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Abstract

de León, Oubiña, Salgado: Characterization of Higher-order Tangent Bundles

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T.angent bundles of higher order are the framework on which the higher order Lagrangian theories are developed. Moreover, almost tangent manifolds of higher order have been shown to be relevant in singular Lagrangian theories. The purpose of this paper is to prove that under some global hypotheses a higher order integrable almost tangent manifold is diffeomorphic to the tangent bundle of the same order of some manifold. An application to the reduction of singular Lagrangians is given.
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1. Introduction

As we know, higher order tangent bundles are the natural framework on which the dynamics of Lagrangian systems of higher order is developed. In fact, if $L : T^kQ \longrightarrow \mathbb{R}$ is a Lagrangian function of order k, where Q is the configuration manifold, then the dynamics is characterized by a suitable vector field X defined on the tangent bundle $T^{2k-1}Q$ of order $2k-1$. In fact, we can define from L the symplectic Poincaré–Cartan form ω_L (supposed L to be regular) and the energy E_L associated to L. Then there exists a unique vector field ξ_L on $T^{2k-1}Q$ such that $i_{\xi_L}\omega_L = dE_L$. The vector field ξ_L is a $2k$–th order differential equation such that the projections of its integral curves onto Q are the solutions of the Euler–Lagrange equations for L. These results give a global symplectic formulation for higher order mechanics (see [11, 12]); for the corresponding local formulation see [20], for instance.

If L is singular then ω_L is not symplectic (we say that ω_L is presymplectic) and thus the equation $i_X\omega_L = dE_L$ has no solution in general, and even if a solution exists it will be neither

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unique nor a $2k$–th order differential equation. In [9], de León, Giraldo & Rodrigues have proved that, under some hypotheses on L, the space $T^{2k-1}Q$ can be reduced to an integrable almost tangent manifold of the same order on which there exists a local regular Lagrangian gauge equivalent to the original one. Then the two Lagrangians contain the same dynamical information. To do this we project all the geometric ingredients of $T^{2k-1}Q$ onto the quotient manifold $\hat{T}^{2k-1}Q = T^{2k-1}Q/\ \mathrm{Ker}\ \omega_L$. The projected canonical almost tangent structure of $T^{2k-1}Q$ gives an integrable almost tangent structure on $\hat{T}^{2k-1}Q$. Since de León & Rodrigues [14] have proved that the integrable almost tangent manifolds are sufficient to derive the Euler-Lagrange equations, then we obtain a local regular Lagrangian gauge equivalent to L (see [1, 2] for the order one).

From this discussion we see that the study of integrable almost tangent manifolds of higher order is relevant for Mechanics. Particularly, an interesting problem is to characterize when an integrable higher order almost tangent manifold is a higher order tangent bundle of some manifold. The local equivalence is given from the integrability (see [4, 8] for instance). The aim of this paper is to obtain a global result. In fact, we prove that if (M, J) is an integrable almost tangent manifold of order k satisfying some global hypothesis, then M is diffeomorphic to the tangent bundle T^kS of order k of some manifold S. Moreover, the diffeomorphism may be choosen in such a way that the almost tangent structure J on M is transported to the canonical almost tangent structure of T^kS. This result generalizes those of Crampin & Thompson [3] and Thompson & Schwardmann [17] for the order one.

For the order one, an early and different approach is due to Nagano [15] in the case of the characterization of cotangent bundles. In fact, Nagano has proved that if M is a differentiable manifold endowed with a vector field satisfying the same properties of those satisfied by the canonical vector field of a vector bundle, then there exists a unique vector bundle structure on M over the singular submanifold S of the vector field. If, moreover, M is an exact symplectic manifold then M is isomorphic to the cotangent bundle T^*S, indeed as symplectic vector bundles. In [10] we have used the ideas of Nagano to give a characterization of tangent bundles. In fact, we prove that if M is an integrable almost tangent manifold endowed with a vector field C which satisfies the properties of the Liouville vector field of a tangent bundle, then there exists a unique vector bundle structure on M isomorphic to TS (S being the singular submanifold) and such that the Liouville vector field and the canonical almost tangent structure of TS are transported via the isomorphism to C and the almost tangent structure on M, respectively. This result was obtained independently by Filippo, Landi, Marmo & Vilasi [5], by using a different procedure. In [10] we have proved that the reduced manifold $\hat{T}Q = TQ/\ker\omega_L$ is in fact a *bona fide* tangent bundle. The same is true in the higher order case. In fact, if $L : T^kQ \to \mathbb{R}$ is a singular Lagrangian of order k admitting a global dynamics, then the projected almost tangent manifold of order $2k-1$, $\hat{T}^{2k-1}Q = T^{2k-1}Q/\ker\omega_L$, is a tangent bundle of order $2k-1$ over the singular submanifold defined by the projected higher order Liouville vector field.

The paper is structured as follows. In §2 we recall the notations and main results which will be used in the sequel. In §3, we obtain the characterization of tangent bundles of order 2. This result is extended to the arbitrary higher order case in §4. In §5 we give an alternative characterization by using the ideas of Nagano and Filippo et al. By using the later characterization, in §6 we prove that $\hat{T}^{2k-1}Q$ is a true higher tangent bundle when L

admits a global dynamics.

2. Background

Let Q be a differentiable manifold of dimension m. The tangent bundle of order k of Q is the $(k+1)m$–dimensional manifold T^kQ whose points are k–jets at $0 \in \mathbb{R}$ of differentiable curves in Q. If $k = 0$ we identify T^0Q with Q and, if $k = 1$, $T^1Q = TQ$ is the tangent bundle of Q. If $r < k$ we denote by $\rho^k_r : T^kQ \to T^rQ$ the canonical projection given by

$$\rho^k_r(j^k_0\sigma) = j^r_0\sigma,$$

for all $j^k_0\sigma \in T^kQ$.

We shall now describe the local coordinates in T^kQ. Let (U, q^i) be a coordinate neighborhood of Q and $\sigma : \mathbb{R} \to Q$ a curve in Q such that $\sigma(0) \in U$. Then the k-jet $j^k_0\sigma$ is represented in T^kU by $(z^i_0, z^i_1, \cdots, z^i_k)$ where $z^i_r = \frac{1}{r!}(d^r/dt^r)_{|t=0}(q^i \circ \sigma)$, $1 \leq r \leq k$, $1 \leq i \leq m$.

We denote by J_1 the canonical tensor field of type (1, 1) on T^kQ locally given by

$$J_1 = \sum_{r=0}^{k} \frac{\partial}{\partial z^i_{r+1}} \otimes dz^i_r \; . \tag{1}$$

As $J_1^{k+1} = 0$ and $\operatorname{rank} J_1 = km$, the tensor field J_1 defines a *canonical almost tangent structure of order* k on T^kQ (see [12, 4, 8, 21]). For each r, $2 \leq r \leq k$, we set $J_r = {J_1}^r$, the composition of J_1 r-times.

Also, let C_1 be the canonical vector field on T^kQ which is locally given by

$$C_1 = \sum_{r=1}^{k} r\, z^i_r \frac{\partial}{\partial z^i_r} \; ; \tag{2}$$

C_1 is called the *Liouville vector field* of T^kQ and it is the infinitesimal generator of the positive homothetias of T^kQ. A direct computation shows that

$$L_{C_1} J_1 = -J_1 \; , \qquad J_k C_1 = 0 \; .$$

We set $C_r = J_{r-1}C_1$, $2 \leq r \leq k$.

As we know [6, 16] $\rho^k_{k-1} : T^kQ \to T^{k-1}Q$ is an affine bundle modelled on the vector bundle $(\rho^{k-1}_0)^*(TQ) = T^{k-1}Q \times_Q TQ$, i.e., the induced vector bundle defined by the following commutative diagram:

$$\begin{tikzcd}
T^{k-1}Q \times_Q TQ \arrow[r] \arrow[d] & TQ \arrow[d, "\rho^1_0"] \\
T^{k-1}Q \arrow[r, "\rho^{k-1}_0"] & Q
\end{tikzcd}$$

We know that this vector bundle is canonically isomorphic to the vertical vector bundle $\operatorname{Ker} T\rho^{k-1}_{k-2} = V^{\rho^{k-1}_{k-2}}(T^{k-1}Q)$ (see [12]).

3. Characterization of tangent bundles of order 2

Let (M, J) be an almost tangent manifold of order 2 with almost tangent structure J, i.e., M is a $3n$–dimensional manifold and J is a tensor field of type (1,1) and of constant rank $2n$ such that $J^3 = 0$. Then we have

$$\mathrm{Im}\ J^2 \subset \mathrm{Im}\ J\ , \qquad \mathrm{Ker}\ J \subset \mathrm{Ker}\ J^2\ ,$$

$$\mathrm{Im}\ J^2 = \mathrm{Ker}\ J\ , \qquad \mathrm{Im}\ J = \mathrm{Ker}\ J^2\ .$$

We assume that J is integrable. Then the Nijenhuis tensor of J vanishes identically and hence $\mathrm{Im}\ J$ and $\mathrm{Im}\ J^2$ are integrable distributions. Consider the spaces of leaves $M_0 = M/\mathrm{Im}\ J$ and $M_1 = M/\mathrm{Im}\ J^2$ determined by the corresponding foliations, also denoted by $\mathrm{Im}\ J$ and $\mathrm{Im}\ J^2$. We obtain the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\pi_1^2"] \arrow[d, "\pi_0^2"'] & M_1 \arrow[dl, "\pi_0^1"] \\ M_0 & \end{tikzcd}$$

where $\pi_0^2 : M \to M_0$ and $\pi_1^2 : M \to M_1$ are the canonical projections. We denote by $\pi_0^1 : M_1 \to M_0$ the induced projection which maps the leaf of $\mathrm{Im}\ J^2$ passing through a point $x \in M$ onto the leaf of $\mathrm{Im}\ J$ passing through x.

We shall consider an example of manifold M with an integrable almost tangent structure of order 2 such that the quotient space $M_0 = M/\operatorname{Im} J$ is a quotient manifold of M but not so is $M_1 = M/\operatorname{Im} J^2$. Let M be the 3–torus $\mathbb{T}^3$ obtained as the quotient manifold of $\mathbb{R}^3$ by the integral lattice $\mathbb{Z}^3$. Let (x, y, z) be the natural coordinates of $\mathbb{R}^3$, which may be regarded as local coordinates on $\mathbb{T}^3$. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then

$$J = \frac{\partial}{\partial x} \otimes dx - \frac{1}{\alpha}\frac{\partial}{\partial x} \otimes dy + \alpha\frac{\partial}{\partial y} \otimes dx - \frac{\partial}{\partial y} \otimes dy + \frac{\partial}{\partial x} \otimes dz$$

determines an integrable almost tangent structure of order 2 on $\mathbb{T}^3$. Moreover, $J(\partial/\partial x) = X$, $J(\partial/\partial y) = (-1/\alpha)X$ and $J(\partial/\partial z) = \partial/\partial x$, where X is the vector field on $\mathbb{T}^3$ given by $X = \partial/\partial x + \alpha\,\partial/\partial y$. Then $\operatorname{Im} J = \langle \partial/\partial x, \partial/\partial y\rangle = \langle X, \partial/\partial x\rangle$ and $\operatorname{Im} J^2 = \langle X\rangle$. If α is irrational, each leaf of $\operatorname{Im} J^2$ is dense in $\mathbb{S}^1 \times \mathbb{S}^1 \times \{y\}$, for some $y \in \mathbb{S}^1$, and so the space $\mathbb{T}^3/\operatorname{Im} J^2$ is not a quotient manifold of $\mathbb{T}^3$.

Now, we return to the general situation and suppose that $\mathrm{Im}\ J$ and $\mathrm{Im}\ J^2$ define fibrations, i.e., there exist differentiable structures on M_0 and M_1, respectively such that $\pi_0^2 : M \to M_0$ and $\pi_1^2 : M \to M_1$ are submersions. In other words, we are assuming that M_0 and M_1 have the structure of quotient manifolds. Then M_1 and M_0 have dimension $2n$ and n, respectively. Further, $\pi_0^1 : M_1 \to M_0$ is a differentiable submersion and M_0 is also a quotient manifold of M.

Proposition 3.1 *J projects onto an integrable almost tangent structure J_1 on M_1.*

Proof. Let $X_1 \in T_{x_1}M_1$, $x_1 \in M_1$. Then we define

$$(J_1)_{x_1}X_1 = T\pi_1^2(JX), \tag{3}$$

where $X \in T_xM$ is such that $\pi_1^2(x) = x_1$, $T\pi_1^2 X = X_1$.

We shall prove that (3) does not depend on the choice of x and X. In fact, if $Y \in T_xM$ projects also onto X_1, then we have $T\pi_1^2(Y - X) = 0$ which implies $Y - X \in \operatorname{Im} J^2$ and then $JY = JX$. On the other hand, (3) does not depend on the choice of the point x. In fact it is sufficient to check (see [1]) that $\operatorname{Im}(L_X J) \subset \operatorname{Im} J^2$, for all $X \in \operatorname{Im} J^2$, which is a matter of routine.

Finally, we have

$$J_1^2(X_1) = J_1(T\pi_1^2(JX)) = (T\pi_1^2(J^2X)) = 0\ ,$$

and hence $J_1^2 = 0$. Thus, $\operatorname{Im} J_1 \subset \operatorname{Ker} J_1$. On the other hand, let $X_1 \in \operatorname{Ker} J_1$. Then $X_1 = T\pi_1^2 X$, where $JX \in \operatorname{Im} J^2$. Therefore there exists a tangent vector Y on M such that $JX = J^2Y$. Then $X - JY \in \operatorname{Ker} J = \operatorname{Im} J^2 = \ker T\pi_1^2$. Hence

$$X_1 = T\pi_1^2(X) = T\pi_1^2(JY) = J_1(T\pi_1^2(JY)) \in \operatorname{Im} J_1.$$

Since $\operatorname{Ker} J_1 = \operatorname{Im} J_1$ we deduce that J_1 is an almost tangent structure on M_1. The integrability of J_1 follows since the Nijenhuis tensor of J projects onto the Nijenhuis tensor of J_1. $\square$

From Proposition 3.1 we deduce that (M_1, J_1) is an integrable almost tangent manifold. Moreover, it defines a fibration. In fact, since $\operatorname{Ker}(T\pi_0^1) = \operatorname{Im} J_1$, the space of leaves $M_1/\operatorname{Im} J_1$ may be canonically identified to $M/\operatorname{Im} J$ and then it possesses a differentiable structure of quotient manifold in such a way that the canonical projection $\pi : M_1 \to M_1/\operatorname{Im} J_1$ is identified to the projection π_0^1. Then we obtain the following commutative diagram:

$$\begin{tikzcd}
M \arrow[rd, "\pi_1^2"] \arrow[dd, "\pi_0^2"'] & \\
& M_1 \arrow[ld, "\pi_0^1 \equiv \pi"] \\
M_0 \equiv M_1/\operatorname{Im} J_1 &
\end{tikzcd}$$

Since (M_1, J_1) is integrable then there exists a linear symmetric connection ∇_1 on M_1 such that $\nabla_1 J_1 = 0$. As we know (see [3]) ∇_1 induces a flat connection on the fibres of π_0^1. According to Crampin & Thompson [3] we have the following

Proposition 3.2 *Suppose that each leaf of the foliation defined by* $\mathrm{Im}\, J_1$ *is simply connected and geodesically complete with respect to the flat connection induced on it by* ∇_1. *Then* $\pi_0^1 : M_1 \to M_0$ *is an affine bundle modelled on* TM_0. *Moreover, if* $M_1 \to M_0$ *admits a global section then* M_1 *is isomorphic as a vector bundle to* TM_0, *this isomorphism depending on the choice of the section.*

For each π_0^1-vertical tangent vector $X \in T_{x_1}M_1$ we define its *vertical lift* X^v to T_xM, $\pi_1^2(x) = x_1$, as follows:

$$X^v = J\tilde{X} ,$$

where $\tilde{X} \in T_xM$ is such that $T\pi_1^2\tilde{X} = X$. Then we have

$$T\pi_1^2 X^v = T\pi_1^2 J(\tilde{X}) = J_1(X) = 0 ,$$

since $T\pi_0^1 X = 0$ implies that $X \in \mathrm{Ker}\, T\pi_0^1 = \mathrm{Im}\, J_1 = \mathrm{Ker}\, J_1$. Thus, X^v is π_1^2-vertical. Moreover, every π_1^2–vertical vector is a vertical lift.

Remark 3.1 We can consider a differentiable subbundle S of TM complementary to $\mathrm{Ker}\, T\pi_1^2 = \mathrm{Im}\, J^2$ with respect to a Riemannian metric. Then, if X is a vector field on M_1, there exists a unique vector field $\tilde{X}$ on M π_1^2–related to X such that $\tilde{X}_x \in S_x$ for all $x \in M$. Since S is a differentiable subbundle of TM, $\tilde{X}$ is a differentiable vector field on M. If the vector field X is π_0^1–vertical then $X^v = J\tilde{X}$ is a π_1^2–vertical differentiable vector field. Further, the vector field $\tilde{X}$ is π_0^2–vertical and then $\tilde{X} \in \mathrm{Im}\, J$.

Lemma 3.1 *We have:*

$$[X^v, Y^v] = 0 ,$$

for all π_0^1*-vertical vector fields* X *and* Y *on* M_1.

Proof. In fact, we have

$$[X^v, Y^v] = [J\tilde{X}, J\tilde{Y}] = J[J\tilde{X}, \tilde{Y}] + J[\tilde{X}, J\tilde{Y}] = J[X^v, \tilde{Y}] + J[\tilde{X}, Y^v] .$$

But $T\pi_1^2[X^v, \tilde{Y}] = [T\pi_1^2 X^v, T\pi_1^2\tilde{Y}] = [0, Y] = 0$. Also, $T\pi_1^2[\tilde{X}, Y^v] = 0$. Hence $[X^v, \tilde{Y}]$ and $[\tilde{X}, Y^v]$ belong to $\mathrm{Ker}\, T\pi_1^2 = \mathrm{Im}\, J^2$, which implies the result. □

Since J is integrable, then there exists a symmetric linear connection ∇ on M such that $\nabla J = 0$.

Proposition 3.3 *The connection* ∇ *induces a flat connection on each fibre of the fibration* $\pi_1^2 : M \to M_1$.

Proof. From $\nabla J = 0$ and the fact that ∇ is symmetric, it follows

$$\begin{aligned} \nabla_{X^v} Y^v &= \nabla_{X^v} J\tilde{Y} = J(\nabla_{X^v} \tilde{Y}) = J(\nabla_{\tilde{Y}} X^v) + [X^v, \tilde{Y}]) \\ &= J(\nabla_{\tilde{Y}} X^v) = J(\nabla_{\tilde{Y}} J\tilde{X}) = \nabla_{\tilde{Y}} J^2\tilde{X} = 0 , \end{aligned}$$

since $\tilde{X} \in \mathrm{Im}\, J$. Now, the vertical lifts X^v span the vertical bundle $\mathrm{Ker}\, T\pi_1^2$. Then we have the result. □

Consider the vector bundle $V(M_1) \to M_1$ consisting of the tangent vectors to M_1 which are vertical with respect to the fibration $\pi_0^1 : M_1 \to M_0$.

Proposition 3.4 *Suppose that each leaf of the foliation defined by* $\operatorname{Im} J^2$ *is simply connected and geodesically complete with respect to the flat connection induced on it by* ∇. *Then* $\pi_1^2 : M \to M_1$ *is an affine bundle modelled on* $V(M_1)$.

Proof. We define a morphism

$$\Phi : M \times_{M_1} V(M_1) \to M,$$

as follows. Let $x \in M$, $X \in V(M_1)$ such that $X \in T_{x_1}M_1$, $x_1 = \pi_1^2(x)$, $T\pi_0^1(X) = 0$. Consider the vertical lift X^v of X as a vector field on the fibre $(\pi_1^2)^{-1}(x_1)$.

By Proposition 3.3, we have $\nabla_{X^v} X^v = 0$ and hence X^v is a geodesic vector field. From the assumptions of completeness of the fibres, we deduce that X^v is complete. Denote by $\phi_X(t, x)$ the integral curve of X^v on $(\pi_1^2)^{-1}(x_1)$ such that $\phi_X(0, x) = x$. Now we define

$$\Phi_{x_1}(x, X) = \phi_X(1, x) ,$$

and we have a commutative diagram

$$\begin{tikzcd} M \times_{M_1} V(M_1) \arrow[rr, "\Phi"] \arrow[dr] & & M \\ & M_1 \arrow[ur] & \end{tikzcd}$$

Only rest to prove that Φ_{x_1} defines a transitive, free action of $V(M_1)_{x_1}$ on $(\pi_1^2)^{-1}(x_1)$. To do this we proceed as in Crampin & Thompson [3, pp. 69–70] (see also [13]). □

Theorem 3.1 *Suppose that:*

(1) each leaf of the foliation defined by $\operatorname{Im} J^2$ *is simply connected and geodesically complete with respect to the flat connection induced on it by* ∇;

(2) each leaf of the foliation defined by $\operatorname{Im} J_1$ *is simply connected and geodesically complete with respect to the flat connection induced on it by* ∇_1.

Then we have:

(1) $\pi_0^1 : M_1 \to M_0$ *is an affine bundle modelled on* TM_0;

(2) $\pi_1^2 : M \to M_1$ *is an affine bundle modelled on* $V(M_1)$.

Moreover, if $\pi_0^2 : M \to M_0$ *and* $\pi_1^2 : M \to M_1$ *admit global sections then* M_1 *is isomorphic as a vector bundle to* TM_0 *and* M *is diffeomorphic to* T^2M_0. *Further, if we make the identification* $M_1 \equiv TM_0$, *then the fibre bundles* $\pi_0^2 : M \to M_0$ *and* $\pi_1^2 : M \to M_1$ *may be identified to* $\rho_0^2 : T^2M_0 \to M_0$ *and* $\rho_1^2 : T^2M_0 \to TM_0$, *respectively.*

Proof. (1) and (2) are a direct consequence of Propositions 3.2 and 3.4. If the fibration $\pi_0^2 : M \to M_0$ admits a global section $s : M_0 \to M$ then $\pi_0^1 : M_1 \to M_0$ admits the global section $\pi_1^2 \circ s$. Therefore there exists a vector bundle isomorphism $\varphi : M_1 \to TM_0$. The restriction $(T\varphi)_{|V(M_1)} : V(M_1) \to V(TM_0)$ is also a vector bundle isomorphism. Moreover, if π_1^2 admits a global section, there exist a diffeomorphism $M \to V(M_1)$ over M_1. Since

$\rho_1^2 : T^2M_0 \to TM_0$ is a locally trivial bundle with fibre $\mathbb{R}^n$, there exists a global section of ρ_1^2 and so we also have a diffeomorphism $T^2M_0 \to V(TM_0)$ over TM_0. Summing up we have the following commutative diagram, where the top arrows define a diffeomorphism $M \to T^2M_0$.

$$\begin{tikzcd}
M \arrow[rr] \arrow[drr, "\pi_1^2"'] & & V(M_1) \arrow[rr, "(T\varphi)_{|V(M_1)}"] \arrow[d] & & V(TM_0) \arrow[d] & & T^2M_0 \arrow[ll] \arrow[dll, "\rho_1^2"] \\
& & M_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_0^1"'] & & TM_0 \arrow[dl, "\rho_0^1"] & & \\
& & & M_0 & & &
\end{tikzcd}$$

□

4. Characterization of tangent bundles of higher-order

In this section we shall generalize the results of the preceedent one for an arbitrary order k.

Let (M, J) be an integrable almost tangent manifold of order k with almost tangent structure J. Then M is a $(k+1)n$-dimensional manifold, $J^{k+1} = 0$ and J has constant rank kn. Further we have

$$\mathrm{Im}\ J^k \subset \mathrm{Im}\ J^{k-1} \subset \cdots \subset \mathrm{Im}\ J^2 \subset \mathrm{Im}\ J\ ,$$

where $\mathrm{Im}\ J^r$ has dimension $(k-r+1)n$. Also we have

$$\mathrm{Im}\ J^r = \mathrm{Ker}\ J^{k-r+1}\ .$$

We put $M_k = M$ and denote by $M_r = M/\mathrm{Im}\ J^{r+1}$, $0 \le r \le k-1$, the corresponding spaces of leaves and by $\pi_s^r : M_r \to M_s$ the canonical projections, $0 \le s < r \le k$.

We assume that all the spaces of leaves $M_{k-1}, M_{k-2}, \cdots, M_1, M_0$ have a structure of quotient manifold of M. Then M_r, $0 \le r \le k-1$, has dimension $(r+1)n$ and we have a family of fibrations

$$\pi_r^k : M_k \longrightarrow M_r\ , \quad 0 \le r \le k-1\ .$$

Proposition 4.1 *J induces an integrable almost tangent structure J_r of order r on M_r, for all r, $1 \le r \le k-1$.*

Proof. We define J_r by

$$J_rX = T\pi_r^k(J\tilde{X}), \quad X \in T_{x_r}M_r,\ x_r \in M_r\ , \tag{4}$$

where $\tilde{X} \in T_x M$, $x_r = \pi_r^k(x)$ and $T\pi_r^k(\tilde{X}) = X$.

Now, we shall prove that (4) is well defined. In fact, let $\tilde{X}, \tilde{Y} \in T_x M$ such that $T\pi_r^k(\tilde{X}) = T\pi_r^k(\tilde{Y})$. Then $\tilde{X} - \tilde{Y} \in \operatorname{Ker} T\pi_r^k = \operatorname{Im} J^{r+1}$, and so $J(\tilde{X} - \tilde{Y}) \in \operatorname{Im} J^{r+2} \subset \operatorname{Im} J^{r+1} = \operatorname{Ker} T\pi_r^k$, hence $T\pi_r^k(J\tilde{X}) = T\pi_r^k(J\tilde{Y})$.

On the other hand, (4) does not depend on the choice of the point x. In fact, it is sufficient to check that $\operatorname{Im}(L_X J) \subset \operatorname{Im} J^{r+1}$, for all $X \in \operatorname{Im} J^{r+1}$, which is a matter of routine.

Finally we can directly check that $\operatorname{Im} J_r = \operatorname{Ker} J_r{}^r$, then $J_r^{r+1} = 0$ and J_r has constant rank rn. The integrability of J_r follows from the integrability of J since the Nijenhuis tensor of J projects onto the Nijenhuis tensor of J_r. □

Now, since $\operatorname{Ker}(T\pi_s^r) = \operatorname{Im} J_r^{s+1}$, the quotient space $M_r / \operatorname{Im} J_r^{s+1}$ may be identified to $M / \operatorname{Im} J^{s+1} = M_s$ and then M_s possesses a differentiable structure of quotient manifold such that the canonical projection $M_r \to M_r / \operatorname{Im} J_r^{s+1}$ is identified to π_s^r, and so we have a family of fibrations $\pi_s^r : M_r \to M_s \equiv M_r / \operatorname{Im} J_r^{s+1}$.

Also, the integrable almost tangent structure $(J_r)_s$ of order s induced by J_r on $M_r / \operatorname{Im} J_r^{s+1}$ coincides with the almost tangent structure J_s of order s induced by J on $M / \operatorname{Im} J^{s+1}$.

Since J_r is integrable then there exists a symmetric linear connection ∇_r on M_r such that $\nabla_r J_r = 0$. Also, there exists a symmetric linear connection $\nabla = \nabla_k$ on $M = M_k$ such that $\nabla J = 0$.

Finally, we remark that the connection induced by ∇_r on the fibres of the fibration $\pi_{r-1}^r : M_r \to M_{r-1} = M_r / \operatorname{Im} J_r^r$, $1 \leq r \leq k$, is flat. In fact, we can define the *vertical lift* X^v of a π_{r-2}^{r-1}–vertical tangent vector $X \in T_{x_{r-1}} M_{r-1}$ to $T_{x_r} M_r$, $\pi_{r-1}^r(x_r) = x_{r-1}$, by $X^v = J_r \tilde{X}$, where $\tilde{X} \in T_{x_r} M_r$, $T\pi_{r-1}^r(\tilde{X}) = X$, and we obtain the analogous to Proposition 3.3.

Theorem 4.1 *Suppose that:*

(1) each fibre of the fibration $\pi_{r-1}^r : M_r \to M_{r-1}$, $1 \leq r \leq k$, is simply connected and geodesically complete with respect to the flat connection induced on it by ∇_r;

(2) $\pi_r^k : M \to M_r$, $0 \leq r < k$, admits a global section.

Then we have:

(1) M_1 is isomorphic as a vector bundle to TM_0;

(2) M_r is diffeomorphic to $T^r M_0$ for all r, $2 \leq r \leq k$.

Proof. First, we prove that $\pi_{r-1}^r : M \to M_{r-1}$ is an affine bundle modelled on the vector bundle $V(M_{r-1})$, where $V(M_{r-1})$ denotes the vertical vector bundle over M_{r-1} defined by the fibration $\pi_{r-2}^{r-1} : M_{r-1} \to M_{r-2}$. In other words, $V(M_{r-1}) = \operatorname{Ker} T\pi_{r-2}^{r-1}$. The proof runs in a similar way to that of Proposition 3.4.

Since there exists a global section of $\pi_0^1 : M_1 \to M_0$, we have (1). Suppose that the theorem holds for the case $r-1$. Notice that $\pi_{r-1}^r : M_r \to M_{r-1}$ admits also a global section and hence M_r is diffeomorphic to $V(M_{r-1})$ in such a way that this diffeomorphism is a fibred mapping over $M_{r-1} \equiv T^{r-1} M_0$.

Proceeding as in Theorem 3.1 we can define a diffeomorphism $M_r \to T^r M_0$ given by the following commutative diagram:

$$\begin{tikzcd}
M_r \arrow[r] \arrow[dr, "\pi^r_{r-1}"'] & V(M_{r-1}) \arrow[rr] \arrow[d] & & V(T^{r-1}M_0) \arrow[d] & T^r M_0 \arrow[l] \arrow[dl, "\rho^r_{r-1}"] \\
& M_{r-1} \arrow[rr] \arrow[dr, "\pi^{r-1}_{r-2}"'] & & T^{r-1}M_0 \arrow[dl, "\rho^{r-1}_{r-2}"] & \\
& & T^{r-2}M_0 & &
\end{tikzcd}$$

□

5. Characterization of tangent bundles of order k: an alternative approach

In this section we shall give an alternative approach to the problem of characterizing tangent bundles of second order inspired in the work of Nagano [15] and Filippo et al. [5].

Theorem 5.1 *Let M be a $(k+1)n$–dimensional manifold endowed with an integrable almost tangent structure J of order k and let C be a complete vector field on M satisfying*

$$L_C\, J = -J\,, \qquad J^k C = 0\,.$$

Suppose that for each point $x \in M$, there exists $\lim_{t\to-\infty}(\exp tC)(x)$, where $\exp tC$ denotes the flow of C. Then the set of singular points S of C is an n–dimensional submanifold of M and M is diffeomorphic to T^kS. Further M is fibred over S and there exists a diffeomorphism fibred over the identity of S in such a way that it transports the canonical almost tangent structure J_S and the canonical vector fields $C_1, C_2, \ldots, C_k$ of T^kS to J and $C, JC, \ldots, J^{k-1}C$, respectively.

Proof. Let $(z_0^i, z_1^i, \ldots, z_k^i)$ be a chart adapted to J defined on an open subset U of M, i.e.,

$$J\left(\frac{\partial}{\partial z_r^i}\right) = \frac{\partial}{\partial z_{r+1}^i}\,, \quad J\left(\frac{\partial}{\partial z_k^i}\right) = 0\,, \quad 0 \le r \le k-1, \;\; 1 \le i \le n.$$

Since $J^kC = 0$, C may be locally expressed

$$C = a_1^i \frac{\partial}{\partial z_1^i} + a_2^i \frac{\partial}{\partial z_2^i} + \ldots + a_k^i \frac{\partial}{\partial z_k^i}\,.$$

Now, from $L_C J = -J$, it follows that

$$a_r^i = r\, z_r^i + \phi_r^i(z_0^j, z_1^j, \ldots, z_{r-1}^j)\,, \quad 1 \le r \le k,$$

where ϕ^i_r are functions on U satisfying

$$\frac{\partial \phi^i_k}{\partial z^j_s} = \frac{\partial \phi^i_{k-1}}{\partial z^j_{s-1}} = \ldots = \frac{\partial \phi^i_{k-s}}{\partial z^j_0}, \quad 1 \leq s \leq k-1. \tag{5}$$

We put

$$\begin{aligned} \bar{z}^i_0 &= z^i_0, \\ \bar{z}^i_r &= z^i_r + \psi^i_r(z^j_0, z^j_1, \ldots, z^j_{r-1}), \quad 1 \leq r \leq k, \end{aligned}$$

where

$$\begin{aligned} &\psi^i_1(z^j_0) = \phi^i_1(z^j_0), \\ &\psi^i_r(z^j_0, \ldots, z^j_{r-1}) = \frac{1}{r}\left(\phi^i_r(z^j_0, \ldots, z^j_{r-1}) + \sum_{s=1}^{r-1} a^l_s \frac{\partial \psi^i_{r-s}}{\partial z^l_0}\right), \quad 2 \leq r \leq k. \end{aligned}$$

Obviously, $(\bar{z}^i_0, \bar{z}^i_1, \ldots, \bar{z}^i_k)$ is a coordinate system on U and from equations (5) it follows that it is a chart adapted to J and

$$C = \sum_{r=1}^{k} r\, \bar{z}^i_r \frac{\partial}{\partial \bar{z}^i_r}. \tag{6}$$

Now, the flow $\varphi_t = \exp tC$ of C is locally given by

$$\varphi_t(\bar{z}^i_0, \bar{z}^i_1, \ldots, \bar{z}^i_k) = (\bar{z}^i_0, e^t \bar{z}^i_1, \ldots, e^{kt} \bar{z}^i_k)$$

for sufficiently small t, from which

$$(\varphi_t)_*\left(\frac{\partial}{\partial \bar{z}^i_r}\right) = e^{rt}\frac{\partial}{\partial \bar{z}^i_r}, \quad {\varphi_t}^* d\bar{z}^i_r = e^{rt}\, d\bar{z}^i_r.$$

Also,

$$(\varphi_t)_* J = e^t J (\varphi_t)_* \tag{7}$$

for small t and hence for every $t \in \mathbb{R}$.

For each point in the set S of the singular points of C we can consider a neighborhood U with coordinates $(\bar{z}^i_0, \bar{z}^i_1, \ldots, \bar{z}^i_k)$ and so S is locally defined by the equations $\bar{z}^i_r = 0$, $1 \leq r \leq k$, which implies that S is an n–dimensional regular submanifold of M. We can suppose that the image of U by this chart is of the form $B \times B^{kn}$, where B is an open ball in $\mathbb{R}^n$ and B^{kn} is a zero centred ball in $\mathbb{R}^{kn}$ with small radius. Our purpose is to extend this chart to one in which we replace B^{kn} by $\mathbb{R}^{kn}$.

Denote by $\pi : M \to M$ the mapping $\pi(x) = \lim_{t \to -\infty} \varphi_t(x)$. Clearly, π is continuous and so $W = \{x \in M : \pi(x) \in U\}$ is an open subset of M. We shall transport the chart $(\bar{z}^i_r)$ along the integral curves of C. We first extend the forms $d\bar{z}^i_r$ on U to differential 1–forms α^i_r, $0 \leq r \leq k$, $1 \leq i \leq n$, on W. We put

$$(\alpha^i_r)_x = \frac{1}{e^{rt}} (\varphi_t)^*_x (d\bar{z}^i_r)_{\varphi_t(x)}, \tag{8}$$

where $x \in W$, $\varphi_t(x) \in U$. Equation (8) is satisfied in a neighborhood of each point of M for some $t \in \mathbb{R}$. So, the 1–forms α^i_r are closed and hence exact by Poincaré Lemma, since W is homotopically equivalent to the open ball B of $\mathbb{R}^n$. Then there exist differentiable functions $\tilde{z}^i_r$ on W such that $\alpha^i_r = d\tilde{z}^i_r$, for $0 \leq r \leq k$, $1 \leq i \leq n$, and we can choose them such that $\tilde{z}^i_{r|U} = \bar{z}^i_{r|U}$.

From (8) we obtain that $\alpha^1_0 \wedge \ldots \wedge \alpha^n_0 \wedge \alpha^1_1 \wedge \ldots \wedge \alpha^n_1 \wedge \ldots \ \ldots \wedge \alpha^1_k \wedge \ldots \wedge \alpha^n_k$ is never zero and so the functions $\tilde{z}^i_r$ are independent and define a chart on W.

Moreover, for each $x \in W$, if $\varphi_t(x) \in U$, we have

$$\left(\frac{\partial}{\partial \tilde{z}^i_r}\right)_x = e^{rt}\,(\varphi_{-t})_{*\varphi_t(x)} \left(\frac{\partial}{\partial \bar{z}^i_r}\right)_{\varphi_t(x)}, \tag{9}$$

and, since $C = (\varphi_{-t})_* C$, we get

$$C_x = \sum_{r=1}^{k} r\,\tilde{z}^i_r(x) \left(\frac{\partial}{\partial \tilde{z}^i_r}\right)_x ; \qquad x \in W.$$

By using (7) and that the chart $(\bar{z}^i_r)$ is adapted to J, it follows

$$J\left(\frac{\partial}{\partial \tilde{z}^i_r}\right) = \frac{\partial}{\partial \tilde{z}^i_{r+1}}\,, \quad J\left(\frac{\partial}{\partial \tilde{z}^i_k}\right) = 0\,, \quad 0 \leq r \leq k-1, \quad 1 \leq i \leq n.$$

The integral curves of C passing through an arbitrary point $z \in W$ are contained in W and they are given by $t \to (\tilde{z}^i_0, e^t \tilde{z}^i_1, \ldots, e^{kt}\tilde{z}^i_k)$. Then, the mapping π is locally given by $\pi(\tilde{z}^i_0, \tilde{z}^i_1, \ldots, \tilde{z}^i_k) = (\tilde{z}^i_0, 0, \ldots, 0) \in S$, and, in particular, it is a differentiable submersion.

Corresponding to a chart $(\tilde{z}^i_r)$ in a neighborhood of a point of S, we consider the induced chart $x^i_0 = \tilde{z}^i_0$, $i = 1, \ldots, n$, on S and the associated chart $(x^i_0, x^i_1, \ldots, x^i_k)$ on T^kS. Setting $F(\tilde{z}^i_0, \tilde{z}^i_1, \ldots, \tilde{z}^i_k) = (x^i_0, x^i_1, \ldots, x^i_k)$ we obtain a well defined global diffeomorphism $F : M \to T^kS$ which is fibred over the identity of S and transports the almost tangent structure J and the vector fields $C, JC, \ldots, J^{k-1}C$ of M to the canonical tangent structure J_S and the canonical vector fields C_1, C_2, …, C_k of T^kS, respectively. □

6. An application: singular Lagrangians of higher order

Let $L : T^kQ \to \mathbb{R}$ be a Lagrangian of order k over a configuration manifold Q, $\dim Q = m$. We denote by α_L, ω_L and E_L the Poincaré–Cartan 1–form, the Poincare–Cartan 2–form, and the energy function associated to L, respectively, defined by

$$\alpha_L = \sum_{r=1}^{k} (-1)^r \,\frac{1}{r!}\, d_T^{r-1} d_{J_r} L, \qquad \omega_L = -d\alpha_L,$$

$$E_L = \sum_{r=1}^{k} (-1)^r \,\frac{1}{r!}\, d_T^{r-1}(C_r L) - L,$$

where d_T is the total derivative with respect to the time ([12, 19]) and d_{J_r} is the derivation induced by J_r. Hence they are defined on the phase space of generalized velocities $T^{2k-1}Q$. The global equation of motion is

$$\imath_X \omega_L = dE_L. \tag{10}$$

If L is regular, that is ω_L is symplectic, then there exists a unique vector field ξ_L on $T^{2k-1}Q$ satisfying (10). Furthermore, ξ_L is a $2k$–th differential equation (i.e., $J_1\xi_L = C_1$), and its solutions are just the solutions of the Euler–Lagrange equations of L:

$$\sum_{r=0}^{k}(-1)^r \frac{d^r}{dt^r}\left(\frac{\partial L}{\partial q_r^i}\right) = 0.$$

Now, let $L : T^kQ \to \mathbb{R}$ be a singular Lagrangian of order k. Then ω_L is not symplectic and thus, in general, the global equation of motion $\imath_X\omega_L = dE_L$ has not solution. The obstruction to the regularity is given by the characteristic distribution $\ker\omega_L$, which is called the gauge distribution. In order to avoid it, we need to quotient $T^{2k-1}Q$ by $\ker\omega_L$. We assume that the following conditions are satisfied:

(A1) ω_L is presymplectic, i.e., ω_L has constant even rank.

(A2) $J_r(\ker\omega_L) = \operatorname{Im} J_r \cap \ker\omega_L$, $1 \le r \le 2k-1$.

Hence $\dim\ker\omega_L = 2ka$, where $a = \dim J_{2k-1}(\ker\omega_L)$, and $\ker\omega_L$ defines a foliation on $T^{2k-1}Q$. We denote by

$$\hat{T}^{2k-1}Q = T^{2k-1}Q/\ker\omega_L$$

the corresponding space of leaves, $\pi_L : T^{2k-1}Q \to \hat{T}^{2k-1}Q$ being the canonical projection.

(A3) $\pi_L : T^{2k-1}Q \to \hat{T}^{2k-1}Q$ is a fibration, i.e., $\hat{T}^{2k-1}Q$ has a structure of quotient manifold such that π_L is a surjective submersion.

(A4) L admits a global dynamics, i.e., there exists a vector field X on $T^{2k-1}Q$ such that $\imath_X\omega_L = dE_L$.

In such a case, L admits only primary constraints [7]. From (A2) and (A4) we easily deduce that there exists a $2k$–th ODE ξ in $T^{2k-1}Q$ such that $\imath_\xi\omega_L = dE_L$. Also, from (A1), (A2) and (A3), we deduce that there exists a unique symplectic form $\hat{\omega}_L$ on $\hat{T}^{2k-1}Q$ such that $\pi_L^*\hat{\omega}_L = \omega_L$. Moreover, we have that ξ is a projectable vector field with projection $\hat{\xi}$ and the function E_L projects to a function $\hat{E}_L$ on $\hat{T}^{2k-1}Q$. If, in addition, $\ker\omega_L$ is a tangent distribution, i.e., $\ker\omega_L$ is the complete lift of an involutive distribution D on Q, then the canonical almost tangent structure J_1 of order $2k-1$ and the canonical Liouville vector field C_1 on $T^{2k-1}Q$ projects onto an almost tangent structure of the same order $\hat{J}_1$ and a vector field $\hat{C}_1$ on $\hat{T}^{2k-1}Q$.

In [9] it is proved that there exists a local regular Lagrangian $\hat{L}$ defined on some open subset of $\hat{T}^{2k-1}Q$ such that

— $\hat{L}$ is of order k.

— L and $\hat{L}\circ\pi_L$ are gauge equivalent, i.e., $\hat{L}\circ\pi_L = (L+\tilde{\alpha})\circ\rho_k^{2k-1}$ + constant, where $\rho_k^{2k-1} : T^{2k-1}Q \to T^kQ$ is the canonical projection and α is a 1–form on $T^{k-1}Q$.

Hence L and $\hat{L}$ have the same dynamics. The point now is that $\hat{T}^{2k-1}Q$ is in fact a higher order tangent bundle.

Suppose that L satisfies (A1), (A3), (A4) and $\ker\omega_L$ is a tangent distribution, in which case L satisfies (A2). Then $\hat{T}^{2k-1}Q$ is an almost tangent manifold of order $2k-1$ with almost tangent structure $\hat{J}_1$ and $\hat{C}_1$ is a vector field on $\hat{T}^{2k-1}Q$ such that

$$L_{\hat{C}_1}\hat{J}_1 = \hat{J}_1, \qquad \hat{J}_1^{2k-1}\hat{C}_1 = 0.$$

Since C_1 is complete, so is $\hat{C}_1$, and the flow of $\hat{C}_1$ is just

$$(\exp t\hat{C}_1)(\pi_L(x)) = \pi_L((\exp t\hat{C}_1)x), \quad x \in T^{2k-1}Q.$$

Hence there exists $\lim_{t\to -\infty}(\exp t\hat{C}_1)(x)$ and it is unique. Thus, from Theorem 5.1, we deduce the following

Theorem 6.1 *The set S of singular points of $\hat{C}_1$ is an $(m-a)$–dimensional submanifold of $\hat{T}^{2k-1}Q$. Further, $\hat{T}^{2k-1}Q$ is fibered over the identity of S and there exists a diffeomorphism $T^{2k-1}S \to \hat{T}^{2k-1}Q$ such that it transports the canonical almost structure of higher order $(J_1)_S$ and the canonical Liouville vector field $(C_1)_S$ to $\hat{J}_1$ and $\hat{C}_1$, respectively.*

Remark 6.1 ¿From Theorem 3.3 in [14] and Theorem 6.1 (see also [9]) we deduce that $\hat{L}$ is defined on some open subset of T^kS, α is defined on some open subset of $T^{k-1}S$, and we have

$$(\hat{L} + \tilde{\alpha}) \circ (\rho_k^{2k-1})_S \circ \pi_L = L \circ \rho_k^{2k-1} + \text{constant},$$

where $(\rho_k^{2k-1})_S : T^{2k-1}S \to T^kS$ is the canonical projection.

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