Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 36 (1995), No. 1, 145-148.

On the Radial Ratio in Dense Mixed Sphere Packings

Luis G. De Lucía
Auburn University, Department of Mathematics
P.O. Box 82, 228 Parker Hall, Auburn, AL 36849, U.S.A.
E-mail: deluclg@mail.auburn.edu

Abstract. Packings of E^3 are constructed with balls of two radii, 1 and $r < 1$, with density higher than the sphere lattice packing density ($\delta_L^3 = \frac{\pi}{\sqrt{18}}$) for every r in the interval $(0.482\ldots, 0.577\ldots]$.

Troughout this note, we will work in the Euclidean 3-space, E^3. A *packing* of E^3 is a collection of compact and convex subsets of E^3 with mutually disjoint interiors. The volume of a set K will be denoted by $\mathrm{Vol}(K)$, and the convex hull of a set of points S by $\mathrm{Conv}(S)$.

Intuitively, the density of a packing P is the proportion of space occupied by the elements of the packing. Let $B(\rho)$ be the ball of radius $\rho > 0$ centered at the origin O. Let $\delta(P, B(\rho))$ denote the ratio of the sum of volumes of those elements of P which are contained in $B(\rho)$ to the volume of $B(\rho)$. Then the density of the packing P is given by

$$\delta(P) = \limsup_{\rho \to \infty} \delta(P, B(\rho))$$

For a detailed discussion of densities of packings see [5].

For a fixed $r \in (0, 1]$, a *mixed sphere packing* of E^3, is a packing consisting of balls of radius 1 and balls of radius r. A mixed sphere packing with balls of radius 1 and r will be denoted by P_r.

For each $r \in (0, 1]$ we define the optimum density $\delta(r)$ as the supremum of $\delta(P_r)$ taken over all mixed sphere packings P_r. For completeness' sake, we define

$$\delta(0) = \limsup_{r \to 0} \delta(r)$$

In the plane there are mixed circle packings with density higher than the circle packing density ($\delta^2 = \frac{\pi}{\sqrt{12}}$) for all $r < 0.645\ldots$ [8]. Furthermore it is known that no such packing exists for $r > 0.74299$ [1]. However, much less is known about the analogous problem in space. Let δ^3 denote the density of the densest packing of E^3 with unit balls, and δ_L^3 denote the density of the densest lattice packing with unit balls. It is known that $\delta_L^3 = \frac{\pi}{\sqrt{18}}$, and it is conjectured that $\delta^3 = \delta_L^3$. Furthermore this conjecture may be strengthened to say that $\delta(r) = \delta_L^3$ for values of r sufficiently close to 1, as suggested by the case in two dimensions.

For this reason it is of interest to find the values of $r \in [0,1]$ for which $\delta(r) > \delta_L^3$. It is easy to show that $\delta(r) > \delta_L^3$ for $r \in [0, 0.444\ldots)$. Namely, when $r = 0$, L. Fejes Tóth's result in the plane [7] generalizes to give $\delta(0) = \delta^3(2-\delta^3) > \delta^3 \geq \delta_L^3$. For $r \in (0, \sqrt{2}-1]$, density higher than δ_L^3 is achieved by placing the centers of the small balls in the "deep holes" (see [3]) of the densest lattice packing of balls of radius 1. Leaving all the centers fixed in the previous packing, we can increase the radius of the small balls and decrease the radius of the balls of radius 1 simultaneously so that they do not overlap. Then we notice that the density is larger than δ_L^3 as long as the ratio of these radii remains smaller than $0.444\ldots$.

Recently Vassallo and Wills [12] showed that $\delta(r) > \delta_L^3$ for $r \in [0.577\ldots, 0.623\ldots)$. Although to find mixed sphere packings P_r for $r \geq 0.623\ldots$ for which $\delta(P_r) > \delta_L^3$ remains the main problem, Vassallo and Wills point out that it is not known wether this can be done in the "gap" $0.444\ldots \leq r < \frac{\sqrt{3}}{3} = 0.577\ldots$.

It is the aim of this note to narrow this gap by means of the following

Theorem. *If r is in the interval $((\frac{1}{2}(\sqrt{\frac{3}{2}}-1))^{\frac{1}{3}}, \frac{\sqrt{3}}{3}] = (0.482\ldots, 0.577\ldots]$ then there is a mixed sphere packing with balls of radius 1 and balls of radius r with density higher than δ_L^3.*

Proof: For $h \geq 1$, let $x_1 = (\frac{-1}{\sqrt{3}}, 1, h)$, $x_2 = (\frac{-1}{\sqrt{3}}, -1, h)$, $x_3 = (\frac{2}{\sqrt{3}}, 0, h)$, $x_4 = (\frac{-1}{\sqrt{3}}, 1, -h)$, $x_5 = (\frac{-1}{\sqrt{3}}, -1, -h)$, and $x_6 = (\frac{2}{\sqrt{3}}, 0, -h)$, and let $T_h = \mathrm{Conv}(\{x_1, \ldots, x_6\})$. For each $i \in \{1, 2, \ldots, 6\}$, let B_i denote a sphere of radius 1 centered at the point x_i. Clearly the interiors of these spheres are pairwise disjoint. Let B_7 denote a sphere of variable radius r centered at the origin O.

For $0 < r \leq \sqrt{\frac{7}{3}} - 1$ and $h = 1$, we have that $B_7 \subset T_1$ and B_7 does not overlap any of the other spheres. Consider the intersection of T_1 with the seven spheres, and observe that translates of T_1 and its reflection $-T_1$ tile E^3 in a facet-to-facet manner. This tiling produces a periodic mixed sphere packing P_r.

Since twelve copies of T_1 are placed around a vertex, and there are spheres of radius 1 at each of the six vertices of T_1, we have that $\mathrm{Vol}(T_1 \cap (B_1 \cup \ldots \cup B_6)) = \frac{1}{2}(\frac{4\pi}{3})$. Also, $B_7 \subset T_1$ implies $\mathrm{Vol}(T_1 \cap B_7) = \mathrm{Vol}(B_7) = \frac{4\pi}{3} r^3$. As well, $\mathrm{Vol}(T_1) = 2\sqrt{3}$. Thus we have

$$\delta(P_r) = \frac{\frac{1}{2}(\frac{4\pi}{3}) + \frac{4\pi}{3} r^3}{2\sqrt{3}}$$

Easy calculations then show that $\delta(P_r) > \delta_L^3$ for $(\frac{1}{2}(\sqrt{\frac{3}{2}}-1))^{\frac{1}{3}} < r \leq \sqrt{\frac{7}{3}} - 1$.

For $\sqrt{\frac{7}{3}} - 1 \leq r \leq \frac{\sqrt{3}}{3}$ and $h = ((r+1)^2 - \frac{4}{3})^{\frac{1}{2}} \geq 1$, we have that the sphere B_7 lies in T_h and it does not overlap any of the other spheres. Therefore, using a tiling with translates of T_h and $-T_h$, spheres of radii 1 and r can be arranged in a periodic packing P_r of E^3.

Now we only need to add the observation that $\mathrm{Vol}(T_h) = 2h\sqrt{3} = 2\sqrt{3(r+1)^2-4}$. Hence we have

$$\delta(P_r) = \frac{\frac{1}{2}(\frac{4\pi}{3}) + \frac{4\pi}{3}r^3}{2\sqrt{3(r+1)^2-4}}$$

In finding the values of r for which $\delta(P_r) > \delta_L^3$ we arrive at the inequality

$$8r^6 + 8r^3 - 3r^2 - 6r + 3 > 0$$

Let $f(x) = 8x^6 + 8x^3 - 3x^2 - 6x + 3$. Observe that the line segment for $\frac{1}{2} \leq x < \frac{3}{4}$ tangent to f at the point $(\frac{1}{2}, \frac{3}{8})$ lies above the x-axis. Since $f''(x) > 0$ in this interval, the latter inequality is true for all values of r in the interval $[\frac{1}{2}, \frac{3}{4})$.

In conclusion, $\delta(P_r) > \delta_L^3$, whenever $\sqrt{\frac{7}{3}} - 1 \leq r \leq \frac{\sqrt{3}}{3}$, which completes the proof of the theorem. □

Thus the gap of $0.444\ldots \leq r < 0.577\ldots$ has been narrowed to $0.444\ldots \leq r \leq 0.482\ldots$ where it is still unknown wether $\delta(r) > \delta_L^3$ or not.

References

[1] G. Blind: *Über Unterdeckungen der Ebene durch Kreise,* Journal reine angew. Math. **236** (1969), 145–173.

[2] D.W. Boyd: *The Osculatory Packing of the Three Dimensional Sphere,* Canad. J. Math. **25** (1973), 303–322.

[3] J.H. Conway and N.J.A. Sloane: *Sphere Packings, Lattices and Groups,* Springer-Verlag, New York etc. (1988).

[4] G. Fejes Tóth: *Packing and Covering with Incongruent Circles,* Studia Sci. Math. Hung. **19** (1984), 49–53.

[5] G. Fejes Tóth and W. Kuperberg: *Packing and Covering with Convex Sets,* Handbook of Convex Geometry (Amsterdam) **B** ed. P.M. Gruber and J.M. Wills, North-Holland, Amsterdam (1993).

[6] L. Fejes Tóth: *Regular Figures,* International Series of Monographs in Pure and Applied Mathematics, vol. 48 Pergamon Press Oxford etc. (1964).

[7] L. Fejes Tóth: *Lagerungen in der Ebene, auf der Kugel und im Raum,* Springer, Berlin (1972).

[8] L. Fejes Tóth: *Some Density-Bounds for Packing and Covering with Incongruent Circles,* Studia Sci. Math. Hung. **15** (1980), 63–70.

[9] L. Few: *A Mixed Packing Problem,* Mathematika **7** (1963), 229–242.

[10 P.M. Gruber and C.G. Lekkerkerker: *Geometry of Numbers,* Mathematical Library, vol. 37 North- Holland, Amsterdam etc. (1987).

[11] T.L. Saaty and J.M. Alexander: *Optimization and the Geometry of Numbers: Packing and Covering,* SIAM Review **17** (1975), 475–519.

[12] S. Vassallo and J.M. Wills: *On Mixed Sphere Packings,* Contributions to Algebra and Geometry **35** (1994), 67–71.

Received January 6, 1995.