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A Note on Unions of Ideals and Cosets of Ideals

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Abstract. The paper gives a refinement to a well known and easy-to-prove result of basic ideal theory: If an ideal I is contained in the union of a given finite collection of ideals, with at most two of them not prime, I must be contained in one of them. The proposition, stated in this way, leaves out the special structure of the ring under consideration. We show that the lower bound implicit in the proposition can be replaced by the minimum over cardinalities of residue fields of the given ring.

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Closely related to the fact that an ideal can not be contained in the union of two arbitrary and a finite number of prime ideals unless it is contained in one of them (cf. [1], (1.B)), a vector space over a field k can not be covered by a finite collection of proper subspaces unless their number exceeds the cardinality of k (cf.[2], p.23. A similar proposition holds for residue classes of subspaces.). Since a nontrivial field contains at least two elements, the aforesaid might suggest that the field case is inherent in the general case treated by the above quoted proposition. And indeed, the basic results of this paper are

Proposition A: *Let $I, I_1, ..., I_r$ be ideals, $I_{r+1}, ..., I_{r+s}$ prime ideals of A, where $r, s \in \mathbf{N}$, $r + s \geq 1$. If $r \leq \min\{\operatorname{card}({}^{A}/_{\mathbf{m}}) \,|\, \mathbf{m} \in m\text{-spec}(A)\}$, then $I \subset \bigcup_{i=1}^{r+s} I_i$ implies $I \subset I_i$ for some $i \in \{1, ..., r+s\}$.*

Proposition B: *Let $I, I_1, ..., I_t$ be ideals of A, $t \in \mathbf{N}$, $a_{ij} \in A$, $j = 1, ..., n_i$, $n_i \in \mathbf{N}$ for $i = 1, ..., t$. Suppose $I \not\subset I_i$ for all i and $\sum_{i=1}^{t} n_i < \inf\{\operatorname{card}({}^{A}/_{\mathbf{m}}) \,|\, \mathbf{m} \in m\text{-spec}(A)\}$. Then $I \not\subset \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} (a_{ij} + I_i)$.*

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In the proofs, the above quoted basic lemma will, without special reference, frequently be part of the argument. A will always denote a commutative ring with 1, M will be an arbitrary A-module, m-spec(A) the set of maximal ideals of A, m-supp$(M) :=$ m-spec$(A) \cap$ supp(M). An equality $M = \bigcup\limits_{i=1}^{r} M_i$ or an inclusion $I \subset \bigcup\limits_{i=1}^{r} I_i$ with submodules M_i or ideals I_i, I, respectively,will be called *irredundant* if none of the M_i or I_i can be omitted without the relation under consideration turning false.

I. A preparatory lemma and its consequences

If P is a subset of spec(A), $\overline{P}$ will denote the closure of P in the Zariski topology. If L is an arbitrary set and $\{M_\alpha\}_{\alpha\in L}$ a system of submodules of an A-module M, any mapping $s : L \to \bigsqcup_{\alpha\in L}$ supp$({}^M/_{M_\alpha})$ with $s(\alpha) \in$ supp$({}^M/_{M_\alpha})$ for all $\alpha \in L$ will be called *a section for* $\{M_\alpha\}_{\alpha\in L}$.

The proof of the following lemma is clear.

Lemma 1: *A subset $P \subset$ spec(A) the number of minimal elements of which is finite is disconnected in the Zariski topology if and onlyif there exists a partition $P = P_1 \cup P_2$ such that, for any $\mathbf{p} \in P_1$, $\mathbf{q} \in P_2$, $\mathbf{p} \not\subset \mathbf{q}$ and $\mathbf{q} \not\subset \mathbf{p}$.*

The following lemma and its corollary are crucial to all subsequent proving.

Lemma 2: *Let $r > 1$ be an integer, $M_1, ..., M_r$ submodules of M, $X = \{1, ..., r\}$. If either of the following two conditions is satisfied:*

(i) *There exists a section s for $\{M_i\}_{i\in X}$ such that $s(X)$ is disconnected and $\bigcup\limits_{i=1}^{r}((M_i + s(i)M)_{s(i)} \cap M)$ is irredundant;*

(ii) *there exists a section s for $\{M_i\}_{i\in X}$ such that $\overline{s(X)}$ is disconnected and $\bigcup\limits_{i=1}^{r}(M_i + s(i)M)$ is irredundant,*

then we have $M \neq \bigcup\limits_{i=1}^{r} M_i$.

Proof: (i) Having chosen a partition $X = X_1 \cup X_2$ according to Lemma 1, we can find an element $a_i \in s(i)\setminus \bigcup\limits_{j\in X_2} s(j)$, for each $i \in X_1$. Setting $a = \prod_{i\in X_1} a_i$, we obtain $a \in (\bigcap\limits_{i\in X_1} s(i))\setminus(\bigcup\limits_{j\in X_2} s(j))$, and likewise, we find an element $b \in (\bigcap\limits_{j\in X_2} s(j))\setminus(\bigcup\limits_{i\in X_1} s(i))$. Now let $N_i = (M_i + s(i)M)_{s(i)} \cap M$, for $i \in X$, and let $x_1 \in M\setminus \bigcup\limits_{i\in X_1} N_i$, $x_2 \in M\setminus \bigcup\limits_{i\in X_2} N_i$. Obviously, $ax_2 \in \bigcap\limits_{i\in X_1} N_i$, while $ax_2 \notin N_j$, for all $j \in X_2$. The latter becomes clear by localization at $s(j)$ and by the fact that $\frac{a}{1}$ is a unit in $A_{s(j)}$ and $(N_j)_{s(j)} = (M_j + s(j)M)_{s(j)}$. Thus, we get $ax_2 \in (\bigcap\limits_{i\in X_1} N_i)\setminus(\bigcup\limits_{j\in X_2} N_j)$ and, by analogy, $bx_1 \in (\bigcap\limits_{j\in X_2} N_j)\setminus(\bigcup\limits_{i\in X_1} N_i)$, whereby we have proved $ax_2 + bx_1 \in M\setminus \bigcup\limits_{i\in X} N_i \subset M\setminus \bigcup\limits_{i\in X} M_i$.

(ii) Let $\overline{s(X)} = P_1 \cup P_2$ be a partition according to Lemma 1. For each $i \in X$, we have either $V(s(i)) \subset P_1$ or $V(s(i)) \subset P_2$, thus obtaining a partition $X = X_1 \cup X_2$ with $V(s(i)) \subset P_1$, for all $i \in X_1$,and $V(s(i)) \subset P_2$, for all $i \in X_2$. With $N_i = M_i + s(i)M$, we can find elements $x_1 \in M \setminus \bigcup_{i \in X_1} N_i$, $x_2 \in M \setminus \bigcup_{i \in X_2} N_i$. Quite generally, we will denote the residue class of an element x by $\overline{x}$, the actual meaning of this depending on the factorization under consideration. There are ideals $\mathbf{m}_i \in m\text{-supp}\,({}^M/_{N_i})$ and $\mathbf{m}_j \in m\text{-supp}\,({}^M/_{N_j})$ such that $\overline{x}_1$ does not vanish in $({}^M/_{N_i})_{\mathbf{m}_i}$, nor does $\overline{x}_2$ in $({}^M/_{N_j})_{\mathbf{m}_j}$, for each $i \in X_1$ and $j \in X_2$, respectively. As, by Lemma 1, no element of $s(X_1)$ is contained in any element of $s(X_2)$ and vice versa, we can again find elements $a \in (\bigcap_{i \in X_1} s(i)) \setminus (\bigcup_{j \in X_2} \mathbf{m}_j)$ and $b \in (\bigcap_{j \in X_2} s(j)) \setminus (\bigcup_{i \in X_1} \mathbf{m}_i)$. An argument similar to the proof of (i) leads us to $ax_2 + bx_1 \in M \setminus \bigcup_{i=1}^{r} N_i \subset M \setminus \bigcup_{i=1}^{r} M_i$. □

Corollary: *Let M_{ij} be maximal submodules of M, $i = 1, ..., t$, $t \in \mathbf{N}$, $t > 1$, $j = 1, ..., n_i$, $n_i \in \mathbf{N}$, $n_i > 0$, with ideals $\mathbf{m}_i \in m\text{-spec}\,(A)$ such that $\mathbf{m}_i \neq \mathbf{m}_j$, for $i \neq j$, and such that there exists an A-isomorphism ${}^M/_{M_{ij}} \simeq {}^A/_{\mathbf{m}_i}$, for all i and j. Let $M' = \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} M_{ij}$ be irredundant. Then $M \neq M'$.*

Proof: Since $\text{supp}\,({}^M/_{M_{ij}}) = \{\mathbf{m}_i\}$, for all i and j, the only section s for $\{M_{ij}\}_{i=1}^{t}{}_{j=1}^{n_i}$ is defined by $s((i,j)) = \mathbf{m}_i$. $s\{(i,j) | i = 1, ..., t\,;\, j = 1, ..., n_i\}$ being apparently disconnected, $M_{ij} + \mathbf{m}_i M = M_{ij}$ renders the assertion a simple application of Lemma 1 (ii). □

If $M = \bigcup_{i=1}^{r} M_i$ is irredundant with proper submodules M_i, we can easily find a finitely generated submodule N of M (namely $N = \sum_{i=1}^{r} Ax_i$, where $x_i \in M_i \setminus \bigcup_{j \neq i} M_j$ chosen by virtue of the irredundancy assumption) such that $N = \bigcup_{i=1}^{r} (M_i \cap N)$ is irredundant, too. Note that a finitely generated A-module always contains maximal (proper) submodules containing a given submodule. To prove our main proposition, we need the first part of the following lemma the general boundary of which can be improved upon in case certain submodules of M are sure to occur in a given covering of M (parts (ii) and (iii)).

Lemma 3: *Let $M = \bigcup_{i=1}^{r} M_i$ be irredundant with submodules $M_1, ..., M_r$ of M, $r \in \mathbf{N}$, $r > 1$, and let $X = \{1, ..., r\}$. Then*

(i) *there exists an ideal $\mathbf{m} \in m\text{-supp}\,(M)$ with $\text{card}\,({}^A/_{\mathbf{m}}) < r$;*

(ii) $\sup\{\text{card}\,({}^A/_{\mathbf{m}}) | \mathbf{m} \in m\text{-supp}\,({}^M/_{M_i})$ *for some* $i \in X\} < r$.

(iii) *If $\mathbf{m} \in m\text{-supp}\,({}^M/_{M_i})$ for some $i \in X$, then at least $s = \text{card}\,({}^A/_{\mathbf{m}}) + 1$ among the submodules M_j satisfy $\mathbf{m} \in m\text{-supp}\,({}^M/_{M_j})$.*

Proof: (i) We choose a finitely generated submodule N of M such that $N = \bigcup_{i=1}^{r} (M_i \cap N)$

is irredundant, and a maximal (proper) submodule $N_i \subset N$ satisfying $N \cap M_i \subset N_i$, for each $i \in \{1, ..., r\}$. Let $\mathbf{m}_1, ..., \mathbf{m}_r$ be the corresponding maximal ideals of A, i.e. ${}^{N}/_{N_i} \simeq {}^{A}/_{\mathbf{m}_i}$, $i = 1, ..., r$. Without loss of generality, $N = \bigcup\limits_{i=1}^{r'} N_i$ is irredundant ($r' \leq r$). As an immediate consequence of the Corollary, we obtain $\mathbf{m}_1 = ... = \mathbf{m}_{r'}$, and we will denote this ideal by $\mathbf{m}$. Thus, ${}^{N}/_{N_i} \simeq {}^{A}/_{\mathbf{m}}$, $i = 1, .., r'$, implying $\mathbf{m}N \subset \bigcap\limits_{i=1}^{r'} N_i$. Applying the initially mentioned result concerning vector spaces to ${}^{A}/_{\mathbf{m}}$ and $\overline{N} = \bigcup\limits_{i=1}^{r'} \overline{N}_i$, where

$$\overline{N} = {}^{N}/_{\bigcap_{i=1}^{r'} N_i}, \quad \overline{N}_i = {}^{N_i}/_{\bigcap_{i=1}^{r'} N_i}, \quad ({}^{\overline{N}}/_{\overline{N}_i} \simeq {}^{N}/_{N_i} \neq 0),$$

we get $r \geq r' > \operatorname{card}({}^{A}/_{\mathbf{m}})$, and $\mathbf{m} \in m\text{-supp}\,(N) \subset m\text{-supp}\,(M)$, as asserted.

(ii) W.l.o.g., we consider $\mathbf{m} \in m\text{-supp}\,({}^{M}/_{M_1})$. Our assumptions ensure the existence of elements $x \in M \backslash \bigcup\limits_{i=2}^{r} M_i \subset M_1$ and x' with $ax' \notin M_1 + \mathbf{m}x'$, for all $a \in A \backslash \mathbf{m}$. Let us now examine the finite submodule $N = Ax + Ax'$ more closely.

We can assume that, for each $i \in X$, either $x' \in M_i$, or $\overline{x}'$ is not in the kernel of the canonical A-homomorphism ${}^{M}/_{M_i} \to ({}^{M}/_{M_i})_{\mathbf{m}}$. For otherwise we could replace x' by ax', with $a \in A \backslash \mathbf{m}$ and $ax' \in M_i$, whereby whatever part of the above alternative remains true or untrue, respectively, for any M_j with $j \neq i$. Thus, a partition $X = X_1 \cup X_2$ ensues, where $x' \in M_i$, for $i \in X_1$, and $x' \notin M_i + \mathbf{m}x'$, for $i \in X_2$. The choice of x' implies $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. For $i \in X_1$, we have $M_i \cap N = (M_i \cap Ax) + Ax'$, and there exists a maximal (proper) submodule $N_i \subset N$ with $M_i \cap N \subset N_i$, as well as an ideal $\mathbf{m}_i \in m\text{-supp}\,({}^{Ax}/_{M_i \cap Ax})$ such that ${}^{N}/_{N_i} \simeq {}^{A}/_{\mathbf{m}_i}$. Obviously, $x \notin N_i$, $i \in X_1$. For $i \in X_2$, $\mathbf{m} \in m\text{-supp}\,({}^{N}/_{M_i \cap N})$ since $x' \notin (M_i \cap N) + \mathbf{m}x'$; hence there exists a maximal submodule N_i with $M_i \cap N \subset N_i \subset N$ and ${}^{N}/_{N_i} \simeq {}^{A}/_{\mathbf{m}}$. W.l.o.g., $N = \bigcup\limits_{i=1}^{r'} N_1$ is irredundant ($r' \leq r$). There must be an index $i \in \{1, ..., r'\}$ such that ${}^{N}/_{N_i} \simeq {}^{A}/_{\mathbf{m}}$ as $x \notin N_i$ for any $i \in X_1$. The Corollary in the first section shows that ${}^{N}/_{N_i} \simeq {}^{A}/_{\mathbf{m}}$ for $i = 1, ..., r'$. Thus, we can consider ${}^{N}/_{\bigcap_{i=1}^{r'} N_i}$ as a vector space over ${}^{A}/_{\mathbf{m}}$, and, as above, we obtain $r \geq r' > \operatorname{card}({}^{A}/_{\mathbf{m}})$. This proves (ii).

Since the above construction can be carried out for any maximal ideal $\mathbf{m}$ satisfying our condition, (iii) is settled by a reminder of $m\text{-supp}\,({}^{N}/_{N_i}) \subset m\text{-supp}\,({}^{N}/_{N \cap M_i})$, ${}^{N}/_{N \cap M_i} \simeq {}^{M_i + N}/_{M_i}$ and $m\text{-supp}\,({}^{M_i+N}/_{M_i}) \subset m\text{-supp}\,({}^{M}/_{M_i})$. □

Proof of Proposition A: The case of ${}^{A}/_{\mathbf{m}}$ being infinite for all $\mathbf{m} \in m\text{-spec}\,(A)$ is now easily settled by noting that the irredundancy of $I \subset \bigcup\limits_{i=1}^{r+s} I_i$ is inherited by $I = \bigcup\limits_{i=1}^{r+s} (I_i \cap I)$, and that $I \not\subset I_i$, $i = 1, ...r + s$, leads to a contradiction to Lemma 3(i).

For $s = 0$, the finite-cardinality case is an equally simple consequence of Lemma 3(i). The induction for $s \geq 1$ is carried out in exactly the same way as is usually employed in the case of $r \leq 2$. □

We collect some immediate consequences of Lemma 3 in the following

Corollary: *For $r \in \mathbf{N}$, $r > 1$, let $I, I_1, ..., I_r$ be ideals of A such that $I \subset \bigcup_{i=1}^{r} I_i$ is irredundant. Then*

(i) $r > \sup\{\mathrm{card}\,({}^A/_{\mathbf{m}}) | \mathbf{m} \in m\text{-supp}\,({}^I/_{I \cap I_i}),\ i = 1, ..., r\}$;

(ii) *there exist ideals $\mathbf{m}_i \in V(I_i) \cap m\text{-spec}\,(A)$ such that $r > \max_{i=1}^{r}\{\mathrm{card}\,({}^A/_{\mathbf{m}_i})\}$; in particular, $r > \max_{i=1}^{r} \min_{V_i}\{\mathrm{card}\,({}^A/_{\mathbf{m}_i})\}$, where $V_i = V(I_i) \cap m\text{-spec}\,(A)$;*

(iii) *if $\mathbf{m} \in m\text{-supp}\,({}^I/_{I \cap I_i})$ for some $i \in \{1, ..., r\}$, there are at least $s := \mathrm{card}\,({}^A/_{\mathbf{m}}) + 1$ ideals among $I_1, ..., I_r$, say $I_{i_1}, ..., I_{i_s}$, such that $\mathbf{m} \in V(I_{i_\nu})$, $\nu = 1, ..., s$.*

Proof: (i) is an immediate consequence of Lemma 3(ii). Since, by assumption, ${}^I/_{I \cap I_i} \neq 0$, there exist ideals $\mathbf{m}_i \in m\text{-supp}\,({}^I/_{I \cap I_i}) \subset V(I_i)$. Thus, (ii) follows from (i). Lemma 3(iii) and $m\text{-supp}\,({}^I/_{I \cap I_{i_\nu}}) \subset V(I_{i_\nu})$, for all ν, ensure the existence of ideals $I_{i_1}, ..., I_{i_s}$ in (iii). $\square$

II. Residue classes and proof of Proposition B

To get some results concerning residue classes, we will now make use of what we know about submodules (ideals).

Lemma 4: *For $t \in \mathbf{N}$, $t > 1$, let $M_1, ..., M_t \subset M$ be proper submodules, $x_{ij} \in M$, $j = 1, ..., n_i$, $n_i \in \mathbf{N}$, $i = 1, ..., t$, such that $\{x_{i1}, ..., x_{in_i}\}$ does not contain a full residue system $\bmod M_i$ for any $i \in \{1, ..., t\}$.*

(i) *If* $\sum_{i=1}^{t} n_i < \min\{\mathrm{card}\,({}^A/_{\mathbf{m}}) | \mathbf{m} \in m\text{-supp}\,(M)\}$, *then* $M \neq \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} (x_{ij} + M_i)$.

If, moreover, $M = \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} (x_{ij} + M_i)$ *is irredundant, then*

(ii) $\sum_{k \neq i} n_k \geq \inf\{\mathrm{card}\,({}^A/_{\mathbf{m}}) | \mathbf{m} \in m\text{-supp}\,(M_i)\}$, *for all* $i \in \{1, ..., t\}$;

(iii) $\sum_{k \neq i} n_k \geq \inf\{\mathrm{card}\,({}^A/_{\mathbf{m}}) | \mathbf{m} \in m\text{-supp}\,({}^{M_i}/_{M_i \cap M_k}), k \neq i\}$, *for all* $i \in \{1, ..., t\}$.

Proof: (i) W.l.o.g., we assume the union of residue classes to be irredundant. Lemma 3(i) furnishes an element $x \in M \setminus \bigcup_{i=1}^{t} M_i$. Thus, there exist ideals $\mathbf{m}_i \in m\text{-supp}\,({}^M/_{M_i}) \subset m\text{-supp}\,(M)$ such that $\frac{x}{1} \neq 0$ in $({}^M/_{M_i})_{\mathbf{m}_i}$, $i = 1, ..., t$, which, in particular, means that $a \not\equiv a' \bmod \mathbf{m}_i$, $a, a' \in A$, implies $ax \not\equiv a'x \bmod M_i$. Hence, for each x_{ij}, there is at most one residue class $\bmod\,\mathbf{m}_i$ such that $ax + M_i = x_{ij} + M_i$ for some a in this class. Let $\mathbf{m}_1 = ... = \mathbf{m}_{i_1}, \mathbf{m}_{i_1+1} = ... = \mathbf{m}_{i_2}, ..., \mathbf{m}_{i_{t'-1}+1} = ... = \mathbf{m}_{i_{t'}}$, $0 < i_1 < i_2 < ... < i_{t'} = t$, $t' \leq t$. Since $\sum_{i=i_\nu+1}^{i_\nu+1} n_i < \mathrm{card}\,({}^A/_{\mathbf{m}_{i_{\nu+1}}})$, $\nu = 0, ..., t'-1$ $(i_0 = 0)$, we can find residue classes $a_{i_\nu} \bmod \mathbf{m}_{i_\nu}$, $\nu = 1, ..., t'$, such that none of the residue classes $a_{i_\nu}x + M_i$, $i = i_{\nu-1}+1, ..., i_\nu$, occurs among the $x_{ij} + M_i$. According to the Chinese Remainder Theorem, there is an element $a \in A$ satisfying $a \equiv a_{i_\nu} \bmod \mathbf{m}_{i_\nu}$, $\nu = 1, ..., t'$. But then, none of the $x_{ij} + M_i$ contains ax. (ii) Let us say $i = 1$. Suppose $x_1 \in M$ such that $x_1 + M_1$ does not occur in $\{x_{11} + M_1, ..., x_{1n_1} + M_1\}$. Hence $x_1 + M_1 \subset \bigcup_{i=2}^{t} \bigcup_{j=1}^{n_i} (x_{ij} + M_i)$, i.e. $M_1 \subset \bigcup_{i=2}^{t} \bigcup_{j=1}^{n_i} (x_{ij} - x_1 + M_i)$. If $(x_{ij} - x_1 + M_i) \cap M_1 \neq \emptyset$, then $(x_{ij} - x_1 + M_i) \cap M_1 = x'_{ij} + (M_i \cap M_1)$, with suitable elements $x'_{ij} \in M_1$. Thus, reordering our set of submodules as well as the sets of residue classes in case of necessity, we get $M_1 = \bigcup_{i=2}^{t'} \bigcup_{j=1}^{n_i'} (x'_{ij} + M_i \cap M_1)$, where $t' \leq t, n_i' \leq n_i$, and, by the irredundancy assumption, $M_1 \neq x'_{ij} + M_i \cap M_1$, for $i = 2, ..., t'$. (ii) now follows from (i). The proof of (i) shows that the general lower bound is determined by those ideals of m-supp(M) contained in m-supp$({}^{M}/_{M_i})$ for some $i \in \{1, ..., t\}$. Tracking the proof of (ii), we can apply this to M_1 and ${}^{M_1}/_{M_1 \cap M_i}$, $i \neq 1$, to prove (iii). $\square$

Proof of Proposition B: W.l.o.g., $(a_{ij} + I_i) \cap I \neq \emptyset$, for all i and j. We have $(a_{ij} + I_i) \cap I = a'_{ij} + (I_i \cap I)$ with suitable elements $a'_{ij} \in I$, as is easily checked. Since m-supp$(I) \subset m$-spec(A), the assertion is an immediate consequence of Lemma 4(i). $\square$

Remark: (i) $I \not\subset I_i$ in Proposition B is not actually a restriction as $I \subset I_i$ implies $(a + I_i) \cap I = \emptyset$ for $a \notin I_i$, thus rendering I_i completely irrelevant. (ii) Generally, the lower bound of Lemma 4(i) can not be improved upon since for any $\mathbf{m} \in m$-supp(M), M finite, there exists a maximal submodule $N \subset M$ with ${}^{M}/_{N} \simeq {}^{A}/_{\mathbf{m}}$, and so M is covered by the $s = \text{card}({}^{A}/_{\mathbf{m}})$ residue classes of N. (iii) The application of Lemma 4(ii) and (iii) to the ideal case is straightforward and requires just a fairly clear reformulation.

III. Examples. Exactness of Lower Bounds

To show that the lower bound in Proposition A (Lemma 3) can not be improved upon, we give

Example 1: Let M be a finite A-module, $\mathbf{m} \in m$-supp(M). If ${}^{M}/_{\mathbf{m}M}$ is non-cyclic, there exist submodules $M_0, M_\alpha \subset M$, $\alpha \in {}^{A}/_{\mathbf{m}}$, such that $M = (\bigcup_{\alpha \in {}^{A}/_{\mathbf{m}}} M_\alpha) \cup M_0$ is irredundant.

Proof: Assume that ${}^{M}/_{\mathbf{m}M}$ has an ${}^{A}/_{\mathbf{m}}$-basis $\{x_1, ..., x_n\}$, $n \geq 2$. If $n > 2$, we consider the ${}^{A}/_{\mathbf{m}}$-space $\overline{M} = {}^{M}/_{\mathbf{m}M + \sum_{i=3}^{n} Ax_i}$, otherwise, we set $\overline{M} = {}^{M}/_{\mathbf{m}M}$. With x_1 and x_2 now being considered as elements of $\overline{M}$, we have $\overline{M} = ({}^{A}/_{\mathbf{m}})x_1 \oplus ({}^{A}/_{\mathbf{m}})x_2$. If $\overline{M}_0 = ({}^{A}/_{\mathbf{m}})x_2$ and $\overline{M}_\alpha = ({}^{A}/_{\mathbf{m}})(x_1 + \alpha x_2)$, for each $\alpha \in {}^{A}/_{\mathbf{m}}$, then $\overline{M} = (\bigcup_{\alpha \in {}^{A}/_{\mathbf{m}}} \overline{M}_\alpha) \cup \overline{M}_0$. Indeed, $\beta \neq 0$ implies $\beta x_1 + \gamma x_2 = \beta(x_1 + \beta^{-1}\gamma x_2) \in \overline{M}_{\beta^{-1}\gamma}$, whereas $\gamma x_2 \in \overline{M}_0$ for $\beta = 0$, hence the above equality holds. $\overline{M}_\alpha \cap \overline{M}_0 = 0$, for arbitrary $\alpha \in {}^{A}/_{\mathbf{m}}$, being obvious, let $x \in \overline{M}_\alpha \cap \overline{M}_\beta$, $\alpha \neq \beta$, so that $x = \gamma(x_1 + \alpha x_2) = \gamma(x_1 + \beta x_2)$ implying $\gamma = 0$ and $x = 0$. Thus, the above equality is irredundant. M_0 and M_α, $\alpha \in {}^{A}/_{\mathbf{m}}$, of our initial assertion are obtained by taking inverse images in M of $\overline{M}_0$, $\overline{M}_\alpha$, $\alpha \in {}^{A}/_{\mathbf{m}}$. $\square$

To get an idea how this works, one may consider $\mathbf{Z}$-modules of the form $M = {}^{\mathbf{Z}}/_{n\mathbf{Z}} \oplus {}^{\mathbf{Z}}/_{n\mathbf{Z}}$, with $n \in \mathbf{N}$, $n > 1$. We have m-supp$(M) = \{n_1\mathbf{Z}, ..., n_s\mathbf{Z}\}$ if $n_1, ..., n_s$ are the prime divisors of n. Setting $N = (n_i\mathbf{Z})M$, $i \in \{1, ..., s\}$, we get $M_0 = N + \mathbf{Z}(0,1)$, $M_1 = N + \mathbf{Z}(1,1), ..., M_{n_i} = N + \mathbf{Z}(1, n_i) = N + \mathbf{Z}(1,0)$.

For the ideal case, consider $A = k[x,y]$ with a field k and indeterminates $x, y, I = (x,y)$. We have $(x,y) \in \operatorname{supp}(I)$, and ${}^{I}/_{I^2}$ is non-cyclic.

Finally, an equally simple example proves the lower bounds of Lemma 4 to be exact.

Example 2: Let $A = \mathbf{Z}$, $M = {}^{\mathbf{Z}}/_{6\mathbf{Z}} \oplus {}^{\mathbf{Z}}/_{6\mathbf{Z}}$, $M_1 = (2\mathbf{Z})M$, $M_2 = \{(0,a) | a \in {}^{\mathbf{Z}}/_{6\mathbf{Z}}\}$. No inclusion relation holds between M_1 and M_2, and we have
$M_1 = \{(0,0), (0,2), (0,4), (2,0), ..., (4,4)\}$;
$M_1 + (0,1) = \{(0,1), (0,3), ..., (4,5)\}$;
$M_2 + (2,0) = \{(2,a) | a \in {}^{\mathbf{Z}}/_{6\mathbf{Z}}\}$;
$M_2 + (4,0) = \{(4,a) | a \in {}^{\mathbf{Z}}/_{6\mathbf{Z}}\}$.
$M_1 \cup (M_1 + (0,1)) = M_2 \cup (M_2 + (2,0)) \cup (M_2 + (4,0))$ is easily checked. Hence $M = (M_1 + (1,0)) \cup (M_1 + (1,1)) \cup M_2 \cup (M_2 + (2,0)) \cup (M_2 + (4,0))$ is irredundant. $\operatorname{supp}(M_1) = \{3\mathbf{Z}\}$ and $\operatorname{supp}(M_2) = \{2\mathbf{Z}, 3\mathbf{Z}\}$ show that the present example hits the infimum of Lemma 4(ii).

References

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