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Homology of Singular Cubic Surfaces in P(3,C)

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Abstract. An 1871 paper by Diekmann contains all the information needed to compute the integer singular homology for all classes of not ruled singular cubic surfaces.

Three linearly independent quadrics in projective threespace define a cubic Cremona transformation that associates to any point x the intersection of the polar planes of x with respect to the three quadrics. That intersection is a point y except if x is a vertex of a common polar tetrahedron of two quadrics. In that case, the image of x is a projective line. The locus of the exceptional points is a sextic space curve. Any cubic surface in space that is not ruled (i.e., is neither Cayley's surface nor a cone over an elliptic curve) can be obtained as the image of some projective plane in the map just described, known as Grassmann's generation of cubic surfaces [3], [11], [8]. If the six points do not lie on a conic and no three of them are collinear then the surface is regular; the images of the six conics through any five points and the fifteen diagonals give the 27 straight lines on the surface. Since a cubic surface with a singular line is ruled, the classification of the singular cubic surfaces with isolated singularities is reduced to a study of the admissible configurations of six points in the complex projective plane. There are much more than 21 combinatorially distinct configurations; several configurations can give surfaces of the same Schläfli class. A study of these equivalences and the determination of the nature of the singularities from the plane configuration in the case of one singular point has been given by Diekmann [2]. Diekmann's work leads immediately to a method for the determination of the singular homology over **Z** for all singular cubic surfaces that are not ruled. The method used was developed by the author long ago [4], [5], [6]. The connection with the counting formulas of algebraic geometry is not obvious in all cases.

A modern classification of cubic surfaces is given by J.W. Bruce and C.T.C. Wall [1]; they give the classification in terms of Arnol'd's classification of singularities. In addition, they very carefully determine the moduli of the normal forms for the singularities; these are missing in most of the normal forms given by [10], pp. 324-325. The geometry of the singular cubic surfaces has been studied in detail by G. Salmon [9] from the algebraic point

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of view and by R. Sturm [12] and C. Juel [7] as exercises in order geometry. It turns out that in all cases the homology of a cubic surface obtainable by Grassmann's generation is torsion free in all dimensions (the result is nontrivial only in dimension 2.) It should be noted that the nature of the singularities does play no role in our determination of homology groups, all that is needed is the set on the surface on which the Grassmann relation is not one-to-one. If there are several (≤ 4) singular points, they are necessarily connected by lines that are part of the singular locus. In our discussion, we retain Cayley's notation of the singular points for the convenience of the reader who might want to study Diekmann's original article.

1. Exact sequences associated with the Grassmann relation

Let S be a complex projective plane in $\mathbf{P}(3,\mathbf{C})$ and G the configuration of singular points. The exact sequence of singular homology over $\mathbf{Z}$ is

$$\begin{gathered} 0 \to H_4(S) \to H_4(S,G) \to H_3(G) \to H_3(S) \to H_3(S,G) \to \\ \to H_2(G) \to H_2(S) \to H_2(S,G) \to H_1(G) \to H_1(S) \to H_1(S,G) \to \\ \to H_0(G) \to H_0(S) \to 0 \end{gathered}$$

In terms of real geometry, G is a configuration of points or two-spheres, and $H_{2k}(S) = \mathbf{Z}$, $H_{2k+1}(S) = 0$. The sequence reduces to

$$\begin{gathered} 0 \to \mathbf{Z} \to H_4(S,G) \to 0 \\ 0 \to H_3(S,G) \to H_2(G) \to \mathbf{Z} \to H_2(S,G) \to H_1(G) \to 0 \\ 0 \to H_1(S,G) \to H_0(G) \to \mathbf{Z} \to 0 \end{gathered} \tag{1}$$

Now let Σ be the cubic surface and Γ the singular set of the Grassmann relation. Σ is an orientable pseudomanifold of dimension 4, $H_4(\Sigma) = \mathbf{Z}$. Then $\Sigma \backslash \Gamma$ is an analytic one-to-one image of $S \backslash G$ and $H_k(S,G) \approx H_k(\Sigma,\Gamma)$. Γ is a configuration of points and real two-spheres, $H_3(\Gamma) = 0$. The nontrivial part of the exact sequence for Σ reduces to

$$\begin{gathered} 0 \to H_3(\Sigma) \to H_3(S,G) \to H_2(\Gamma) \to H_2(\Sigma) \to H_2(S,G) \to H_1(\Gamma) \to \\ \to H_1(\Sigma) \to H_1(S,G) \to H_0(\Gamma) \to \mathbf{Z} \to 0. \end{gathered} \tag{2}$$

We shall denote the direct sum of k copies of $\mathbf{Z}$ by $k \times \mathbf{Z}$.

In the nonsingular case which is well known, Γ is the disjoint union of six real 2-spheres and G the union of six distinct points. Hence, $H_k(G) = 0$ for $k > 0$ and $H_0(G) = 6 \times \mathbf{Z}$. From (1) it follows that $H_3(S,G) = 0$, $H_2(S,G) = \mathbf{Z}$, $H_1(S,G) = 5 \times \mathbf{Z}$. The exact sequence for Σ now becomes

$$\begin{gathered} 0 \to H_3(\Sigma) \to 0 \\ 0 \to H_2(\Gamma) \to H_2(\Sigma) \to \mathbf{Z} \to 0 \\ 0 \to H_1(\Sigma) \to 5 \times \mathbf{Z} \to 6 \times \mathbf{Z} \to \mathbf{Z} \to 0. \end{gathered}$$

Hence, $H_3(\Sigma) = 0$. Since $H_2(\Gamma) = 6 \times \mathbf{Z}$, it follows that $H_2(\Sigma) = 7 \times \mathbf{Z}$. From the last sequence it follows that $H_1(\Sigma)$ has no torsion and, hence, is zero. These results are all well known. In the general case, it is still true that $H_k(G) = H_k(\Gamma) = 0$ for $k > 2$ and that $H_2(G)$ and $H_2(\Gamma)$ are torsion free. Singular irreducible cubic surfaces are connected orientable pseudomanifolds of real dimension 4, $H_o(\Sigma) = H_4(\Sigma) = \mathbf{Z}$. We study first the surfaces with only one singular point.

2. One Conical Singular Point C_2 (Schläfli Class 2, Singularity Type A_1).

The six points of intersection of the sextic with the given plane are on a nondegenerate conic. The image of the conic in the transformation is a curve of degree six that contains six concurrent straight lines. Hence, these lines make up the curve. The point of concurrence is a conical singular point. From this point of view, one has to do some work to prove the absence of torsion in dimension two since the inclusion map $i_* : H_2(G) \to H_2(S)$ is multiplication by 2. Now Diekmann [2, p.453] has shown that the same singularity is obtained if three of the points are collinear and the other three in general position. Hence, the exceptional set G is the union of a projective line and three disjoint points:

$$H_2(G) = \mathbf{Z}, H_1(G) = 0, H_0(G) = 4 \times \mathbf{Z}.$$

The generator of $H_2(G)$ is a complex projective line; $i_* : H_2(G) \to H_2(S)$ is the identity operator. From the exact sequence one has

$$H_3(S,G) = H_2(S,G) = 0, H_1(S,G) = 3 \times \mathbf{Z}.$$

The exceptional set Γ is the union of three skew and three concurrent complex projective lines,

$$H_2(\Gamma) = 6 \times \mathbf{Z}, H_1(\Gamma) = 0, H_0(\Gamma) = 4 \times \mathbf{Z}.$$

The exact sequence reduces to

$$0 \to H_3(\Sigma) \to 0 \to 6 \times \mathbf{Z} \to H_2(\Sigma) \to 0$$
$$0 \to H_1(\Sigma) \to 0, 0 \to 3 \times \mathbf{Z} \to 4 \times \mathbf{Z} \to \mathbf{Z} \to 0,$$

hence, $H_2(\Sigma) = 6 \times \mathbf{Z}$, $H_3(\Sigma) = H_1(\Sigma) = 0$.

There is a third Diekmann representation, by one double and four isolated points, which leads to the same result. This representation is convenient in the study of surfaces with more than one singular point.

3. Biplanar Node B_3, Surface of Class 9 (Schläfli Class 3, Singularity Type A_2)

In this case, the tangent cone degenerates into two planes; their line of intersection is not a line of Σ. We choose the Diekmann representation [2, p.458] derived from the one used in the preceding case if two of the collinear points coincide, giving the direction of the line. This does not change the homology of G but Γ now is the union on only five lines, two intersecting and three skew. Hence, $H_2(\Gamma) = 5 \times \mathbf{Z}$ and $H_2(\Sigma) = 5 \times \mathbf{Z}$; the other groups remain unchanged.

Another Diekmann representation is available where G is the union of a triple point of the exceptional curve and three single points.

4. Biplanar Node B_4, Surface of Class 8 (Schläfli Class 5, Singularity Type A_3)

In this case, the intersection of the two planes of the tangent cone is a line on Σ, a part of Γ. We choose the Diekmann representation [2, p.462] where G is formed by two lines, each carrying two singular points, and intersecting at a singular double point representing a direction different from both given lines. In this case, $H_2(G) = 2 \times \mathbf{Z}, H_1(G) = 0$; $H_2(\Gamma) = 5 \times \mathbf{Z}, H_1(\Gamma) = 0$, and a computation as before yields $H_2(\Sigma) = 4 \times \mathbf{Z}$, $H_3(\Sigma) = H_1(\Sigma) = 0$.

5. Biplanar Node B_5, Surface of Class 7 (Schläfli Class 7, Singularity Type A_4)

This is the preceding case, but now the intersection of the two tangent planes at the node belongs the tangent plane at every one of its points. Diekmann has 6 different representations of the Grassmann map, of which we shall study two. The first one is obtained from the representation of the previous case, but now the intersection of the two singular lines is a point of multiplicity 3 in the direction of one of the lines. That line carries another singular point; the other line carries two. The homology of G is unchanged but $H_2(\Gamma) = 4 \times \mathbf{Z}$, giving $H_2(\Sigma) = 3 \times \mathbf{Z}$.

In the second representation, G is formed by a line carrying a point of multiplicity 4 in the direction of the line and two other simple singular points. Then $H_2(G) = \mathbf{Z}, H_3(S, G) = H_2(S, G) = 0, H_2(\Gamma) = 3 \times \mathbf{Z}$ and from the exact sequence $0 \to 3 \times \mathbf{Z} \to H_2(\Sigma) \to 0$ we again get $H_2(\Sigma) = 3 \times \mathbf{Z}$, but from a different computation.

6. Biplanar Node B_6, Surface of Class 6 (Schläfli Class 11, Singularity Type A_5)

Here the intersection of the two tangent planes osculates the surface, i.e., has contact of order > 1 at the node. Again, here we can use two of Diekmann's representations [2, pp. 468-471]. In the first, G is composed of two lines, the point of intersection is of order 4, blowing up in Γ into two lines, one of which is given by a triple point. The other line carries two more points that are blown up into projective lines. In the other representation there is only one line carrying three points of multiplicity 3, one of which determines the direction of the line; the other blowing up into two lines. In both cases, the computations reduce to the two cases given by B_5, $H_2(\Sigma) = 3 \times \mathbf{Z}$.

7. Uniplanar node U_6, Surface of Class 6 (Schläfli Class 12, Singularity Type D_4)

The tangent cone becomes a tangent plane counted twice. The two Diekmann configurations [2, p.471-472] are either three collinear points of multiplicity 2, representing three distinct directions, or one simple point and one point of multiplicity 4 representing two directions. In both cases, $H_2(G) = \mathbf{Z}$, $H_3(S,G) = H_2(S,G) = 0, H_2(\Gamma) = 3 \times \mathbf{Z}$, and $H_2(\Sigma) = 3 \times \mathbf{Z}$, see B_5.

8. Uniplanar node U_7, Surface of Class 5 (Schläfli Class 15, Singularity Type D_5)

This is obtained from U_6 if two of the double points coincide. In all cases, $H_2(G) = \mathbf{Z}$ implies $H_3(S,G) = H_2(S,G) = 0$ and $H_2(\Sigma) = H_2(\Gamma)$. Here $H_2(\Gamma) = 2 \times \mathbf{Z}$ since Γ is composed of two (multiple) lines. There are two other configurations that give the same result.

9. Uniplanar node U_8, Surface of Class 4 (Schläfli Class 20, Singularity Type E_6)

Here the double tangent plane becomes osculating, all points coincide; $H_2(\Sigma) = 2 \times \mathbf{Z}$.

10. Two Conical Singular Points $2C_2$, Surface of Class 8, Schläfli Class 4

From the discussion of one conical point, we see that here G is composed of two lines, each one carrying 3 points that are blown up into lines; none of the points is the intersection of the two lines. Hence, $H_2(G) = 2 \times \mathbf{Z}$, $H_1(G) = 0$. In the exact sequence

$$0 \to H_3(S,G) \to 2 \times \mathbf{Z} \to^{i}_{*} \mathbf{Z} \to H_2(S,G) \to 0$$

the map i_* is of degree one; hence, $H_2(S,G) = 0$ and $H_3(S,G) = \mathbf{Z}$. Now $H_2(\Gamma) = 6 \times \mathbf{Z}$. The interesting part of the exact sequence for Σ is

$$0 \to H_3(\Sigma) \to \mathbf{Z} \to 6 \times \mathbf{Z} \to H_2(\Sigma) \to 0.$$

From the previous sequence, we see that $\partial_* H_3(\Sigma,\Gamma)$ is an isomorphism; hence, $H_3(\Sigma) = 0$ and $H_2(\Sigma) = 5 \times \mathbf{Z}$.

11. Several Singularities

All remaining cases of the Schläfli classification are combinations of the simple singularities discussed above and can be obtained from the previous discussion.

From the combinatorial point of view there is a slight difference in the treatment of cases where $H_1(G) \neq 0$ and those, as in all preceding cases, where $H_1(G) = 0$. For example, the exceptional set G for 3 conical singular points ($3C_2$, Schläfli class 8) is a complete triangle on which the three vertices and another point on each of the sides are blown up. The sides of the triangle are three real 2-spheres that touch one another at one point only. Hence, $H_1(G) = \mathbf{Z}$. The set Γ is the union of a complete triangle and a complex projective line through each one of the vertices; the additional lines are skew to one another. Hence, $H_1(\Gamma) = \mathbf{Z}$ also. As before, the injection map $i_* : H_2(G) \to H_2(S)$ is surjective and one easily obtains $H_2(\Sigma) = 4 \times \mathbf{Z}$.

The case $4C_2$, Schläfli class 16, is treated similarly. Here G is formed by a complete quadrilateral, the singular points are the six vertices. Γ is formed by the six lines of a tetrahedron. Looking at the real configurations, we see that all the closed 1-circuits on the spheres representing the complex projective lines are homotopic, and therefore homologous: $H_1(G) = \mathbf{Z}$, $H_1(\Gamma) = \mathbf{Z}$, and $H_2(\Sigma) = 3 \times \mathbf{Z}$.

The only other case for which $H_1(G) = \mathbf{Z}$ is $3B_3$ where G is formed by a triangle whose vertices are double points representing the direction of a side of the triangle; here $H_2(\Sigma) = \mathbf{Z}$. In all other cases, there are no closed nontrivial lines on G. We give a quick table of the remaining cases:

singularity	B_3C_2	$2B_3$	B_4C_2	B_32C_2	B_5C_2	$2B_3C_2$	B_42C_2	B_6C_2
2nd Betti no.	4	3	3	4	2	2	2	1

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