\dots,n} \frac{r_{st}}{t-s}. \tag{20}$$

Because $r_{st} = r_{sk} + r_{kt}$ for arbitrary s, t, k we look to the polynomial f as a polynomial in $n-1$ variables only. We choose $x_1 = r_{1k}, \dots, x_{k-1} = r_{k-1,k}, x_{k+1} = r_{k,k+1}, \dots, x_n = r_{kn}$ as such variables which are independent. On the right side of (20) we separate the factor

$$2^n \prod_{s=1,\dots,k-1} r_{sk} \prod_{t=k+1,\dots,n} r_{kt} = 2^n \prod_{i=1,\, i\neq k}^{n} x_i$$

from the rest R_{kn} and consider the equation

$$\begin{aligned} f_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) &+ f_k(x_1 - 1, \dots, x_{k-1} - 1, x_{k+1} + 1, \dots, x_n + 1) \\ &= 2^n \prod_{i=1,\, i\neq k}^{n} x_i \end{aligned} \tag{21}$$

and for $k = n$

$$f_n(x_1, \dots, x_{n-1}) + f_n(x_1 - 1, \dots, x_{n-1} - 1) = 2^n \prod_{i=1,\dots,n-1} x_i. \tag{22}$$

It is easy to see that $R_{kn} f_k(r_{1k}, \dots, r_{k-1,k}, r_{k,k+1}, \dots, r_{kn})$ is a solution of (20) where f_k solves (21). Furthermore, the solutions of (20) and (21) resp. are unique by the same arguments as in the proof of the proposition of 3.

Now we compare (21) for arbitrary k with the case $k = n$. If we replace r_{kt} by $-r_{kt} = r_{tk}$ and $r_{kt}(\Theta)$ by $-r_{kt}(\Theta) = r_{tk}(\Theta)$ we see by (18) that must be valid

$$R_{kn} f_k(r_{1k}, \ldots, r_{k-1,k}, r_{k,k+1}, \ldots, r_{kn}) =$$
$$(-1)^{n-k} R_{kn} f_n(r_{1k}, \ldots, r_{k-1,k}, -r_{k,k+1}, \ldots, -r_{kn}).$$

So it is enough to solve (22) obtaining the solution of (20). But (22) is solved by the proposition of 3.

Theorem: *Let $Y_k([m]_{n+1}) = \bar{W}([m]_{n+1})/\bar{I}$ be an atypical finitedimensional irreducible sl(1,n) module with highest weight $[m_1, \ldots, m_k = k-1, \ldots, m_n]^{irr}_{n+1}$. For all possible s,t,k we set*

$$r_{st} = m_s - m_t + t - s$$

and

$$R_{kn} = \prod_{s<t,\, s,t=1,\ldots,n} \frac{1}{t-s} \prod_{i<j, i\neq k, j\neq k, i,j=1,\ldots,n} r_{ij}.$$

The dimension of $Y_k([m]_{n+1})$ we get by the following formula:

$$dim(Y_k([m]_{n+1}) = 2^{n-1} R_{kn} *$$

$$\sum_{l=0}^{n-1} (-1)^{n-k} H_l \sigma_{(n-1)-l}(r_{1k}, \ldots, r_{k-1,k}, -r_{k,k+1}, \ldots, -r_{kn})$$

where the $\sigma_{(n-1)-l}$ are the elementary symmetric functions of the first kind and the H_l are the coefficients determined in 3. More exact, because $m_k = k-1$ we have

$$r_{sk} = m_s - s + 1, -r_{kt} = m_t - t + 1$$

such that

$$dim(Y_k([m]_{n+1})) = 2^{n-1} R_{kn} \sum_{l=0}^{n-1} (-1)^{n-k} H_l *$$

$$\sigma_{(n-1)-l}(m_1, m_2 - 1, \ldots, m_{k-1} - (k-2), m_{k+1} - k, \ldots, m_n - (n-1)).$$

Proof: The general case is proved above so we have only to consider the exceptional case. In this one it is possible to verify that

$$dim(Y_k([m]_{n+1})) + dim(Y_{k-\kappa}([m']_{n+1})) = 2^n \prod_{s<t,\, s,t=1,\ldots,n} \frac{r_{st}}{t-s}$$

($[m']$ is given by (19)) by aid of the corollary of 3.

The remark to this corollary said that the solutions are not unique in this case. We get the uniqueness in the following way:

I. If $m_{k+1} < k-1$ or $k = n$ we can look to the admissible representation $[\bar{m}]^{irr}_{n+1} = [m+1]^{k;irr}_{n+1}$ which is atypical but not exceptional; $[m]^{irr}_{n+1}$ and $[\bar{m}]^{irr}_{n+1}$ are paired in $[\bar{m}]_{n+1}$ and because the dimension formula is correct for $[\bar{m}]_{n+1}$ this must be also for $[m]^{irr}_{n+1}$.

II.If for some $\kappa > 0$ we have $m_{k+1} = \ldots = m_{k+\kappa} = k-1$ (κ maximal chosen) we define $[\bar{m}]^{irr}_{n+1} = [m+\kappa+1]^{k,\ldots,k+\kappa}_{n+1}$. $[\bar{m}]^{irr}_{n+1}$ is atypical with $\bar{m}_{k+\kappa} = k+\kappa-1$ but in general exceptional. We repeat this process; by finitely many steps we come to I. and must apply the arguments on the constructed chain of highest weights in a successive way.

As examples we list the explicit expressions for the dimensions of the atypical irreducible $sl(1,3)$ modules:

$$dimY_3([m_1, m_2, 2]) =$$
$$4 * 1/2(m_1 - m_2 + 1)(H_0\sigma_2(m_1, m_2 - 1) + H_1\sigma_1(m_1, m_2 - 1) + H_2) =$$
$$(m_1 - m_2 + 1)(2m_1m_2 - m_1 + m_2 + 1),$$
$$dimY_2([m_1, 1, m_3]) =$$
$$4 * 1/2 * (-1)(m_1 - m_3 + 2)(H_0\sigma_2(m_1, m_3 - 2) + H_1\sigma_1(m_1, m_3 - 2) + H_2) =$$
$$(m_1 - m_3 + 2)(-2m_1m_3 + 3m_1 - m_3 + 2),$$
$$dimY_1([0, m_2, m_3]) =$$
$$4 * 1/2(m_2 - m_3 + 1)(H_0\sigma_2(m_2 - 1, m_3 - 2) + H_1\sigma_1(m_2 - 1, m_3 - 2) + H_2) =$$
$$(m_2 - m_3 + 1)(2m_2m_3 - 3m_2 - m_3 + 1).$$

References:

[1] Boseck, H.: *Lie superalgebras and Lie supergroups,I,* Seminar Sophus Lie **1**, no. 2 (1991), 101–122.

[2] Gelfond, A.O.: *Differenzenrechnung,* Dt. Verlag der Wissenschaften, Berlin 1958.

[3] Hughes, J.W.B.; King, R.C.; Van der Jeugt, J.: *On the composition factors of Kac-modules for the Lie superalgebras sl(m/n),* J. Math. Phys. **33** (1992), 470–491.

[4] Van der Jeugt, J.; Hughes, J.W.B.; King, R.C.; Thierry-Mieg, J.: *A character formula for singly atypical modules of the Lie superalgebra sl(m/n),* Communications in Algebra **18** (1990), 3453–3480.

[5] Kac, V.G.: *Representations of classical Lie superalgebras,* Lecture Notes Math. **626** (1978), 597–626.

[6] Palev, T.D.: *Finite-dimensional representations of the special linear Lie superalgebra sl(1,n). I. Typical representations,* J. Math. Phys. **28**,10 (1987), 2280–2303.

[7] Palev, T.D.: *Finite-dimensional representations of the special linear Lie superalgebra sl(1,n). II. Nontypical representations,* J. Math. Phys. **29**,12 (1988), 2589–2598.

[8] Schlosser,H.: *Atypical representations of the Lie superalgebra sl(1,n) for* $n > 1$, Seminar Sophus Lie, **3**, no. 1 (1993), 15-24.

Received: January 26, 1994, revised version June 3, 1994.