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Convex Hulls of Spatial Polygons with a Fixed Convex Projection

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Abstract. Let F be a convex n-gon in a horizontal plane of the Euclidean 3-space. Consider its spatial variation under which its vertices move vertically and let F_* be the convex hull of such a variation. In the general position, the boundary of F_* splits naturally into the "bottom" F' and the "top" F''. The polyhedron F' (F'') has triangular faces and no vertices inside. The projection of these faces on F yields a triangulation T' (T'') of F. Obviously T' and (T'') have no common diagonals. Suppose now that T' and (T'') with no common diagonals are prescribed. Guibas conjectured that the appropriate variation exists. The present paper gives a sufficient and a necessary condition of such existence. The necessary condition can fail which disproves the Guibas conjecture.

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A segment connecting two vertices of a flat convex polygon and crossing its interior will be called a *diagonal.*

Let F be a convex n–gon with no 3 vertices collinear. By a *diagonal triangulation* T of F, we will mean a nonempty set of its diagonals which do not intersect in int$F$ and split F into triangles. One can see easily that these triangles are nondegenerate and their number is $n-2$, where n is necessarily ≥ 4. We will call them *triangles of* T.

Let F lie in the xy–plane of E^3 referred to the standard rectangular coordinates x, y, z. Let $A_1, A_2, \ldots, A_n$ be its vertices. Take two diagonal triangulations T_1 and T of F having no common diagonals. Consider an n–vector $J = (z_1, z_2, \ldots, z_n)$ and denote by $R(J)$ the set $\{a_1, a_2, \ldots, a_n\} \subset E^3$ such that the projection of the point a_i onto the xy–plane is A_i and the z-coordinate of a_i is z_i, $i = 1, 2, \ldots, n$. Suppose that the convex polytope

$$F_* = F_*(J) \stackrel{\text{def}}{=} \text{conv}\, R(J) \tag{1}$$

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is nondegenerate (i.e. has interior points). Note that ∂F_* contains no vertical faces since such a face would contain at least three points a_i, a_j, a_k and then A_i, A_j, A_k would be collinear. Therefore the spatial polygon $a_1 a_2 \ldots a_n$ splits the polyhedral surface ∂F_* into two polyhedral surfaces: the "bottom" F' and the "top" F'' homeomorphic to a closed circle. (We assume that the z–axis is vertical and directed upward. The terms like "bottom", "bellow", "under" etc. will be used accordingly.) An edge of F_* lying in F' (F'') but not on its boundary will be called an *interior* edge of F' (F'').

Suppose that each (complete) face of F_* is a triangle. Moreover, assume that the projections of the interior edges of F' (F'') onto the xy–plane form the triangulation T_1 (T). Then the set F_* is called a *realization* of the pair (T_1, T) and the pair (T_1, T) is said to be *realizable.* Note that if (T_1, T) has a realization $F_*(J)$ then $F_*(-J)$ is a realization of (T, T_1). Hence, the realizability does not depend on the order of the triangulations.

Guibas conjectured [4] that each pair of diagonal triangulations of F having no common diagonals is realizable. We established here, in Theorem 1, a necessary and, in Theorem 2, a sufficient condition of such realizability. The necessary condition can fail, see Remark 1, which disproves the Guibas conjecture. (In private communications in August 1993, Guibas mentioned that he was aware of a counterexample to his conjecture. He gave no details.)

Diagonal triangulations of convex polygons seem to be of an independent geometric interest. They appear also as a tool in other areas, see for instance paper [5, §1 and 2] by Sleator, Tarjan and Thurston where the triangulations are used to study tree graphs.

To proceed, we will need a definition arising from the following construction. Let T be a diagonal triangulation of a convex polygon F with no 3 vertices collinear. Let AB be a diagonal which is not in T. Let APQ (BRS) be that unique triangle (shaded in Fig.1) of the triangulation T whose intersection AX (BY) with the diagonal AB has a positive length. Denote by L the line determined by AB. Since $AB \notin T$, none of the vertices P, Q, R, S can lie on L. Hence one may assume that P and R lie on one side of L while Q and S lie on the other side. Suppose that the segments AQ and BR are diagonals from T (i.e. not sides of F) and let AQU and BRV be triangles of the triangulation T different from APQ and BRS respectively, see Fig.1. One may assume that either the segments AU and BV are parallel or the lines determined by them intersect at the point W which lies on the same side of L where Q and S lie. (Otherwise replace the notation P by Q and R by S.) To cover both of these cases, we allow the value ∞ for the distance BW.

Denote by M the line through Q and R. (These points are the ends of the diagonals AQ and BR singled out above. Note also that the points U and V are in different relations to W: the pair U, V is separated by L while the pair U, W is not.) Suppose that the ray (not the line) coming from B in the direction of the segment VB is either parallel to M or crosses M at a point Z. To cover both of these cases, we allow the value ∞ for the distance BZ. Finally, assume that

$$BZ \leq BW, \tag{2}$$

see Fig.1. We will say then that the diagonal AB is *rejected* by the triangulation T.

Note that if $BZ > BW$, then the diagonal AB still can be rejected by T through consideration of the pair (AP, BS) instead of (AQ, BR). (In Fig.1, the pair (AP, BS) however cannot be considered since BS is not a diagonal.) If each of the pairs (AP, BS)

Figure 1

Figure 2

and $(AQ,\ BR)$ contains the side of F, then the diagonal AB is not rejected.

Theorem 1. *Let F be a convex polygon with no 3 vertices collinear. Let T_1 and T be its diagonal triangulations having no common diagonals. If a diagonal of one of them is rejected by the other one, then the pair $(T_1,\ T)$ is not realizable.*

Remark 1. The simplest case of rejection is shown in Fig. 2. This is a hexagon $UAPVBQ$ in the xy–plane whose triangulation $T = \{AQ, QP, PB\}$ rejects the diagonal AB. (Here, the points R, S introduced above coincide with P, Q and BV is parallel to AU.) Hence, the pair $(T_1,\ T)$ is not realizable for the triangulation $T_1 = \{VA,\ AB,\ BU\}$. The same is true of the regular hexagon with $U = (-2, 0)$ and $V = (2, 0)$. This disproves the Guibas conjecture.

According to Barnette's theorem [1], every 3-polytope with a Hamiltonian circuit has a realization such that the Hamiltonian circuit is a shadow boundary. This implies that the Guibas conjecture is true up to a combinatorial deformation of the projection of the circuit. However, by Theorem 1, if the projection is fixed, it might be that the polytope has no realization.

Grünbaum [3] asked if, given a circuit of n vertices in a 3-polytope P and an n-gon A, there is an isomorphic 3-polytope P' for which the corresponding circuit projects orthogonally onto A. Barnette [2] gave a negative answer to this question by constructing a counterexample. Note in this connection that each case of rejection mentioned in Theorem 1 constitutes such a counterexample as well. In particular, this is true of the case in Fig. 2. The appropriate counterexample has thus 6 vertices while Barnette's one has 8, see [2].

A natural question regarding Theorem 1 is this. Suppose a pair of diagonal triangulations of the polygon F is not realizable. Is there a diagonal in one of them rejected by the other one? We do not know the answer.

Proof of Theorem 1. Denote by AB a diagonal from T_1 rejected by T. Supose to the contrary that a pair $(T_1,\ T)$ has a realization F_*. Introduce the notation P, Q, R, S, U, V, W, Z, L, M, F' and F'' as above. Let a, b, p, q, r, s, u, v be the points of $R(J)$ in (1) whose projections onto the xy–plane are denoted by the same capital letters. Denote by α, α', β, β' the planes of the triangles aqu, aqp, brv, brs respectively. Let h be the line $\alpha \cap \beta$. One can see easily that the projection H of the line h onto the xy–plane crosses the open segment $(AR) \stackrel{\text{def}}{=} AR \setminus \{A,\ R\}$, see Fig.1. To that end, one may consider the crossection of our construction by the vertical plane γ that contains a, r, A, R. The polygon $\gamma \cap F_*$ in that plane is clearly supported from below by the lines $\alpha \cap \gamma$, and $\alpha' \cap \gamma$ at the point a, see Fig.3. These two lines are distinct since the planes α and α' are distinct by the definition of realization. Obviously the line $\alpha' \cap \gamma$ is above the line $\alpha \cap \gamma$ within the open halfplane of γ bounded by the vertical line through a and containing the point r. Since r is not below $\alpha' \cap \gamma$, the point r is above the line $\alpha \cap \gamma$. Similarly, the point a is above the line $\beta \cap \gamma$. Therefore the intersection point $\alpha \cap \beta \cap \gamma = h \cap \gamma$ of the last two lines should be inside the open strip bounded by the vertical lines through A and R, see Fig.3. Hence the projection $H \cap \gamma$ of this point onto xy–plane should lie in the open segment (AR).

One can check similarly that H crosses the open segment (BQ). Then H crosses the open diagonal (QR) of the quadrangle $ARBQ$. Hence, H crosses the open segment (BZ)

Figure 3

at a point G, see Fig.1. (This point is finite even when the distance $BZ = \infty$.) Due to (2), the length BG satisfies

$$0 < BG < BZ \leq BW. \tag{3}$$

Therefore one can choose a point C on the segment AG which lies in the interior of the quadrangle $AXQU$, see Fig.1. Obviously, the segment CV crosses the diagonal AB at a point $D \in \operatorname{int} F$.

Let $c_1 \in \alpha$ have the projection C. Since $C \in \operatorname{int} AXQU$, the point C lies in $\operatorname{int} F$. Therefore one can choose a point $c_2 \in \operatorname{int} F_*$ with the same projection C. The open segment $(c_2 v)$ lies in $\operatorname{int} F_*$. Denote by d_1 and d_2 the points of $(c_1 v)$ and $(c_2 v)$ respectively whose projections coincide with D. (They exist since $D \in CV$.) Since c_2 is above c_1, the point d_2 lying in $\operatorname{int} F_*$ satisfies

$$d_2 \text{ is above } d_1. \tag{4}$$

To get a contradiction to (4), denote by g the point of the plane α having projection G. Then c_1 lies on the segment ga. Obviously g lies in β, too. Then b lies on the segment gv. The relations $c_1 \in ga$ and $b \in gv$ imply that the segments $c_1 v$ and ab cross at the point d_1. Since $ab \subset F''$, one has $d_1 \in F''$. Then the point d_2 cannot be above d_1 contrary to (4). □

To state Theorem 2, we need yet another definition based on an elaborate geometric construction. Let F be a convex polygon with no 3 collinear vertices. Let ab be a (directed) diagonal of F. Fix a number axis x in the plane of F containing ab and having the direction of ab. Points of the plane lying on the x–axis will be identified with their x–coordinates. Denote by $P_0 = a, P_1, P_2, \ldots, P_m = b$, $m \geq 2$, the vertices of F lying on one side of the x–axis and following each other on ∂F in this order. Let u_i, $i = 1, 2, \ldots, m-1$, be the

x–intercept of the line determined by the side P_iP_{i+1}. (If P_iP_{i+1} is parallel to the x–axis, we write $u_i = \omega$.) A similar construction on the other side of x–axis yields vertices $Q_0 = a, Q_1, Q_2, \ldots, Q_n = b$, $n \geq 2$, and the x–intercepts $v_1, v_2, \ldots, v_{n-1}$. The polygon F has thus $m + n \geq 4$ vertices.

Obviously the numbers $u_i, v_j \in S \stackrel{\text{def}}{=} (-\infty, a) \cup [b, \infty) \cup \{\omega\}$ and $u_{m-1} = v_{n-1} = b$. We introduce now a (non–standard) relation $\leq_R$ of linear order in the set S as follows. For $x_1 \in (-\infty, a)$, we assume $x_1 \leq_R x_2$ if $x_2 \in [x_1, a)$ and we assume $x_2 \leq_R x_1$ if $x_2 \in (-\infty, x_1] \cup \{\omega\} \cup [b, \infty)$. For $x_1 = \omega$, assume $x_1 \leq_R x_2$ if $x_2 \in (-\infty, a) \cup \{\omega\}$ and $x_2 \leq_R x_1$ if $x_2 \in \{\omega\} \cup [b, \infty)$. For $x_1 \in [b, \infty)$, assume $x_2 \leq_R x_1$ if $x_2 \in [b, x_1]$ and $x_1 \leq_R x_2$ if $x_2 \in [x_1, \infty) \cup \{\omega\} \cup (-\infty, a)$. The relations $\geq_R$, $=_R$, $\neq_R$, $<_R$, and $>_R$ are introduced in the standard way.

Let T be a diagonal triangulation of F such that $ab \notin T$ and each diagonal of T crosses ab in int F. Such a diagonal is thus of the form P_iQ_j, with $i \in \{1, \ldots, m-1\}$ and $j \in \{1, \ldots, n-1\}$. One can observe easily that

$$P_1Q_1 \in T,\ P_{m-1}Q_{n-1} \in T. \tag{5}$$

Note that if a diagonal $P_iQ_j \in T$ differs from $P_{m-1}Q_{n-1}$, then either $Q_jP_{i+1} \in T$ or $P_iQ_{j+1} \in T$ (but not both).

Suppose that the triangulation T has in addition the following property (α)

(α) If a diagonal $P_iQ_j \in T$ differs from $P_{m-1}Q_{n-1}$ then $u_i \neq_R v_j$. Moreover, if $u_i >_R v_j$, then $Q_jP_{i+1} \in T$. If $v_j >_R u_i$, then $P_iQ_{j+1} \in T$.

We will say that such a triangulation T is *generated by the diagonal ab.*

Remark 2. The triangulation T can be constructed, starting from the diagonal P_1Q_1, step by step by means of the property (α) (unless $T = \{P_1Q_1\}$). At a certain stage of this process, when a diagonal P_iQ_j has been just constructed, one of the vertices P_{i+1}, Q_{j+1} (but not both) will coincide with b. Suppose, for instance, that $Q_{j+1} = b = Q_n$. Thus one has at this stage the diagonal $P_iQ_j = P_iQ_{n-1}$. This implies $v_{n-1} = b$ and hence $u_i >_R v_{n-1}$. Now, by (α), the diagonal $Q_{n-1}P_{i+1}$ is added to the construction. Next we add $Q_{n-1}P_{i+2}, \ldots, Q_{n-1}P_{m-1}$. This is obviously the end of the construction. The described process introduces a linear order on the set T under which P_1Q_1 is the first and $P_{m-1}Q_{n-1}$ the last element of T.

The number of vertices involved in the process grows by one with each new diagonal. After the first diagonal, two vertices are involved : P_1 and Q_1. After the last diagonal $P_{m-1}Q_{n-1}$, all vertices but a and b are involved. Their number is $m + n - 2$. Thus the number of diagonals in T is

$$|T| = (m + n - 2) - 2 + 1 = m + n - 3. \tag{6}$$

Remark 3. The above definition obviously does not depend on the position of zero on the x–axis. One also can check easily that if T is generated by ab, then T is generated by ba. A triangulation T generated by ab is, of course, a very special case of diagonal triangulation: since each diagonal in T should cross ab in int F, no three such diagonals can form a triangle.

Theorem 2. *Let F be a convex polygon with no 3 vertices collinear. Let T and T_1 be its diagonal triangulations. Suppose that T is generated by a diagonal from T_1. Then the pair (T_1, T) is realizable.*

Remark 4. Let all the vertices $V_0, V_1, \ldots, V_k$ of the polygon F of Theorem 2 follow each other on ∂F in this order. Put $T = \{V_0V_2, V_0V_3, \ldots, V_0V_{k-1}\}$. Suppose a diagonal triangulation T_1 has no common diagonals with T. Let us show that Theorem 2 implies realizability of the pair (T_1, T). (This particular case of guaranteed realizability was known to Guibas prior to this paper.) Consider a triangle of the triangulation T_1 with the vertex V_0. None of the sides of this triangle coming from V_0 can be a diagonal of F. Hence, they are V_0V_1 and V_0V_k. Therefore $T_1 \ni V_1V_k$. One can check easily that T is generated by V_1V_k. By Theorem 2, the pair (T_1, T) is realizable.

Remark 5. To test realizability of a pair (T_1, T_2) by Theorem 2, one might first inspect if there is any "special" diagonal in the triangulation T_1 or T_2 that crosses all the diagonals of the other triangulation in int F. If not, then Theorem 2 cannot be applied. (This is always the case when each of T_1, T_2 contains a tripple of diagonals forming a triangle.) If yes, then one can check whether or not one of the "special" diagonals generates the triangulation skipping this diagonal. If not, the test is inconclusive. If yes, the pair (T_1, T_2) is realizable by Theorem 2.

To prove Theorem 2 we will need the following

Lemma. *Let F be as in Theorem 2 and T_1 be its diagonal triangulation. Then there exists a diagonal triangulation T_2 of F such that the pair (T_1, T_2) is realizable.*

The Lemma will be proved at the end of the paper.

Remark 6. Let F_* be a realization of (T_1, T_2) and $L : E^3 \to E^3$ be an invertible linear transformation that maps each directed vertical line into itself. Obviously, $L(F_*)$ is also a realization of (T_1, T_2). One can check easily that, for any three vertices of F, any (non–vertical) plane α in E^3 and any $\varepsilon > 0$, the transformation L can be chosen so that the corresponding three vertices of $L(F_*)$ lie in α and each other vertex of $L(F_*)$ is at a distance $< \varepsilon$ from α.

Proof of Theorem 2.
A. Let ab be a diagonal from T_1 that generates T. We assume that F lies in the xy–plane of E^3 and ab lies on the x–axis having its direction. We use the notations P_i, Q_j, u_i, v_j introduced before Remark 2 and $J, F_*(J)$ mentioned in the beginning of the paper.

Let J be the $(m+n)$–vector whose components $d(P_i), d(Q_j) \geq 0$ corresponding to the vertices P_i, Q_j are the distances from P_i, Q_j respectively to the x–axis. In particular

$d(P_0) = d(Q_0) = d(P_m) = d(Q_n) = 0$. The vertices of $F_*(J)$ corresponding to P_i, Q_j will be denoted by p_i, q_j.

We construct a desired realization of the pair (T_1, T) in two stages. The polytope $F_* = F_*(J)$ is the first stage. As it was explained in the beginning, its boundary ∂F_* is split by the polygonal line $\Gamma = ap_1 \ldots p_{m-1} b q_{n-1} q_{n-2} \ldots q_1 a$ into the bottom F' and the top F''. The bottom F' consists obviously of the flat convex polygons $M_p = ap_1 \ldots p_{m-1} b$ and $M_q = aq_1 \ldots q_{n-1} b$:

$$F' = M_p \cup M_q. \tag{7}$$

It has only one interior edge $ab = M_p \cap M_q$.

Denote by E the set of all interior edges of the top F''. Since F'' has no vertices in $F'' \setminus \Gamma$, each edge from E connects a pair of vertices of Γ. Those vertices however cannot be both p's or both q's because the appropriate edge would lie then in the bottom. Thus each edge from E is necessarily of the form $p_i q_j$, $i \in \{1, \ldots, m-1\}$, $j \in \{1, \ldots, n-1\}$.

B. Denote by α, β, and γ the planes of the polygons M_p, M_q, and the triangle ap_1q_1, respectively. If γ does not support F_*, then γ separates one of the vertices $p_2, p_3, \ldots, p_{m-1}$, $q_2, q_3, \ldots, q_{n-1}$ from b. Let such a vertex be p_i. Then the line $\gamma \cap \alpha$ containing the side ap_1 of M_p separates its vertices b and p_i contrary to convexity of M_p. Hence γ supports F_* and the triangle $ap_1q_1 \subset F''$.

If p_1q_1 is not an edge of F_*, then γ should contain a vertex of F_*, say q_j, different from a, p_1, q_1. Then a, q_1, q_j are collinear and hence a, Q_1, Q_j are collinear contrary to the assumption about F. Thus $p_1q_1 \in E$ and the triangle ap_1q_1 is a (complete) face of F''. Similarly one can see that $p_{m-1}q_{n-1} \in E$ and the triangle $bp_{m-1}q_{n-1}$ is a face of F''. In summary,

$$p_1q_1 \in E, \;\; p_{m-1}q_{n-1} \in E. \tag{8}$$

C. We show now that

(*) each face f of F'' other than ap_1q_1 and $bp_{m-1}q_{n-1}$ is either a triangle of one of the two forms $p_iq_jq_{j+1}$, $p_iq_jp_{i+1}$ or a quadrangle of the form $p_iq_jq_{j+1}p_{i+1}$. Here $i \in \{1, \ldots, m-1\}$, $j \in \{1, \ldots, n-1\}$ and

$$\text{either } i \neq m-1 \text{ or } j \neq n-1. \tag{9}$$

The quadrangle $p_iq_jq_{j+1}p_{i+1}$ does not degenerate into the triangle $p_{m-1}q_{n-1}b$ due to (9). Denote by δ the plane of f. If f contains more than two p's, then these p's lie on the line $\delta \cap \alpha$. Accordingly, their projections P's are collinear, contrary to our assumption about the polygon F. Thus f contains not more than two p's and similarly not more than two q's. Obviously f contains at least one p and at least one q: otherwise δ would coincide with α or β. If there are two p's in f, they should be neighbouring vertices of M_p: otherwise the supporting plane δ would cut int M_p. Similarly, two q's are neighbours. This means that the face f is as described in (*).

D. Each edge $p_iq_j \in E$ splits F'' into two polyhedral surfaces: F''_a containing a and F''_b containing b. As one augments F''_a by the face adjacent to it along p_iq_j, F''_a becomes one face bigger and F''_b one face smaller. Due to (*), the new F''_a and F''_b are separated by another edge from E which will be called the next to p_iq_j. Clearly, each edge from E will be involved in such a process that starts from p_1q_1. Thus E becomes linearly ordered set with p_1q_1 being the first and $p_{m-1}q_{n-1}$ the last element. A counting similar to (6) shows that

$$|E| \leq m + n - 3. \tag{10}$$

E. We show now that

(•) the ordered set T (see Remark 2) consists of projections of all edges from E onto the xy–plane. Moreover, the mapping $\pi : E \to T$ determined by the projection is an order isomorphism.

Denote by T_k, E_k the subsets of T and E respectively containing k first elements each. Note that T_k is defined for any $k \in \{1, \cdots, m+n-3\}$, see(6), and $T_{m+n-3} = T$. The set E_1 is also defined, see (8). Due to (5) and (8), the statement (•) holds for $k = 1$, i.e. with T and E replaced by T_1 and E_1.

Let us show that if (•) holds for some $k \in \{1, \cdots, m+n-4\}$, then E_{k+1} is defined and (•) holds for $k+1$. Let the k–th element in E_k be an edge p_iq_j. By (•), its projection $\pi(p_iq_j) = P_iQ_j$ lies in T_k and is the last k–th element there. According to the property (α), one may assume for instance that

$$u_i >_R v_j. \tag{11}$$

Then the $(k+1)$–st element in T is Q_jP_{i+1}. Note that condition (9) holds since otherwise $k = m+n-3 > m+n-4$. Therefore p_iq_j is not the last element $p_{m-1}q_{n-1}$ in E and the $(k+1)$–st element exists. Due to (*), it is either q_jp_{i+1}, or p_iq_{j+1}, or $p_{i+1}q_{j+1}$.

In the last case, the face f of F'' adjacent to F''_a is the quadrangle $p_iq_jq_{j+1}p_{i+1}$. Denote by σ_p and σ_q the vertical planes of the trapezoids $p_iP_iP_{i+1}p_{i+1}$, and $q_jQ_jQ_{j+1}q_{j+1}$ respectively and let σ be the xy–plane. Obviously

$$\sigma \cap \sigma_p \cap \alpha = u_i, \;\; \sigma \cap \sigma_q \cap \beta = v_j, \tag{12}$$

see the definition preceding Remark 2. Since the line $\sigma_p \cap \alpha$ containing the segment p_ip_{i+1} lies in the plane δ of f, the point $u_i \in \delta$ due to (12). (If $u_i = \omega$, this means that δ is parallel to the x–axis.) Similarly $v_j \in \delta$. Since $u_i \neq_R v_j$, see (11), δ should contain the x–axis which is clearly impossible. Thus the case of $p_{i+1}q_{j+1}$ is ruled out.

Consider now the case of p_iq_{j+1}. Then the face f of F'' adjacent to F''_a is the triangle $p_iq_jq_{j+1}$. Denote by ξ the plane of f. Due to (12), the line $\sigma_q \cap \beta$ that passes through q_j and q_{j+1} contains v_j. (See Fig.4 for $j = 1$.) Hence $v_j \in \xi$. Since $v_j \in \alpha$, the line

$$\xi \cap \alpha \ni v_j. \text{ Obviously } \xi \cap \alpha \ni p_i. \tag{13}$$

By (12), the line $\sigma_p \cap \alpha$ containing the side p_ip_{i+1} of M_p satisfies

$$\sigma_p \cap \alpha \ni u_i. \tag{14}$$

A simple consiedration in the plane α based on relations (13), (14), and (11) shows that the line $\xi \cap \alpha$ crosses int M_p. Then the supporting plane ξ crosses int M_p which is impossible.

The only remaining possbility is that the $(k+1)$–st element in E is q_jp_{i+1}. Then E_{k+1} is defined and (•) clearly holds for $k+1$. By induction, E_{m+n-3} is defined and (•) holds for $k = m+n-3$. Due to (10), $E_{m+n-3} = E$. Thus (•) has been proved.

Figure 4

F. Note that sufficiently small vertical variations of the vertices of F_* do not change the combinatorial structure of the top F'' and the relation

$$\pi(E) = T \tag{15}$$

implied by ($\bullet$) still holds. The faces M_p and M_q of the bottom F', see (7), can break however into smaller pieces. At the second stage of our construction of a desired realization of (T_1, T), we choose such variations as to break F' into the appropriate triangles.

The diagonal $ab \in T_1$ crosses F into nondegenerate convex polygons F_p and F_q which are projections of M_p and M_q respectively. We first consider the case when none of them is a triangle. Each other diagonal from T_1 crosses either int F_p or int F_q but not both. Let $T_{1p}\,(T_{1q})$ be the collection of those diagonals from T_1 which cross int F_p (int F_q). Thus

$$T_1 = T_{1p} \cup T_{1q} \cup \{ab\}. \text{ (Disjoint union.)} \tag{16}$$

By the Lemma, there exists a realization F_{p*} of the pair (T_{1p}, T_{2p}) where T_{2p} is some other triangulation of F_p. The vertex of F_{p*} corresponding to a vertex P_i of F_p will be denoted by $\tilde{p}_i$. By Remark 6, for any $\varepsilon > 0$, the polytope F_{p*} can be chosen such that

$$\tilde{p}_0 = a,\ \tilde{p}_m = b,\ \tilde{p}_1 = p_1,\ |\tilde{p}_i - p_i| < \varepsilon\sqrt{2},\ i \in \{2, 3, \ldots, m-1\}. \tag{17}$$

A similar construction carried out for the polygon F_q yields a realization F_{q*} of a pair (T_{1q}, T_{2q}) whose vertices $\tilde{q}_j$ satisfy

$$\tilde{q}_0 = a,\ \tilde{q}_n = b,\ \tilde{q}_1 = q_1,\ |\tilde{q}_j - q_j| < \varepsilon\sqrt{2},\ j \in \{2, 3, \ldots, n-1\}. \tag{18}$$

Note that the vertices $\tilde{p}_i$ and $\tilde{q}_j$ in (17) and (18) depend on ε.

G. We prove now that the polytope

$$F_*(\varepsilon) = \mathrm{conv}\{a, b, \tilde{p}_1, \dots, \tilde{p}_{m-1}, \tilde{q}_1, \dots, \tilde{q}_{n-1}\} \tag{19}$$

is a desired realization of (T_1, T) for a sufficiently small ε. According to (15), the projections of the interior edges of the top of $F_*(\varepsilon)$ form the triangulation T.

Consider the vertical line directed upward and passing through a point $A \in F_p$. Denote by $a_1\,(a_2)$ the point at which this line meets $F_*(\varepsilon)$ (F_{p*}) for the first time. Let z_1 and z_2 be the z–coordinates of a_1 and a_2. Obviously $F_{p*} = \mathrm{conv}\{a, b, \tilde{p}_1, \dots, \tilde{p}_{m-1}\}$ lies in $F_*(\varepsilon)$ due to (19). Therefore $z_1 \leq z_2$. If $z_1 < z_2$, then a_1 can be separated from F_{p*} by a non–vertical plane $\alpha(\varepsilon)$ such that

$$a_1 \text{ is under and } F_{p*} \text{ above } \alpha(\varepsilon). \tag{20}$$

Due to (17), one may choose $\alpha(\varepsilon)$ such that it approaches α as $\varepsilon \to 0$. Since the vertices q_i, $i = 1, 2, \dots, n-1$, are above α, see Fig.4, and due to (18), the points $\tilde{q}_i$ are above $\alpha(\varepsilon)$ for sufficiently small ε. Therefore and by (20), all vertices of $F_*(\varepsilon)$ are above $\alpha(\varepsilon)$. Then $a_1 \in F_*(\varepsilon)$ should be above $\alpha(\varepsilon)$ contrary to (20). Thus $z_1 = z_2$ and $a_1 = a_2$.

The equality $a_1 = a_2$ means that the bottom F'_p of F_{p*} lies in the bottom $F'(\varepsilon)$ of $F_*(\varepsilon)$. Similarly $F'_q \subset F'(\varepsilon)$. Obviously

$$F'(\varepsilon) = F'_p \cup F'_q. \tag{21}$$

Due to (17) and (18), the segment $ab = F'_p \cap F'_q$ is an edge of $F'(\varepsilon)$ when ε is small enough. By (21), the set of the interior edges of the bottom $F'(\varepsilon)$ is the union of interior edges from F'_p, F'_q and the edge ab. Due to (16), the polytope $F_*(\varepsilon)$ is a desired realization of the pair (T_1, T).

H. We now come back to the case skipped in part F when M_p or M_q is a triangle. If both are, then necessarily $T_1 = \{ab\}$, $T = \{P_1Q_1\}$ and a desired realization is the tetrahedron abp_1q_1. If only, say, M_q is a triangle, then repeat the construction above replacing F_{q*} and F'_q by M_q. In this process, (16) turns into $T_1 = T_{1p} \cup \{ab\}$, and (21) into $F'(\varepsilon) = F'_p \cup M_q$. The desired realization (19) becomes then $F_*(\varepsilon) = \mathrm{conv}\{a, b, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{m-1}, q_1\}$. $\square$

Proof of the Lemma. We use induction by the number n of the vertices of F. The Lemma is obvious for $n = 4$. Suppose it holds for some $n \geq 4$ and let F have $n+1$ vertices. Among the triangles of the triangulation T_1, one can easily find a triangle ABC whose sides CA and CB lie in ∂F. Cutting this triangle from F along its side $AB \in T_1$, one gets a polygon f with n vertices and a triangulation $t_1 = T_1 \setminus \{AB\}$. By the Lemma, there exists a triangulation t_2 of f such that the pair (t_1, t_2) has a realization f_*. Let ab be the edge of f_* having the projection AB. The bottom f' of f_* has exactly one face containing ab. Denote by δ the plane of that face.

Take now a point c that has projection C and lies above δ. Put $F_* = \mathrm{conv}\{c, f_*\}$. Each vertical line directed upward and crossing F meets F_* for the first time either in the triangle abc or at a point $m \in f'$ above δ. Hence the bottom $F' = abc \cup f'$ and ab

is an interior edge of F'. By our construction, the projections of the interior edges of F' form the triangulation T_1. The projections of the interior edges of the top F'' of F_* do not necessarily form any triangulation T_2 of F because some faces of F'' can contain more than three vertices. This problem however can be easily corrected by a sufficiently small vertical variation of the vertices of F_*. After that, F_* becomes a desired realization of (T_1, T_2). □

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