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Vector Spaces of Matrices of Low Rank and Vector Bundles on Projectic Spaces:

an addendum to a paper by Eisenbud and Harris

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Abstract. Here we give a quick affirmative answer to a linear algebra question raised in D. Eisenbud and J. Harris, Vector spaces of matrices of low rank, Adv. in Math. 70 (1988), 135-155. In the quoted paper methods coming from algebraic geometry (essentially the theory of vector bundles and reflexive sheaves on $\mathbf{P}^N$) were shown to be powerful tools for the study of vector spaces of matrices. Fix a positive integer k. Let V and W be vector spaces; set $v := \dim(V)$, $w := \dim(W)$, $s := \min(v, w)$. Let $M \subseteq Hom(V, W)$ be a vector space of linear maps such that the general $f \in M$ has rank k. Here we show essentially that if k is fixed and "M does not come from a simpler situation", then s cannot be arbitrarily large. With the notations of the quoted paper we show the existence of $s_0(k)$ such that if $s \geq s_0(k)$, then M is not strongly indecomposable.

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In this paper we give a quick affirmative answer to a linear algebra question raised at the end of the paper [7] by D. Eisenbud and J. Harris. In the quoted paper methods coming from algebraic geometry (essentially the theory of vector bundles and reflexive sheaves on $\mathbf{P}^N$) were shown to be very powerful tools for the study of vector spaces of matrices. Even more important; the geometric set up of [7], §2, suggests in a natural way several linear algebra problems. Vice versa, natural linear algebra problems may single out for further study very restrictive classes of sheaves on projective spaces. Hence this seems to be a very good place for interaction between algebraic geometers and specialists of linear algebra. As recent references where a similar game with 3 vector spaces (here they will be V, W and M) with as aim very nice "pure" projective geometry, we mention [4], [5], [6, Remark 3.12] and [3].

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We work over an algebraically closed field, although it is easy to reduce the case in which the base field is any infinite field to the case with an algebraically closed base field. Fix a positive integer k. Let V and W be finite dimensional vector spaces; set $v := \dim(V)$, $w := \dim(W)$, $s := \min(v, w)$. Let $M \subseteq Hom(V, W)$ be a vector space of linear maps such that the general $f \in M$ has rank k; set $m := \dim(M)$. Here we show essentially that if k is fixed and "M does not come from a simpler situation", then s cannot be arbitrarily large.With the notations of the quoted paper (to be recalled below) we prove the following result.

Theorem *Fix a positive integer k. Let V and W be finite dimensional vector spaces over an algebraically closed field of characteristic 0; set $v := \dim(V)$, $w := \dim(W)$ and $s := \min(v, w)$. Let $M \subseteq Hom(V, W)$ be a non degenerate vector space of linear maps such that the general $f \in M$ has rank k. Then there exists an integer r, $0 \leq r \leq k$, such that (V, W, M, k) is an extension of a primitive datum (V'', W'', M'', r) and a compressed space of rank $k - r$. Furthermore, setting $v'' := \dim(V'')$ and $w'' := \dim(W'')$, either $v'' = r + 1$ and $w'' \leq r(r+1)/2$ or $w'' = r + 1$ and $v'' \leq r(r+1)/2$ or there are two integers c, d, with $r = c + d$, $c \geq 2$, $d \geq 2$, $v'' \leq c + 1 + d(d+1)/2$ and $w'' \leq d + 1 + c(c+1)/2$. In particular, if $s > k(k+1)$, then the datum (V, W, M, k) is not strongly indecomposable.*

We recall some of the definitions given in [7], warning the reader that in the linear algebra literature the same concept may have a different name. We will say that the datum (V, W, M, k) is *non degenerate* if 0 is the only vector in V in the kernel of all $f \in M$ and no proper subspace of W contains the images of all $f \in M$. This means that we cannot find a base of V and a base of W such that with respect to these bases either all matrices representing linear maps in M have 0 as first column or all matrices representing linear maps in M have 0 as first row. We recall that, according to [1], [2] and [7] the notion of primitivity is defined in the following way. If $V' \subseteq V$ and $W' \subseteq W$ are vector spaces, let $\pi_{V',W'}$ be the natural linear map $\pi_{V',W'} : Hom(V, W) \longrightarrow Hom(V', W/W')$ sending each $g \in Hom(V, W)$ to the composition of $g|V'$ with the quotient map $W \longrightarrow W/W'$. A vector space $M \subseteq Hom(V, W)$ such that the general $f \in M$ has rank k is called *primitive* if it is non degenerate and there is no such pair $(V', W') \neq (V, W)$, such that k is the rank of a general member of $\pi^{-1}_{V',W'}(\pi_{V',W'}(M))$, i.e. M is primitive if it is not equivalent to a vector space of matrices in such a way that some space of submatrices M'' accounts entirely for the low rank of the matrices of M. This means that we cannot find bases of V and W and integers e, f with $1 \leq e \leq v$, $1 \leq f \leq w$, $e + f < v + w$, and such that the set M' of submatrices $b := (b_{x,y})$, $1 \leq x \leq w - f$, $1 \leq y \leq v - e$, with $a := (a_{i,j})$ varying in M and $b_{x,y} = a_{x+f,y+e}$ for all x, y, is a vector space of matrices with general rank k. See [1], [2] and [7], §1, for further explanations of this concept and for many examples. It is very easy to prove (see [2] or [1] or [7], §1), that for every non degenerate datum (V, W, M, k) there is a minimal integer $z \geq 0$ such that there is a primitive datum $(V', W/W', M', k - z)$ with $z = \dim(V') + \dim(W')$; here V' and W' are not uniquely determined (see [7], p. 140). According to [7], §1, a non degenerate datum (V, W, M, k) is called *compressed* if there are integers $r' > 0$ and $r'' > 0$ with $r' + r'' = r$ and such that the datum is equivalent to a linear space of matrices whose upper left corner of order $(\dim(V) - r') \times (\dim(W) - r'')$ is the 0 matrix. A non degenerate datum is compressed if and only if its primitive part is the zero $r' \times r''$ matrix ([7], p. 139). A datum

(V, W, M, k) is called an extension of the datum (A, B, C, a) by the datum $(A', B', C', k-a)$ if $\dim(V) = \dim(A)+\dim(A')$, $\dim(W) = \dim(C)+\dim(C')$ and (V, W, M, k) is equivalent to a space of matrices which may be partitioned into 4 rectangular blocks: an upper left block equivalent to (A, B, C, a), a lower right block equivalent to $(A', B', C', k-a)$ and one of the two remaining rectangular blocks is 0. Recall from [7], p. 152, that a datum (V, W, M, k) is *strongly indecomposable* if it is not a projection of the direct sum of two data, i.e. that the corresponding family of matrices is not the projection of a family of the same rank and dimension which may be simultaneously put in a block diagonal form. For example a family F of matrices for the numerical datum (v, w, m, k) is not strongly indecomposable if there are two families of matrices F' and F'' with numerical data (v, w', m', k'), $(v, w-w', m'', k-k')$ such that each matrix $Q \in F$ is obtained from $Q' \in F'$ and $Q'' \in F''$ first the rows of Q' and then the rows of Q''. An extension of data is not strongly indecomposable ([7], p.152). The aim of these definitions is to single out the "building blocks" for the vector spaces of matrices with fixed generic rank. Then there are two natural problems. First, give inequalities for the integers v, w, m and k when (V, W, M, k) is a "building block". Then, classify all "building blocks" with integers v, w, m and k for which some of these inequalities are almost sharp. An interesting particular case of this problem is the classification of the "building blocks" with low generic rank.

Now we will give the interpretation (due to Eisenbud and Harris and contained in [7]) of these concepts in terms of sheaves and vector bundles on projective spaces. Recall from [7], §2, that the datum (V, W, M, k) can be translated in the following way. Let $\mathbf{P} := \mathbf{P}(M^*)$ be the projective space of one-dimensional linear subspaces of M. Then the datum (V, W, M, k) is equivalent to a suitable linear map $\phi : V \otimes \mathbf{O_P} \longrightarrow W \otimes \mathbf{O_P}(1)$, such that $E := \mathrm{Im}(\phi)$ has rank k. We will use in an essential way the criteria of primitivity and compressibility given in [7], Prop. 2.3 and Prop. 2.1: the datum is primitive if and only if E^{**} has no factor $\mathbf{O_P}$ or $\mathbf{O_P}(1)$ while the datum is compressed if and only if E^{**} is the direct sum of copies of $\mathbf{O_P}$ and $\mathbf{O_P}(1)$.

Proof of the theorem. The furthermore part of the theorem is just the bound on v'' and w'' proved for primitive data in [2]. Assume that E^{**} has as factors exactly a' copies of $\mathbf{O_P}$ and a'' copies of $\mathbf{O_P}(1)$ with $a' \geq 0$, $a'' \geq 0$. Set $r := k - a' - a''$. Define the rank r coherent sheaf F by the relation $E^{**} \cong F \oplus \mathbf{O}_{\mathbf{P}^{a'}} \oplus \mathbf{O}_{\mathbf{P}^{a''}}(1)$; F is uniquely determined (hence reflexive) by the Krull - Remak - Schmidt property of coherent sheaves on projective varieties. Let V' be the kernel of the composition of the map $V \otimes \mathbf{O_P} \longrightarrow E^{**}$ with the projection onto F. Take a linear subspace V'' of V with $V' \cap V'' = \{0\}$ and $V' + V'' = V$. Let W'' be the minimal subspace of W such that $W'' \otimes \mathbf{O_P}(1)$ contains the image of the restriction to $V'' \otimes \mathbf{O_P}$ of the multiplication map $V \otimes \mathbf{O_P} \longrightarrow W'' \otimes \mathbf{O_P}(1)$. Let G be the image of the map $W'' \otimes \mathbf{O_P}(1) \longrightarrow F$. We have $F \cong G^{**}$. By [7], Prop. 2.3, the datum $(V'', W'', M'', k - a' - a'')$ corresponding to F (for suitable $M'' \subseteq M$, just to preserve the non degeneracy condition) is primitive. Let W' be the minimal subspace of W such that $W' \otimes \mathbf{O_P}(1)$ contains the image of the restriction to $V' \otimes \mathbf{O_P}$ of the multiplication map $V \otimes \mathbf{O_P} \longrightarrow W \otimes \mathbf{O_P}(1)$. Let T be the image of the map from $V' \otimes \mathbf{O_P}(1)$ to $\mathbf{O}_{\mathbf{P}^{a'}} \oplus \mathbf{O}_{\mathbf{P}^{a''}}(1)$ induced by the projection from E^{**} to $\mathbf{O}_{\mathbf{P}^{a'}} \oplus \mathbf{O}_{\mathbf{P}^{a''}}(1)$. Note that V' injects into $H^0(M, T)$. The datum $(V', W', M, a' + a'')$ corresponding to the pair (V', T) corresponds to a compressed datum. By construction V' maps to 0 into $H^0(M', T)$ and

$H^0(M', G)$. Hence the original datum is an extension of the datum (V'', W'', M'', r) by the datum $(V', W', M', a' + a'')$ (for suitable $M' \subseteq M$) corresponding to T and the given inclusion of V' into $H^0(M', T)$. By [7], Prop. 2.1, the datum (V', W', M', k) corresponding to T is compressed, proving the theorem. □

References

[1] Atkinson, M. D.: *Primitive spaces of matrices of bounded rank. II*, J. Austral. Math. Soc. (Series A) **34** (1983), 306 – 315.

[2] Atkinson, M. D. and Lloyd, S.: *Primitive spaces of matrices of bounded rank*, J. Austral. Math. Soc. (Series A) **30** (1981), 473 – 482.

[3] Ballico, E., Giovanetti, F. and Russo, B.: *On logarithmic bundles in positive characteristic*, in: Projective Geometry with Applications, 75 – 79, Lecture Notes in Pure Appl. Math. 166, Marcel Dekker, 1994.

[4] Dolgachev, I. and Kapranov, M.: *Arrangements of hyperplanes and vector bundles*, preprint.

[5] Dolgachev, I. and Kapranov, M.: *Schur quadrics and their generalizations in the theory of vector bundles on* $\mathbf{P}^2$, preprint.

[6] Geramita, A. V., Gimigliano, A. and Pitteloud, Y.: *Graded Betti numbers of some embedded rational n-folds*, preprint Queen's University, October 1993. Math. Preprint 1993-12.

[7] Eisenbud, D. and Harris, J.: *Vector spaces of matrices of low rank*, Adv. in Math. **70** (1988), 135 – 155.

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