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Minimal Surfaces in a Complex Projective Space whose Mean Curvature Vectors are Eigenvectors

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Abstract. B.Y. Chen [Null 2-type surfaces in Euclidean space, Algebra, Analysis and Geometry, Taipei, World Scientific 1989, 1-18] initiated the study of submanifolds whose mean curvature vector H is an eigenvector of the Laplacian Δ. In this article, we study this problem for minimal surfaces in CP^n. Two classification theorems are obtained.

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1. Introduction

Let $\widetilde{x}: M \to CP^n$ be an isometric immersion of a connected Riemannian manifold M into a complex projective space CP^n, and $\Phi: CP^n \to E^N$ the first standard imbedding of CP^n into a Euclidean space E^N. Then the composition $x = \Phi \circ \widetilde{x}: M \to E^N$ can be decomposed as $x = \sum_{k=0}^{\infty} x_k$, where x_k is the k-th eigenfunction of the Laplacion Δ of M. The immersion x is said to be of finite Chen type if

$$x = x_0 + x_1 + \cdots + x_k, \tag{1}$$

where x_0 is a constant vector, $x_1, ..., x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$, $i = 1, ..., k$. If in particular all eigenvalues $\lambda_1, ..., \lambda_k$ are mutually different, then the immersion x (or the submanifold M) is said to be of k-type and if in particular one of $\lambda_1, ..., \lambda_k$ is zero, then x is said to be of null k-type. It is clear that every submanifold of null k-type is non-compact (for details see [1]).

In the study of finite Chen type submanifolds in CP^n, Ros [13] and Dimitric [10] classified 1-type compact CR-minimal submainifolds and 1-type compact submanifolds with parallel mean curvature vector, respectively, and Shen [14] classified 2-type compact non-holomorphic minimal surfaces.

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In [2] Chen initiated the study of submanifolds whose mean curvature vector H is an eigenvector of the Laplacian Δ and proved that such submanfolds are either biharmonic or of 1-type or of null 2-type. The classification of surfaces with $\Delta H = \lambda H$ in a Euclidean 3-space was first done by Chen in 1988 [3] and, independently, by W.H. Chen in 1994 [8]. Moreover, in [4] and [5] Chen classified such submanifolds in hyperbolic spaces and de Sitter space-time, respectively.

In this paper, we study minimal surfaces in CP^n whose mean curvature vector H in E^N satisfies

$$\Delta H = \lambda H, \tag{2}$$

where λ is a nonzero constant. According to Chen's result obtained in [2], (2) implies that such surfaces are of 1-type or null 2-type [2]. Actually we will mainly prove the following.

Theorem 1. *Let $\widetilde{x} : M \to CP^n(4)$ be an isometric minimal immersion of a connected (not necessary compact) surface M into a complex projective space $CP^n(4)$ with the Fubini-Study metric of constant holomorphic sectional curvature 4 and $\Phi : CP^n(4) \to E^N$ the first standard imbedding of $CP^n(4)$ into a Euclidean space E^N. Then the mean curvature vector H of $x = \Phi \circ \widetilde{x}$ satisfies (2) if and only if M is one of the following:*

i) a portion of $CP^1(4)$;

ii) a totally real isotropic minimal surface of $CP^2(4)$.

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2. Preliminaries

Let $HM(n+1) = \{A \in GL(n+1) \mid \overline{A} = A^t\}$ be the set of all Hermitian matrices of degree (n+1). Equipped with the metric

$$\langle A, B\rangle = \frac{1}{2} trace(AB), \quad \text{for any } A, B \in HM(n+1), \tag{3}$$

it becomes the usual Euclidean space E^N of dimension $N = (n+1)^2$. Consider the submanifold $CP^n = \{A \in HM(n+1) \mid A^2 = A \,\text{and}\, trace\, A = 1\}$. It is known that CP^n, with the metric induced by the one on $HM(n+1)$, is isometric to the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4. We denote the isometry by $\Phi : CP^n(4) \to HM(n+1)$.

Let $D, \widetilde{\nabla}, \widetilde{\sigma}, \widetilde{\nabla}^{\perp}, \widetilde{\Lambda}$ and $\widetilde{H}$ be the Levi-Civita connection of $HM(n+1)$, the induced connection in CP^n, the second fundamental form, the normal connection , the Weingarten map and the mean curvature vector of Φ, respectively. Then we have

$$\widetilde{\sigma}(JX, JY) = \widetilde{\sigma}(X, Y), \quad \overline{\nabla}\widetilde{\sigma} = 0. \tag{4}$$

$$\begin{aligned} \langle \widetilde{\sigma}(X,Y), \widetilde{\sigma}(V,W)\rangle =& 2\langle X,Y\rangle \langle V,W\rangle + \langle X,W\rangle \langle Y,V\rangle + \langle X,V\rangle \langle Y,W\rangle \\ &+ \langle X,JW\rangle \langle Y,JV\rangle + \langle X,JV\rangle \langle Y,JW\rangle, \end{aligned} \tag{5}$$

where $X, Y, V, W \in T(CP^n)$, J is the complex structure of CP^n and $\overline{\nabla}\widetilde{\sigma}$ the covariant derivative of $\widetilde{\sigma}$ (for details see [13]). We recall the curvature tensor $\widetilde{R}$ of CP^n for later use:

$$\begin{aligned}\widetilde{R}(X,Y)Z = &< Y, Z > X - < X, Z > Y + < JY, Z > JX \\ &- < JX, Z > JY + 2 < X, JY > JZ,\end{aligned} \tag{6}$$

for any $X, Y, Z \in T(CP^n)$.

Let $\widetilde{x} : M \to CP^n$ be an immersion of a real surface M into CP^n. We choose a local field of orthonormal frames $\{e_1, e_2\}$ on M. Define the Kaehler angle α of $\widetilde{x}$ by $\cos\alpha =< Je_1, e_2 >$. In particular, if $\widetilde{x}$ is minimal, then $\sin\alpha$ vanishes identically (then $\widetilde{x}$ is $\pm$holomorphic) or $\sin\alpha$ has only isolated zeros. For the second case, in the open subset where $\sin\alpha \neq 0$, we may extend $\{e_i, Je_i\}$ to a neighborhood U of CP^2 such that $\mathcal{D}_U = span\{e_i, Je_i\}$ is a real 4-dimensional distribution on U and put

$$\begin{aligned}(\sin\alpha)e_{1^*} &= -(\cos\alpha)e_1 - Je_2, \\ (\sin\alpha)e_{2^*} &= Je_1 - (\cos\alpha)e_2.\end{aligned} \tag{7}$$

It is easy to see that $\{e_i, e_{i^*}\}$ is an orthonormal basis on D_U. Take $\{e_\alpha, e_{\alpha^*}\}$ as an orthonormal basis in CP^n which extends $\{e_i, e_{i^*}\}$ and satisfies $e_{\tau^*} = Je_\tau$, for $\tau \geq 3$ (for details see [9]). Here and in the sequal we employ the following index conventions: $i, j, k, ... = 1, 2$; $\tau, \zeta, ... = 3, ..., n$; $\alpha, \beta, \gamma, ... = 1, ..., n$; and shall agree that repeated indices are summed over the respective ranges.

3. Computation of the Laplacian ΔH

Let M be a connected (not necessary compact) surface and $\widetilde{x} : M \to CP^n$ an isometric minimal immersion. Denote by $\sigma, \nabla^\perp, \Lambda$ and H' the second fundamental form, normal connection, the Weingarten map and the mean curvature vector of $\widetilde{x}$, respectively. Let ∇ be the induced connection on M and H the mean curvature vector of $x = \Phi \circ \widetilde{x}$. Then we have

$$H = H' + \widetilde{H}, \;\; H' = 0 \;\text{ and }\; \widetilde{H} = \frac{1}{2}\Sigma\,\widetilde{\sigma}(e_i, e_i). \tag{8}$$

We assume that $\nabla_{e_i} e_j \mid_A = 0$ for any i, j at a point $A \in x(M)$ .By (4) and (5), we have

$$D_j H = \frac{1}{2}\Sigma D_j \widetilde{\sigma}(e_i, e_i) = \Sigma\widetilde{\sigma}(\sigma(e_i, e_j), e_i) - 3e_j - \Sigma < Je_i, e_j > Je_i \tag{9}$$

$$\begin{aligned}\Delta H = &- \Sigma D_j D_j H = -\Sigma\widetilde{\nabla}_j^\perp \widetilde{\sigma}(\sigma(e_i, e_j), e_i) + 6H \\ &+ \Sigma\widetilde{\Lambda}_{\widetilde{\sigma}(\sigma(e_i, e_j), e_i)} e_j + \Sigma D_j(< Je_i, e_j > Je_i),\end{aligned} \tag{10}$$

where we put $D_j = D_{e_j}$ and $\tilde{\nabla}_j = \tilde{\nabla}_{e_j}$. And by using the Codazzi equation of $\widetilde{x}$ and (4)-(7) the terms of (10) become

$$\Sigma\tilde{\Lambda}_{\tilde{\sigma}(\sigma(e_i, e_j), e_i)} e_j = \Sigma < J\sigma(e_i, e_j), e_j > Je_i, \tag{11}$$

$$\Sigma D_j(< Je_i, e_j > Je_i) = \Sigma < J\tilde{\sigma}(e_i, e_j), e_j > Je_i + 2(\cos\alpha)\tilde{\sigma}(Je_1, e_2), \tag{12}$$

$$\begin{aligned}\Sigma\widetilde{\nabla}_j^{\perp}\tilde{\sigma}(\sigma(e_i, e_j), e_i) =& \Sigma\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) - \Sigma\tilde{\sigma}(\Lambda_{\sigma(e_i,e_j)}e_j, e_i)\\ &-6(\cos\alpha)\tilde{\sigma}(Je_1, e_2) + 6(\cos^2\alpha)H.\end{aligned} \tag{13}$$

Since we also have, with the help of the Gauss equation of $\widetilde{x}$,

$$\Sigma < J\sigma(e_i, e_j), e_j > Je_i = e_2(\cos\alpha)Je_1 - e_1(\cos\alpha)Je_2, \tag{14}$$

$$\Sigma\tilde{\sigma}(\Lambda_{\sigma(e_i,e_j)}e_j, e_i) = 2[(1 + 3\cos^2\alpha) - K]H, \tag{15}$$

combining (10)-(15) together, we may obtain

$$\begin{aligned}\Delta H =& -\Sigma\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) + 8(\cos\alpha)\tilde{\sigma}(Je_1, e_2)\\ &+ 2(4 - K)H + 2e_2(\cos\alpha)Je_1 - 2e_1(\cos\alpha)Je_2.\end{aligned} \tag{16}$$

Lemma 3.1. *Let M be a connected minimal surface in CP^n. If the mean curvature vector H of M in E^n satisfies (2), then $\alpha = const$, i.e., M is a slant surface in CP^n.*

Proof. In (16), we see that the last two terms belong to $T(CP^n)$ and (2) implies that they must vanish, which means that $\alpha = const$. □

Lemma 3.2. *Let M be a connected $\pm$holomorphic minimal surface in $CP^n(4)$. Then the mean curvature vector H of M in E^N satisfies (2) if and only if M is a portion of $CP^1(4)$.*

Proof. Let M be a connected minimal surface in $CP^n(4)$. If M is $\pm$holomorphic, then $\cos\alpha = \pm 1$, it follows that $Je_1 = \pm e_2$, $Je_2 = \mp e_1$ in (7) and hence (16) becomes

$$\Delta H = -\Sigma\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) + 2(8 - K)H. \tag{17}$$

If H satisfies (2), then we have $< \Delta H, H >= \lambda < H, H >$, which implies from (8) and (17) that $\lambda = 12 - K$. On the other hand, we may also have $< \Delta H, \tilde{\sigma}(\sigma(e_1, e_1), \sigma(e_1, e_1)) >= \lambda < H, \tilde{\sigma}(\sigma(e_1, e_1), \sigma(e_1, e_1)) >$, which gives that $\lambda = 2K$. Consequently, we may obtain that $K = 4$ and hence $\| \sigma \|^2 = 8 - 2K = 0$, i.e., M is totally geodesic. Thus M is a portion of $CP^1(4)$. The converse is obvious. □

4. Non ±holomorphic minimal surfaces

In this section, we assume that M is a non $\pm$holomorphic minimal surface in $CP^n(4)$. Then, according to Section 2, we may take the basis $\{e_\alpha, e_{a^*}\}$ on M. Thus by (7) we have

$$(\sin^2\alpha)\Sigma\tilde{\sigma}(e_{i^*}, e_{i^*}) = 2(1 + \cos^2\alpha)H - 4(\cos\alpha)\tilde{\sigma}(Je_1, e_2). \tag{18}$$

Applying it to (16), we obtain

$$\begin{aligned}\Delta H =& -\Sigma\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) + 2[2(3 + \cos^2\alpha) - K]H\\ &- 2(\sin^2\alpha)\Sigma\tilde{\sigma}(e_{i^*}, e_{i^*}) + 2e_2(\cos\alpha)Je_1 - 2e_1(\cos\alpha)Je_2.\end{aligned} \tag{19}$$

If H satisfies (2), then from (19) and Lemma 3.1 we have

$$\begin{aligned} &\Sigma\tilde{\sigma}(\sigma(e_i,e_j),\sigma(e_i,e_j)) - [4(3+\cos^2\alpha) - 2K - \lambda]H \\ &+ 2(\sin^2\alpha)\Sigma\tilde{\sigma}(e_{i^*},e_{i^*}) = 0. \end{aligned} \tag{20}$$

Lemma 4.1. *Let M be a minimal surface of $CP^n(4)$. If M is non $\pm$holomorphic, then we have the following relations:*

$$< H, H >= 3 + \cos^2\alpha, \tag{21}$$

$$< H, \Sigma\tilde{\sigma}(\sigma(e_i,e_j),\sigma(e_i,e_j)) >= 2 \parallel \sigma \parallel^2 + (\sin^2) \parallel \sigma_* \parallel^2 \tag{22}$$

$$< H, \Sigma\tilde{\sigma}(e_{i^*},e_{i^*}) >= 4 + 2(\sin^2\alpha), \tag{23}$$

$$< H, \Sigma\tilde{\sigma}(e_k,e_{k^*}) >= 2(\sin\alpha)(\cos\alpha), \tag{24}$$

$$< \Sigma\tilde{\sigma}(e_{k^*},e_{k^*}), \Sigma\tilde{\sigma}(\sigma(e_i,e_j),\sigma(e_i,e_j)) >= 4 \parallel \sigma \parallel^2 + 2(1+\cos^2\alpha) \parallel \sigma_* \parallel^2, \tag{25}$$

$$< \Sigma\tilde{\sigma}(e_{k^*},e_{k^*}), \Sigma\tilde{\sigma}(e_{i^*},e_{i^*}) >= 4(3+\cos^2\alpha), \tag{26}$$

$$< \Sigma\tilde{\sigma}(e_k,e_{k^*}), \Sigma\tilde{\sigma}(\sigma(e_i,e_j),\sigma(e_i,e_j)) >= -2(\sin\alpha\cos\alpha) \parallel \sigma_* \parallel^2, \tag{27}$$

$$< \Sigma\tilde{\sigma}(e_k,e_{k^*}), \Sigma\tilde{\sigma}(e_{i^*},e_{i^*}) >= -4(\sin\alpha\cos\alpha), \tag{28}$$

where $\parallel \sigma \parallel^2 = \Sigma < \sigma(e_i,e_j), \sigma(e_i,e_j) >$ *and* $\parallel \sigma_* \parallel^2 = \Sigma < \sigma(e_i,e_j), e_{k^*} >^2$.

The proof can be obtained by using (5) and (7) repeatedly.

Applying $< H, - >$, $< \Sigma\tilde{\sigma}(e_{k^*},e_{k^*}), - >$ and $< \Sigma\tilde{\sigma}(e_k,e_{k^*}), - >$ to (20) one after another and using Lemma 4.1, we may get the following.

$$\begin{aligned} &2 \parallel \sigma \parallel^2 + (\sin^2\alpha) \parallel \sigma_* \parallel^2 - [4(3+\cos^2\alpha) - 2K - \lambda](3+\cos^2\alpha) \\ &+ 4(\sin^2\alpha)(2+\sin^2\alpha) = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &2 \parallel \sigma \parallel^2 + (1+\cos^2\alpha) \parallel \sigma_* \parallel^2 - [4(3+\cos^2\alpha) - 2K - \lambda](2+\sin^2\alpha) \\ &+ 4(\sin^2\alpha)(3+\cos^2\alpha) = 0, \end{aligned} \tag{30}$$

$$(\sin\alpha\cos\alpha)[\parallel \sigma_* \parallel^2 + 16 - 2K - \lambda] = 0. \tag{31}$$

Since we have assumed that M is non $\pm$holomorphic, (31) implies that either cosα vanishes identically or the following relation:

$$\parallel \sigma \parallel^2 = 2K + \lambda - 16 \tag{32}$$

holds on the open subset of M where cos$\alpha \neq 0$. We now claim that the second case can not heppen. Since, otherwise, from the Gauss equation and (6), we have

$$\parallel \sigma \parallel^2 = 2(1 + 3\cos^2\alpha) - 2K, \tag{33}$$

adding (29) and (30) together and using (33), we may get

$$\| \sigma_* \|^2 = -2K - 3\lambda + 20 + 12\cos^2\alpha. \tag{34}$$

Combining (32) and (34), we might deduce the following

$$K = 9 + 3\cos^2\alpha - \lambda, \tag{35}$$

$$\| \sigma_* \|^2 = 2 + 6\cos^2\alpha - \lambda. \tag{36}$$

Substituting (35) into (33), we would have

$$\| \sigma \|^2 = 2\lambda - 16, \tag{37}$$

which implies that $\lambda \geq 8$. Applying this to (36), we would obtain

$$\| \sigma_* \|^2 \leq 2 + 6\cos^2\alpha - 8 = -6\sin^2\alpha < 0, \tag{38}$$

which is a contradiction. Thus cosα must vanish identically, i.e., M is totally real and we have proved the following.

Lemma 4.2. *Under the same assumption as in Lemma 3.1. If in addition M is non $\pm$holomorphic, then M must be totally real in $CP^n(4)$.*

In the following we assume that M is totally real. Then we may take

$$e_{1^*} = -Je_2, \quad e_{2^*} = Je_{1.} \tag{39}$$

Consequently, (33), (34) and (20) become the following, respectively,

$$\| \sigma \|^2 = 2 - 2K, \tag{40}$$

$$\| \sigma_* \|^2 = 20 - 2K - 3\lambda, \tag{41}$$

$$\Sigma\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) - (8 - 2K - \lambda)H = 0. \tag{42}$$

Apply $< \Sigma\tilde{\sigma}(e_\tau, e_\tau), - >$ to (42), since we have

$$< \Sigma\tilde{\sigma}(e_\tau, e_\tau), \Sigma\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) >= 2(n-1) \| \sigma \|^2 - 2 \| \sigma_* \|^2, \tag{43}$$

$$< \Sigma\tilde{\sigma}(e_\tau, e_\tau), H >= 2(n-2), \tag{44}$$

we may get, with the help of (40) and (41),

$$(n+1)(\lambda - 6) = 0, \tag{45}$$

which implies that $\lambda = 6$ and it follows that

$$\| \sigma \|^2 = \| \sigma_* \|^2 \tag{46}$$

$$\Sigma\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) - (2 - 2K)H = 0. \tag{47}$$

Proof of Theorem 1. If M is $\pm$holomorphic, the theorem holds by Lemma 3.2. If M is non $\pm$holomorphic, Lemma 4.2 shows that M must be totally real and, moreover, (46) and (47) hold on M. Since (46) implies

$$< \sigma(e_i, e_j), e_\tau >= 0 \quad \text{and} \quad < \sigma(e_i, e_j), Je_\tau >= 0, \tag{48}$$

for any i, j, τ, thus by Theorem 2 of [6], M is in a totally geodesic CP^2.

Applying $< \Sigma\tilde{\sigma}(\sigma(e_k, e_l), \sigma(e_k, e_l)), - >$ to (47), we get

$$\| \sigma \|^4 + \Sigma < \sigma(e_i, e_j), \sigma(e_k, e_l) >^2 - 3(1 - K) \| \sigma \|^2 = 0. \tag{49}$$

Substituting (40) into (49), we may find that

$$(| \sigma(e_1, e_1) |^2 - | \sigma(e_1, e_2) |^2) + 4 < \sigma(e_1, e_1), \sigma(e_1, e_2) >^2 = 0, \tag{50}$$

which implies that

$$| \sigma(e_1, e_1) |^2 = | \sigma(e_2, e_2) |^2 = | \sigma(e_1, e_2) |^2 \tag{51}$$

$$< \sigma(e_1, e_1), \sigma(e_1, e_2) >= 0 \quad \textit{and} \quad < \sigma(e_2, e_2), \sigma(e_1, e_2) >= 0. \tag{52}$$

From these we may conclude that M is isotropic in CP^2 in the sense of O'Neil [12].

Conversely, if M is a totally real isotropic minimal surface of CP^2, then by Theorem 2.8 of [13], M is a minimal surface in a sphere of $HM(3)$ and thus (2) holds on M according to the well known Takahashi theorem, in fact M is of 1-type. □

From the proof of Theorem 1, we have the following immediately.

Corollary. *There are no minimal surfaces in CP^n which is of null 2-type via the first standard imbedding of CP^n into E^N.*

5. Minimal surfaces with constant Gauss curvature

In Theorem 1, if in addition M has constant Gauss curvature, then we will get the result more precisely and thoroughly. Actually we have the following.

Theorem 2. *Let $\tilde{x} : M \to CP^n(4)$ and $\Phi : CP^n \to E^N$ be as in Theorem 1. If $\tilde{x}$ is non $\pm$holomorphic and the Gauss curvature K of M is constant, then the mean curvature vector H of $x = \Phi \circ \tilde{x}$ satisfies (2) if and only if M is an open portion of one of the following surfaces:*

i) the real projective plane $RP^2(1)$ isometrically immersed in $CP^2(4)$ as a totally real totally geodesic surface;

ii) the totally real minimal flat surface of $CP^2(4)$ defined by $\{[z_0, z_1, z_2] \in CP^2(4) : z_0\overline{z}_0 = z_1\overline{z}_1 = z_2\overline{z}_2\}$, where $[z_0, z_1, z_2]$ denotes the homogeneous coordinates of $CP^2(4)$, *and, with respect to a suitable orthonormal coframe field* $\{\omega^1, \omega^2, \omega^{1^*}, \omega^{2^*}\}$ *in* $CP^2(4)$, *the connection form* (ω^A_B) *of* $CP^2(4)$, *restricted to* M, *is given by*

$$\begin{pmatrix} 0 & 0 & -\mu\omega^2 & -\mu\omega^1 \\ 0 & 0 & -\mu\omega^1 & \mu\omega^2 \\ \mu\omega^2 & \mu\omega^1 & 0 & 0 \\ \mu\omega^1 & -\mu\omega^2 & 0 & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}}. \tag{53}$$

Proof. If $K = const.$, we recall from [7] the following formula for totally real minimal surfaces in $CP^2(4)$:

$$\frac{1}{2}\Delta \parallel \sigma \parallel^2 = \parallel \overline{\nabla}\sigma \parallel^2 + 2\Sigma trace(\Lambda_{i^*}\Lambda_{j^*})^2 - 3\Sigma(trace\Lambda_{i^*}\Lambda_{j^*})^2 + 3 \parallel \sigma \parallel^2 . \tag{54}$$

Since M is isotropic, we may choose a local orthonormal frame field $\{e_1, e_2; e_{1^*}, e_{2^*}\}$ on $CP^2(4)$ such that ,restricted to M, e_1, e_2 are tangent to M and have

$$\Lambda_{1^*} = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad \Lambda_{2^*} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}. \tag{55}$$

Then $\parallel \sigma \parallel^2 = 4\mu^2$ and $\mu^2 = \parallel \sigma \parallel^2 /4 = (1-K)/2 = const.$ Thus from (54) and (55) we have

$$\parallel \overline{\nabla}\sigma \parallel^2 = 12\mu^2(2\mu^2 - 1), \tag{56}$$

which implies that either $\mu = 0$ or $\mu^2 \geq \frac{1}{2}$. If μ vanishes identically, then $\parallel \sigma \parallel^2 = 0$ and $K = 1$, thus M is a totally geodesic surface of $CP^2(4)$ with constant curvature $K = 1$, i. e., M is an open portion of i). Otherwise, we have that $K = 1 - 2\mu^2 \leq 0$, then by Theorem A of [11], $K = 0$ and $\mu^2 = \frac{1}{2}$. Moreover, it is easy to see that (53) follows, which implies that M is an open portion of ii).

The converse is clear. □

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