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The Trace Decomposition Problem

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Abstract. The problem of decomposition of mixed tensor spaces by the trace operation is considered. It is shown that a tensor $A = \left(A^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_q}\right)$ can always be expressed as the sum of a traceless term and a linear combination of the Kronecker's δ-tensor, with traceless coefficients. The uniqueness of this decomposition is discussed. Particular cases of the trace decomposition of tensor spaces of type $(1,2)$, $(1,3)$, and $(2,2)$ including the cases of the torsion and curvature tensors, are considered, and explicit decomposition formulas are given. This analysis enables us to derive the Weyl projective, and Weyl conformal curvature tensors.

Key words: Trace, traceless tensor, tensor space decomposition, torsion tensor, curvature tensor.

MSC 1991: 15A72, 53A55

1. Introduction

Let E be a real, n-dimensional vector space, E^p_q the vector space of tensors of type (p,q) on E. We always use a fixed basis of E, and the induced basis of E^p_q. Thus, if $A \in E^p_q$ is a tensor, we write for simplicity $A = \left(A^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\right)$. We say that a tensor $A \in E^p_q$ is *traceless*, if its traces are all zero. We formulate the *trace decomposition problem* as the problem of existence and uniqueness of a decomposition of a tensor in which one term is traceless, and the remaining terms are a linear combination of the Kronecker's δ-tensor.

The problem of decomposition of tensors by the trace operation was studied by H. Weyl as a part of representation theory of the orthogonal group [6]. This paper is partially motivated by Weyl's theory.

However, main reason for our study consists in investigation of the structure of modules of differential forms on jet prolongations of fibered manifolds which appear in global variational theory. The decompositions apply to components of forms, and result in criteria of contactness and strong contactness (see [1], [4]). Within this context, the trace

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decomposition problem appears as a problem of solving some *systems of linear algebraic equations*, with no reference to the group representation theory.

In this paper we adopt this point of view. It is easily seen, however, that the resulting trace decompositions of *tensors* are automatically invariant with respect to the tensor representations of the *general linear group*. The corresponding invariant subspaces are not, in general, indecomposable. In fact, complete decompositions of a tensor space can be obtained by applying the Young-Kronecker theory to both contravariant, and covariant indices of every summand in its trace decomposition. It is not our aim in this paper to combine the trace decomposition theory and the Young-Kronecker theory (see [5]).

In Sec. 2 we prove two basic theorems providing a complete solution of the trace decomposition problem in its algebraic setting. The methods of the proof do not give us a general trace decomposition formula. In Sec. 3 we consider some particular cases which are important in differential geometry. We give *explicit decomposition formulas* for tensor spaces of types $(1,1)$, $(1,2)$, $(1,3)$, and $(2,2)$. As one would expect, the formulas describing the trace decomposition strongly depend on the dimension of the underlying vector space. In Sec. 4 we apply these formulas to the *torsion* and *curvature tensors* on a manifold endowed with a linear connection or a metric. Extensive literature on this subject from different perspectives can be found. For example, in [2], the case of a torsionfree connection is considered on a vector space endowed with a positive definite metric, and a *complete* decomposition of the underlying tensor space, invariant with respect to the group $\mathrm{SO}(n)$, is obtained. Our approach consists in a direct analysis of linear algebraic equations, defining the trace decompositions. We show that these decompositions lead to the *Weyl projective* and *conformal curvature tensors* well-known in differential geometry of linear connections and metric fields. These tensors are obtained as curvatures which are invariant under projective and conformal transformations respectively (see [7]). In this paper we show that the trace decomposition theory *defines* these tensors uniquely as *traceless components* in the trace decompositions of the corresponding tensor spaces. For complete proofs of the trace decomposition formulas for tensors of types $(1,2)$ and $(1,3)$ we refer to [3].

2. The trace decomposition

Let p, q, and n be positive integers such that $p \leq q$, and let E^p_q denote the vector space of systems of real numbers $A = \left(A^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_q}\right)$, where $1 \leq i_1, i_2, \dots, i_p, k_1, k_2, \dots, k_q \leq n$. A system $A \in E^p_q$ is said to be of type (p,q).

A system $A = \left(A^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_q}\right) \in E^p_q$ is called *traceless*, if its traces are all zero. A is called δ^1*-generated* if there exist systems

$$B^{(r)}_{(s)} = \left(B^{(r)\, i_1 i_2 \dots i_{p-1}}_{(s)\, k_1 k_2 \dots k_{q-1}}\right) \in E^{p-1}_{q-1},\ 1 \leq r \leq p,\ 1 \leq s \leq q,\qquad (1)$$
$$1 \leq i_1, i_2, \dots, i_{p-1}, k_1, k_2, \dots, k_{q-1} \leq n$$

such that every element $A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q}$ can be expressed in the form

$$\begin{aligned} A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q} = {} & \delta^{i_1}_{k_1} B^{(1)\, i_2 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} + \delta^{i_1}_{k_2} B^{(1)\, i_2 i_3 \ldots i_p}_{(2)\, k_1 k_3 \ldots k_q} + \ldots + \delta^{i_1}_{k_q} B^{(1)\, i_2 i_3 \ldots i_p}_{(q)\, k_1 k_2 \ldots k_{q-1}} + \\ & \delta^{i_2}_{k_1} B^{(2)\, i_1 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} + \delta^{i_2}_{k_2} B^{(2)\, i_1 i_3 \ldots i_p}_{(2)\, k_1 k_3 \ldots k_q} + \ldots + \delta^{i_2}_{k_q} B^{(2)\, i_1 i_3 \ldots i_p}_{(q)\, k_1 k_2 \ldots k_{q-1}} + \\ & \ldots \\ & \delta^{i_p}_{k_1} B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(1)\, k_2 k_3 \ldots k_q} + \delta^{i_p}_{k_2} B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(2)\, k_1 k_3 \ldots k_q} + \ldots + \delta^{i_p}_{k_q} B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(q)\, k_1 k_2 \ldots k_{q-1}}. \end{aligned}$$

The systems (1) are then called the δ^1-*components* of the the δ^1-*generated* system A. It is easily seen that the components as introduced above need not be defined uniquely. A δ^1-generated system $A = \left(A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q}\right)$ is called δ^2-*generated* if it has δ^1-components that are δ^1-generated (in E^{p-1}_{q-1}). Proceeding on induction, we define a δ^r-*generated* system, where $1 \le r \le p$, as a δ^{r-1}-generated system whose δ^1-components are all δ^{r-1}-generated. To complete the definition, we call any system $A = \left(A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q}\right)$ to be δ^0-*generated*. The *component* of a δ^0-generated system is unique, and coincides with this system. Clearly, a δ^r-generated system is always δ^{r-1}-generated. It is now evident how to introduce the δ^r-*components* of a δ^r-generated system.

A δ^r-generated system $A = \left(A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q}\right)$, where $0 \le r \le p-1$, whose δ^r-components are traceless (in E^{p-r}_{q-r}) is said to be *relatively traceless*. A δ^0-generated, relatively traceless system is traceless.

Let $A \in E^p_q$. The *trace decomposition problem* can now be formulated as the problem of existence and uniqueness of δ^r-generated, relatively traceless systems $A^{(r)} \in E^p_q$, where $0 \le r \le p-1$, and a δ^p-generated system $A^{(p)} \in E^p_q$ such that $A = A^{(0)} + A^{(1)} + A^{(2)} + \ldots + A^{(p)}$. To solve this problem, we begin by proving the following result.

Theorem 1. *Let p, q, n be positive integers, $p \le q$, and let $A = \left(A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q}\right) \in E^p_q$. There exist a system $B = \left(B^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q}\right)$ and systems $B^{(r)}_{(s)} = \left(B^{(r)\, i_1 i_2 \ldots i_{p-1}}_{(s)\, k_1 k_2 \ldots k_{q-1}}\right)$, where $1 \le r \le p$, $1 \le s \le q$, such that*

$$\begin{aligned} A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q} = {} & B^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q} + \delta^{i_1}_{k_1} B^{(1)\, i_2 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} + \delta^{i_1}_{k_2} B^{(1)\, i_2 i_3 \ldots i_p}_{(2)\, k_1 k_3 \ldots k_q} + \ldots + \delta^{i_1}_{k_q} B^{(1)\, i_2 i_3 \ldots i_p}_{(q)\, k_1 k_2 \ldots k_{q-1}} + \\ & \delta^{i_2}_{k_1} B^{(2)\, i_1 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} + \delta^{i_2}_{k_2} B^{(2)\, i_1 i_3 \ldots i_p}_{(2)\, k_1 k_3 \ldots k_q} + \ldots + \delta^{i_2}_{k_q} B^{(2)\, i_1 i_3 \ldots i_p}_{(q)\, k_1 k_2 \ldots k_{q-1}} + \\ & \ldots \\ & \delta^{i_p}_{k_1} B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(1)\, k_2 k_3 \ldots k_q} + \delta^{i_p}_{k_2} B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(2)\, k_1 k_3 \ldots k_q} + \ldots + \delta^{i_p}_{k_q} B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(q)\, k_1 k_2 \ldots k_{q-1}} \end{aligned} \tag{2}$$

and

$$\begin{aligned} & B^{s i_2 \ldots i_p}_{s k_2 \ldots k_q} = 0, \quad B^{s i_2 \ldots i_p}_{k_1 s k_3 \ldots k_q} = 0, \quad \ldots, \quad B^{s i_2 \ldots i_p}_{k_1 k_2 \ldots k_{q-1} s} = 0, \\ & B^{i_1 s i_3 \ldots i_p}_{s k_2 \ldots k_q} = 0, \quad B^{i_1 s i_3 \ldots i_p}_{k_1 s k_3 \ldots k_q} = 0, \quad \ldots, \quad B^{i_1 s i_3 \ldots i_p}_{k_1 k_2 \ldots k_{q-1} s} = 0, \\ & \ldots \\ & B^{i_1 i_2 \ldots i_{p-1} s}_{s k_2 \ldots k_q} = 0, \quad B^{i_1 i_2 \ldots i_{p-1} s}_{k_1 s k_3 \ldots k_q} = 0, \quad \ldots, \quad B^{i_1 i_2 \ldots i_{p-1} s}_{k_1 k_2 \ldots k_{q-1} s} = 0. \end{aligned} \tag{3}$$

The system B is unique. If $p+q \le n+1$, the systems $B^{(r)}_{(s)}$ are also unique.

Proof. Let $g = (g_{ij})$ be any regular, positive definite matrix of dimension $n \times n$ defining a scalar product on $\mathbf{R}^n$. g defines a scalar product on E^p_q by the formula

$$(A|B) = g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_p j_p} g^{k_1 l_1} g^{k_2 l_2} \dots g^{k_q l_q} A^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_q} B^{j_1 j_2 \dots j_p}_{l_1 l_2 \dots l_q}. \tag{4}$$

Consider the subspace of E^p_q generated by the systems $X = \left(X^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \right)$ of the form

$$\begin{aligned} X^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_q} = {} & \delta^{i_1}_{k_1} X^{(1)\, i_2 i_3 \dots i_p}_{(1)\, k_2 k_3 \dots k_q} + \delta^{i_1}_{k_2} X^{(1)\, i_2 i_3 \dots i_p}_{(2)\, k_1 k_3 \dots k_q} + \dots + \delta^{i_1}_{k_q} X^{(1)\, i_2 i_3 \dots i_p}_{(q)\, k_1 k_2 \dots k_{q-1}} + \\ & \delta^{i_2}_{k_1} X^{(2)\, i_1 i_3 \dots i_p}_{(1)\, k_2 k_3 \dots k_q} + \delta^{i_2}_{k_2} X^{(2)\, i_1 i_3 \dots i_p}_{(2)\, k_1 k_3 \dots k_q} + \dots + \delta^{i_2}_{k_q} X^{(2)\, i_1 i_3 \dots i_p}_{(q)\, k_1 k_2 \dots k_{q-1}} + \\ & \dots \\ & \delta^{i_p}_{k_1} X^{(p)\, i_1 i_2 \dots i_{p-1}}_{(1)\, k_2 k_3 \dots k_q} + \delta^{i_p}_{k_2} X^{(p)\, i_1 i_2 \dots i_{p-1}}_{(2)\, k_1 k_3 \dots k_q} + \dots + \delta^{i_p}_{k_q} X^{(p)\, i_1 i_2 \dots i_{p-1}}_{(q)\, k_1 k_2 \dots k_{q-1}}. \end{aligned}$$

The orthogonal complement of this subspace is formed by all systems $B = \left(B^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \right)$ such that $(B|X) = 0$ for all X. Expressing the left side explicitly, we easily see that the orthogonal complement consists of the *traceless* systems $B = \left(B^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \right) \in E^p_q$.

If now $A = \left(A^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \right)$ is an arbitrary element of E^p_q, we can express it as the sum of its projections onto these complementary orthogonal subspaces. This proves the existence of the desired decomposition. The uniqueness of the system $B = \left(B^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \right)$ is obvious.

Now we prove that if $p+q \le n+1$, then the systems $B^{(r)}_{(s)} = \left(B^{(r)\, i_1 i_2 \dots i_{p-1}}_{(s)\, k_1 k_2 \dots k_{q-1}} \right)$ are unique. We wish to show that under this assumption, the system (2), (3) has a unique solution. It is sufficient to show that the homogeneous system

$$\begin{aligned} & \delta^{i_1}_{k_1} B^{(1)\, i_2 i_3 \dots i_p}_{(1)\, k_2 k_3 \dots k_q} + \delta^{i_1}_{k_2} B^{(1)\, i_2 i_3 \dots i_p}_{(2)\, k_1 k_3 \dots k_q} + \dots + \delta^{i_1}_{k_q} B^{(1)\, i_2 i_3 \dots i_p}_{(q)\, k_1 k_2 \dots k_{q-1}} + \\ & \delta^{i_2}_{k_1} B^{(2)\, i_1 i_3 \dots i_p}_{(1)\, k_2 k_3 \dots k_q} + \delta^{i_2}_{k_2} B^{(2)\, i_1 i_3 \dots i_p}_{(2)\, k_1 k_3 \dots k_q} + \dots + \delta^{i_2}_{k_q} B^{(2)\, i_1 i_3 \dots i_p}_{(q)\, k_1 k_2 \dots k_{q-1}} + \\ & \dots \\ & \delta^{i_p}_{k_1} B^{(p)\, i_1 i_2 \dots i_{p-1}}_{(1)\, k_2 k_3 \dots k_q} + \delta^{i_p}_{k_2} B^{(p)\, i_1 i_2 \dots i_{p-1}}_{(2)\, k_1 k_3 \dots k_q} + \dots + \delta^{i_p}_{k_q} B^{(p)\, i_1 i_2 \dots i_{p-1}}_{(q)\, k_1 k_2 \dots k_{q-1}} = 0 \end{aligned} \tag{5}$$

has the trivial solution only. Choose a basis $\xi_1, \xi_2, \dots, \xi_n$ of the vector space $\mathbf{R}^n$, and write in components (with respect to the canonical basis) $\xi_j = \left(\xi^i_j \right)$. Analogously, denote by $\omega^1, \omega^2, \dots, \omega^n$ elements of the dual basis of the dual vector space $\mathbf{R}^{n*}$, and write $\omega^i = \left(\omega^i_k \right)$. By definition,

$$\xi^j_i \omega^i_k = \delta^j_k, \quad 1 \le j, k \le n. \tag{6}$$

Choose an index j such that $1 \le j \le n$. Since $p+q \le n+1$, one can always choose indices $r_1, r_2, \dots, r_{p-1}$ and $s_1, s_2, \dots, s_{q-1}$, all mutually different, such that $j \ne r_1, r_2, \dots, r_{p-1}$,

$s_1, s_2, \ldots, s_{q-1}$. Let for example j be equal to 1. Then $r_1 = 2, r_2 = 3, \ldots, r_{p-1} = p$ and $s_1 = p+1, s_2 = p+2, \ldots, s_{q-1} = p+q-1$, where by hypothesis, $p+q-1 \le n$. Multiplying (5) by

$$w^{k_1 k_2 \ldots k_q}_{i_1 i_2 \ldots i_p} = \xi^{k_1}_1 \xi^{k_{p+1}}_{p+1} \xi^{k_{p+2}}_{p+2} \ldots \xi^{k_{p+q-1}}_{p+q-1} \omega^1_{i_1} \omega^2_{i_2} \ldots \omega^p_{i_p}$$

we obtain

$$\begin{gathered} B^{(1)\, i_2 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} w^{s k_2 \ldots k_q}_{s i_2 i_3 \ldots i_p} + B^{(1)\, i_2 \ldots i_p}_{(2)\, k_1 k_3 \ldots k_q} w^{k_1 s \ldots k_q}_{s i_2 i_3 \ldots i_p} + \ldots + B^{(1)\, i_2 \ldots i_p}_{(q)\, k_1 k_2 \ldots k_{q-1}} w^{k_1 k_2 \ldots k_{q-1} s}_{s i_2 i_3 \ldots i_p} + \\ B^{(2)\, i_1 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} w^{s k_2 \ldots k_q}_{i_1 s i_3 \ldots i_p} + B^{(2)\, i_1 i_3 \ldots i_p}_{(2)\, k_1 k_3 \ldots k_q} w^{k_1 s \ldots k_q}_{i_1 s i_3 \ldots i_p} + \ldots + B^{(2)\, i_1 i_3 \ldots i_p}_{(q)\, k_1 k_2 \ldots k_{q-1}} w^{k_1 k_2 \ldots k_{q-1} s}_{i_1 s i_3 \ldots i_p} + \\ \ldots \\ B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(1)\, k_2 k_3 \ldots k_q} w^{s k_2 \ldots k_q}_{i_1 i_2 \ldots i_{p-1} s} + B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(2)\, k_1 k_3 \ldots k_q} w^{k_1 s \ldots k_q}_{i_1 i_2 \ldots i_{p-1} s} + \ldots + B^{(p)\, i_1 i_2 \ldots i_{p-1}}_{(q)\, k_1 k_2 \ldots k_{q-1}} w^{k_1 k_2 \ldots k_{q-1} s}_{i_1 i_2 \ldots i_{p-1} s} = 0. \end{gathered}$$

Using (6) this equation reduces to

$$B^{(1)\, i_2 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} \xi^{k_2}_{p+1} \xi^{k_3}_{p+2} \ldots \xi^{k_q}_{p+q-1} \omega^2_{i_2} \ldots \omega^p_{i_p} = 0. \tag{7}$$

Since (7) holds for all $\xi^{k_2}_{p+1}, \xi^{k_3}_{p+2}, \ldots, \xi^{k_q}_{p+q-1}, \omega^2_{i_2}, \omega^3_{i_3}, \ldots, \omega^p_{i_p}$, we have

$$B^{(1)\, i_2 i_3 \ldots i_p}_{(1)\, k_2 k_3 \ldots k_q} = 0.$$

Proceeding in the same way we obtain

$$B^{(r)\, i_1 i_2 \ldots i_{p-1}}_{(s)\, k_1 k_2 \ldots k_{q-1}} = 0, \quad 1 \le r \le p, \quad 1 \le s \le q,$$

which means that the system (5) has the trivial solution only.

Formulas (2) and (3) of Theorem 1 state the existence and uniqueness of the *first trace decomposition* of a system $A = \left(A^{i_1 i_2 \ldots i_p}_{k_1 k_2 \ldots k_q}\right) \in E^p_q$.

Remark. Since the *traceless component* $B \in E^p_q$ of $A \in E^p_q$ is unique, its complementary component is also unique. However, this fact does not imply, in general, the uniqueness of the systems $B^{(r)}_{(s)} \in E^{p-1}_{q-1}$. Example $p = 1$, $q = 3$, $n = 2$ shows that if the inequality $p + q \le n + 1$ is violated, then the systems $B^{(r)}_{(s)}$ might not be unique (see Sec. 3).

Now we are in a position to formulate the following result.

Theorem 2. *Let p, q, and n be positive integers such that $p \le q$, and let $A \in E^p_q$. There exist unique δ^r-generated, relatively traceless systems $A^{(r)} \in E^p_q$, where $0 \le r \le p-1$, and a $\delta^{(p)}$-generated system $A^{(p)} \in E^p_q$, such that*

$$A = A^{(0)} + A^{(1)} + A^{(2)} + \ldots + A^{(p)}. \tag{8}$$

Proof. Consider a system $A = \left(A^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_q}\right) \in E^p_q$. By Theorem 1, we express $A^{i_1 i_2 \dots i_p}_{k_1 k_2 \dots k_q}$ in the form (2). Then we can apply the decomposition (2) to each of the systems $B^{(r)}_{(s)} = \left(B^{(r)\, i_1 i_2 \dots i_{p-1}}_{(s)\, k_1 k_2 \dots k_{q-1}}\right)$. Repeating this procedure p times we obtain the desired decomposition (8). This proves existence.

To prove uniqueness, it is sufficient to show that for every pair of indices m, t such that $0 \le m < t \le p$, the components $A^{(m)}, A^{(t)}$ of A are orthogonal (with respect to the scalar product (4) on E^p_q). If $m = 0$, the orthogonality has already been shown in the proof of Theorem 1. Let $m, t \ge 1$. Then the system $A^{(m)}$ is a sum of terms of the form

$$\delta^{i_{r_1}}_{k_{s_1}} \delta^{i_{r_2}}_{k_{s_2}} \dots \delta^{i_{r_m}}_{k_{s_m}} A^{(r_1 r_2 \dots r_m)\, i_{r_{m+1}} i_{r_{m+2}} \dots i_{r_p}}_{(s_1 s_2 \dots s_m)\, k_{s_{m+1}} k_{s_{m+2}} \dots k_{s_q}},$$

and $A^{(t)}$ is a sum of terms of the form

$$\delta^{i_{r_1}}_{k_{s_1}} \delta^{i_{r_2}}_{k_{s_2}} \dots \delta^{i_{r_t}}_{k_{s_t}} A^{(r_1 r_2 \dots r_t)\, i_{r_{t+1}} i_{r_{t+2}} \dots i_{r_p}}_{(s_1 s_2 \dots s_t)\, k_{s_{t+1}} k_{s_{t+2}} \dots k_{s_q}}.$$

Since $A^{(m)}$ is traceless, we get after some calculation

$$\left(A^{(m)} \middle| A^{(t)}\right) = 0 \tag{9}$$

for all m and t proving the uniqueness.

Formula (8) of Theorem 2 states the *second trace decomposition* of a system $A \in E^p_q$.

3. The trace decomposition of systems of type (1,1), (1,2), (1,3), and (2,2)

In this section we give trace decomposition formulas of a system $A \in E^p_q$ in the following four cases: 3.1. $p = 1$, $q = 1$, 3.2. $p = 1$, $q = 2$, 3.3. $p = 1$, $q = 3$, and 3.4. $p = 2$, $q = 2$. We discuss the uniqueness of the decomposition.

The proofs are based on a direct analysis of the corresponding systems of linear equations, and are tedious and long. Therefore, we omit the proofs. In cases 3.1, 3.2, and 3.3, we proceed in three steps: 1) we derive the *first contracted system* by forming all possible contractions of the given system of linear equations, 2) we solve the first contracted system, and 3) we substitute the result into the initial system (see [3]). In the case 3.4, we have one more step: 1) we derive the *first contracted system*, 2) contracting the first contracted system we derive the *second contracted system*, 3) we solve the second contracted system and substitute the result into the first one, 4) we solve the first contracted system and substitute the result into the initial system.

Note that if $n = 1$, the trace decomposition is trivial: any trace of a system consisting of one element coincides with this element. Consequently, traceless systems are all zero, and the trace decomposition is not unique unless $p = q = 1$. In what follows we always assume $n \ge 2$.

3.1. In the case $p = q = 1$ we have the following evident decomposition formulas presented here for completeness.

Theorem 3. *Let $p = q = 1$, $n \geq 2$, and let $A = \left(A^i_k\right) \in E^1_1$. Then there exist a unique traceless system $B = \left(B^i_k\right) \in E^1_1$ and a unique number $C \in \mathbf{R}$ such that*

$$A^i_k = B^i_k + \delta^i_k C.$$

These systems are defined by

$$C = \frac{1}{n} A^s_s, \quad B^i_k = A^i_k - \frac{1}{n} \delta^i_k A^s_s.$$

3.2. If $p = 1$, $q = 2$, then the trace decomposition problem has a unique solution.

Theorem 4. *Let $p = 1$, $q = 2$, $n \geq 2$, and let $A = \left(A^i_{kl}\right) \in E^1_2$. Then there exist a unique traceless system $B = \left(B^i_{kl}\right) \in E^1_2$ and unique systems $C = (C_k)$, $D = (D_k) \in E^0_1$ such that*

$$A^i_{kl} = B^i_{kl} + \delta^i_k C_l + \delta^i_l D_k.$$

These systems are defined by

$$\begin{aligned} C_l &= \frac{1}{n^2 - 1}(nA^t_{tl} - A^t_{lt}), \quad D_k = \frac{1}{n^2 - 1}(-A^t_{tk} + nA^t_{kt}), \\ B^i_{kl} &= A^i_{kl} - \frac{1}{n^2 - 1}\big(\delta^i_k (nA^t_{tl} - A^t_{lt}) + \delta^i_l(-A^t_{tk} + nA^t_{kt})\big). \end{aligned}$$

3.3. Let $p = 1$, $q = 3$. In this case the trace decomposition problem has always a solution; if $n \geq 3$, the solution is unique.

Theorem 5. *Let $p = 1$, $q = 3$, and let $A = \left(A^i_{klm}\right) \in E^1_3$.*
(a) *Assume that $n = 2$. Then there exist a unique traceless system $B = \left(B^i_{klm}\right) \in E^1_3$ and systems $C = (C_{lm})$, $D = (D_{km})$, $E = (E_{kl}) \in E^0_2$ such that*

$$A^i_{klm} = B^i_{klm} + \delta^i_k C_{lm} + \delta^i_l D_{km} + \delta^i_m E_{kl}.$$

These systems are defined by

$$\begin{aligned} C_{21} &= -\frac{1}{3}(A^t_{t12} - \lambda) + \frac{2}{3}A^t_{1t2} - A^t_{12t} + \frac{4}{3}A^t_{t21} - \frac{2}{3}(A^t_{2t1} - \lambda), \\ C_{22} &= \frac{1}{4}(3A^t_{t22} - A^t_{2t2} - A^t_{22t}), \\ D_{11} &= \frac{1}{4}(-A^t_{t11} + 3A^t_{1t1} - A^t_{11t}), \\ D_{12} &= -\frac{2}{3}(A^t_{t12} - \lambda) + \frac{4}{3}A^t_{1t2} - A^t_{12t} + \frac{2}{3}A^t_{t21} - \frac{1}{3}(A^t_{2t1} - \lambda), \\ D_{21} &= \frac{1}{6}(A^t_{t12} - \lambda) - \frac{1}{3}A^t_{1t2} + \frac{1}{2}A^t_{12t} - \frac{2}{3}A^t_{t21} + \frac{5}{6}(A^t_{2t1} - \lambda), \end{aligned}$$

$$D_{22} = \frac{1}{4}(-A^t_{t22} + 3A^t_{2t2} - A^t_{22t}),$$
$$E_{11} = \frac{1}{4}(-A^t_{t11} - A^t_{1t1} + 3A^t_{11t}),$$
$$E_{12} = \frac{1}{2}(A^t_{t12} - \lambda) - A^t_{1t2} + \frac{3}{2}A^t_{12t} - A^t_{t21} + \frac{1}{2}(A^t_{2t1} - \lambda),$$
$$E_{21} = \lambda,$$
$$E_{22} = \frac{1}{4}(-A^t_{t22} - A^t_{2t2} + 3A^t_{22t}),$$

where $\lambda \in \mathbf{R}$ *is a parameter.*
(b) *Assume that* $n \geq 3$. *Then there exist a unique traceless system* $B = (B^i_{klm}) \in E^1_3$ *and unique systems* $C = (C_{lm})$, $D = (D_{km})$, $E = (E_{kl}) \in E^0_2$ *such that*

$$A^i_{klm} = B^i_{klm} + \delta^i_k C_{lm} + \delta^i_l D_{km} + \delta^i_m E_{kl}.$$

These systems are defined by

$$C_{kl} = \frac{n(n^2-3)A^t_{tkl} - (n^2-2)A^t_{ktl} + nA^t_{klt} - 2A^t_{tlk} + nA^t_{ltk} - (n^2-2)A^t_{lkt}}{(n^2-1)(n^2-4)},$$
$$D_{kl} = \frac{-(n^2-2)A^t_{tkl} + n(n^2-3)A^t_{ktl} - (n^2-2)A^t_{klt} + nA^t_{tlk} - 2A^t_{ltk} + nA^t_{lkt}}{(n^2-1)(n^2-4)},$$
$$E_{kl} = \frac{nA^t_{tkl} - (n^2-2)A^t_{ktl} + n(n^2-3)A^t_{klt} - (n^2-2)A^t_{tlk} + nA^t_{ltk} - 2A^t_{lkt}}{(n^2-1)(n^2-4)},$$

and

$$B^i_{klm} = A^i_{klm} - \delta^i_k C_{lm} - \delta^i_l D_{km} - \delta^i_m E_{kl}.$$

3.4. Let $p = 2$, $q = 2$. In this case the trace decomposition problem has always a solution; if $n \geq 3$, the solution is unique.

Theorem 6. *Let* $p = 2$, $q = 2$, *and let* $A = (A^{ij}_{kl}) \in E^2_2$.
(a) *Assume that* $n = 2$. *Then there exist a unique traceless system* $B = (B^{ij}_{kl}) \in E^2_2$, *traceless systems* $C = (C^j_l)$, $D = (D^j_k)$, $E = (E^i_l)$, $F = (F^i_k) \in E^1_1$, *and unique numbers* $G, H \in \mathbf{R}$ *such that*

$$A^{ij}_{kl} = B^{ij}_{kl} + \delta^i_k C^j_l + \delta^i_l D^j_k + \delta^j_k E^i_l + \delta^j_l F^i_k + \delta^i_k \delta^j_l G + \delta^i_l \delta^j_k H$$

for all $i, j, k, l = 1, 2$. *These systems are defined by*

$$G = \frac{1}{6}\left(2A^{st}_{st} - A^{st}_{ts}\right), \quad H = \frac{1}{6}\left(-A^{st}_{st} + 2A^{st}_{ts}\right),$$
$$C^i_l = A^{si}_{sl} - \frac{1}{2}A^{si}_{ls} - \frac{1}{2}A^{is}_{sl} - \frac{1}{2}\delta^i_l(A^{st}_{st} - A^{st}_{ts}) + F^i_l,$$
$$D^j_k = -\frac{1}{2}A^{sj}_{sk} + \frac{3}{4}A^{sj}_{ks} + \frac{1}{4}A^{js}_{sk} + \frac{1}{4}\delta^j_k(A^{st}_{st} + 2A^{st}_{ts}) - F^j_k,$$
$$E^i_l = -\frac{1}{2}A^{si}_{sl} + \frac{1}{4}A^{si}_{ls} + \frac{3}{4}A^{is}_{sl} + \frac{3}{4}\delta^i_l(A^{st}_{st} - 2A^{st}_{ts}) - F^i_l,$$

and

$$B^{ij}_{kl} = A^{ij}_{kl} - \delta^i_k\left(A^{sj}_{sl} - \frac{1}{2}A^{sj}_{ls} - \frac{1}{2}A^{js}_{sl}\right) - \delta^i_l\left(-\frac{1}{2}A^{sj}_{sk} + \frac{3}{4}A^{sj}_{ks} + \frac{1}{4}A^{js}_{sk}\right)$$
$$-\delta^j_k\left(-\frac{1}{2}A^{si}_{sl} + \frac{1}{4}A^{si}_{ls} + \frac{3}{4}A^{is}_{sl}\right) + \frac{1}{6}\delta^i_k\delta^j_l A^{st}_{st} - \frac{1}{3}\delta^i_l\delta^j_k A^{st}_{st} - \frac{1}{3}\delta^i_k\delta^j_l A^{st}_{ts} + \frac{2}{3}\delta^i_l\delta^j_k A^{st}_{ts},$$

where either $F^1_1, F^1_2, F^2_1 \in \mathbf{R}$ or $F^2_2, F^1_2, F^2_1 \in \mathbf{R}$ are parameters such that $F^1_1 + F^2_2 = 0$.
(b) *Assume that $n \geq 3$. Then there exist unique traceless systems $B = (B^{ij}_{kl}) \in E^2_2$, $C = (C^j_l)$, $D = (D^j_k)$, $E = (E^i_l)$, $F = (F^i_k) \in E^1_1$, and unique numbers $G, H \in \mathbf{R}$ such that*

$$A^{ij}_{kl} = B^{ij}_{kl} + \delta^i_k C^j_l + \delta^i_l D^j_k + \delta^j_k E^i_l + \delta^j_l F^i_k + \delta^i_k\delta^j_l G + \delta^i_l\delta^j_k H.$$

These systems are defined by

$$G = \frac{1}{n(n^2-1)}(nA^{st}_{st} - A^{st}_{ts}), \quad H = \frac{1}{n(n^2-1)}(-A^{st}_{st} + nA^{st}_{ts}),$$
$$C^j_l = \frac{1}{n(n^2-4)}((n^2-2)A^{sj}_{sl} - nA^{sj}_{ls} - nA^{js}_{sl} + 2A^{js}_{ls} - n\delta^j_l A^{st}_{st} + 2\delta^j_l A^{st}_{ts}),$$
$$D^j_l = \frac{1}{n(n^2-4)}(-nA^{sj}_{sl} + (n^2-2)A^{sj}_{ls} + 2A^{js}_{sl} - nA^{js}_{ls} + 2\delta^j_l A^{st}_{st} - n\delta^j_l A^{st}_{ts}),$$
$$E^j_l = \frac{1}{n(n^2-4)}(-nA^{sj}_{sl} + 2A^{sj}_{ls} + (n^2-2)A^{js}_{sl} - nA^{js}_{ls} + 2\delta^j_l A^{st}_{st} - n\delta^j_l A^{st}_{ts}),$$
$$F^j_l = \frac{1}{n(n^2-4)}(2A^{sj}_{sl} - nA^{sj}_{ls} - nA^{js}_{sl} + (n^2-2)A^{js}_{ls} - n\delta^j_l A^{st}_{st} + 2\delta^j_l A^{st}_{ts}),$$

and

$$B^{ij}_{kl} = A^{ij}_{kl} - \frac{1}{n(n^2-4)}\delta^i_k((n^2-2)A^{sj}_{sl} - nA^{sj}_{ls} - nA^{js}_{sl} + 2A^{js}_{ls})$$
$$-\frac{1}{n(n^2-4)}\delta^i_l(-nA^{sj}_{sk} + (n^2-2)A^{sj}_{ks} + 2A^{js}_{sk} - nA^{js}_{ks})$$
$$-\frac{1}{n(n^2-4)}\delta^j_k(-nA^{si}_{sl} + 2A^{si}_{ls} + (n^2-2)A^{is}_{sl} - nA^{is}_{ls})$$
$$-\frac{1}{n(n^2-4)}\delta^j_l(2A^{si}_{sk} - nA^{si}_{ks} - nA^{is}_{sk} + (n^2-2)A^{is}_{ks})$$
$$-\frac{1}{(n^2-4)(n^2-1)}\delta^i_k\delta^j_l(-(n^2+2)A^{st}_{st} + 3nA^{st}_{ts})$$
$$-\frac{1}{(n^2-4)(n^2-1)}\delta^i_l\delta^j_k(3nA^{st}_{st} - (n^2+2)A^{st}_{ts}).$$

4. Applications: The trace decomposition of torsion and curvature tensors

In this section, X is a smooth n-dimensional manifold, where $n \geq 3$. We apply the trace decomposition formulas of Theorems 4, 5, and 6 to tensors of type (1,2), (1,3), and (2,2) having the same symmetry properties as the torsion and curvature tensors of a linear connection on X.

4.1. The torsion tensor of a linear connection

Consider a *torsion tensor* T on X. T is a tensor field of type (1,2) written in a chart on X with coordinates $(x^1, x^2, \dots, x^n)$ as

$$T = T^i_{jk} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k,$$

where the components T^i_{jk} satisfy $T^i_{jk} + T^i_{kj} = 0$. The trace decomposition formula of T is given by Theorem 4. We get $T^i_{kl} = B^i_{kl} + \delta^i_k C_l + \delta^i_l D_k$, where

$$C_l = \frac{1}{n-1} T^t_{tl}, \quad D_k = -\frac{1}{n-1} T^t_{tk}, \quad B^i_{kl} = T^i_{kl} - \frac{1}{n-1}\left(\delta^i_k T^t_{tl} - \delta^i_l T^t_{tk}\right).$$

4.2. The curvature tensor of a linear connection

A *curvature tensor* R on X is a tensor field of type (1,3) written in a chart with coordinates $(x^1, x^2, \dots, x^n)$ as

$$R = R^i_{jkl} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l, \tag{10}$$

where the components R^i_{jkl} satisfy $R^i_{klm} + R^i_{kml} = 0$. Recall that the *Ricci tensor* of R is defined as the (1,2)-trace of R, i.e., $R_{kl} = R^s_{ksl} = -R^s_{kls}$.

Using Theorem 5, the trace decomposition of R is obtained by a straightforward, but tedious calculation. We get $R^i_{klm} = B^i_{klm} + \delta^i_k C_{lm} + \delta^i_l D_{km} + \delta^i_m E_{kl}$, where

$$C_{kl} = \frac{n-1}{(n+1)(n-2)} R^t_{tkl} - \frac{1}{(n+1)(n-2)}(R_{kl} - R_{lk}),$$
$$D_{kl} = \frac{1}{(n^2-1)(n-2)}\left(-(n-1)R^t_{tkl} + (n^2-n-1)R_{kl} - R_{lk}\right),$$
$$E_{kl} = -D_{kl},$$

and

$$\begin{aligned} B^i_{klm} = R^i_{klm} &- \frac{1}{(n+1)(n-2)} \delta^i_k \left((n-1)R^t_{tlm} - R_{lm} + R_{ml}\right) \\ &- \frac{1}{(n^2-1)(n-2)} \delta^i_l \left(-(n-1)R^t_{tkm} + (n^2-n-1)R_{km} - R_{mk}\right) \\ &- \frac{1}{(n^2-1)(n-2)} \delta^i_m \left((n-1)R^t_{tkl} - (n^2-n-1)R_{kl} + R_{lk}\right). \end{aligned}$$

4.3. The Weyl projective curvature tensor

Now consider a tensor field R (10) whose components satisfy $R^i_{klm} + R^i_{kml} = 0$, $R^i_{klm} + R^i_{mkl} + R^i_{lmk} = 0$. These symmetry conditions correspond to the *curvature tensor* of a *torsionfree* connection on X.

The (1,1)-trace of R is expressed by means of the Ricci tensor as $R^s_{skl} = R_{kl} - R_{lk}$. The trace decomposition of R has the form $R^i_{klm} = B^i_{klm} + \delta^i_k C_{lm} + \delta^i_l D_{km} + \delta^i_m E_{kl}$, where

$$C_{kl} = \frac{1}{n+1}(R_{kl} - R_{lk}),\; D_{kl} = \frac{1}{n^2-1}(R_{lk} + nR_{kl}),\; E_{kl} = -\frac{1}{n^2-1}(R_{lk} + nR_{kl}),$$

and

$$B^i_{klm} = R^i_{klm} - \frac{1}{n+1}\delta^i_k(R_{lm} - R_{ml}) - \frac{1}{n^2-1}\delta^i_l(R_{mk} + nR_{km}) + \frac{1}{n^2-1}\delta^i_m(R_{lk} + nR_{kl}).$$

This formula defines the well-known *Weyl projective curvature tensor.*

4.4. The Weyl projective curvature tensor of the Levi-Civita connection

Now let X be endowed with a regular, symmetric tensor field g of type (0,2) (the *metric tensor*); g may be a *Riemann metric*, but one may also consider *indefinite metrics*. In a chart on X with coordinates $(x^1, x^2, \ldots, x^n)$, g is expressed as $g = g_{ij}dx^i \otimes dx^j$, where $g_{ij} = g_{ji}$ and $\det(g_{ij}) \neq 0$. As usual, we denote by g^{ij} the components of the inverse matrix to the matrix (g_{ij}). Then $g^{ij} = g^{ji}$, $\det(g^{ij}) \neq 0$, and $g_{ij}g^{jk} = g^{kj}g_{ji} = \delta^k_i$.

Consider a tensor R (10), and the tensor field of type (0,4)

$$R' = R_{ijkl}dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \tag{11}$$

where

$$R_{ijkl} = g_{im}R^m_{jkl}, \quad R^i_{jkl} = g^{is}R_{sjkl}. \tag{12}$$

Assume that the following identities are satisfied:

$$R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = R_{klij}, \quad R_{ijkl} + R_{iljk} + R_{iklj} = 0. \tag{13}$$

Because of these identities, the Ricci tensor is symmetric,

$$R_{jl} = R^s_{jsl} = g^{sk}R_{kjsl} = g^{sk}R_{sjkl} = g^{sk}R_{klsj} = R_{lj}, \tag{14}$$

and

$$R^t_{tkl} = g^{st}R_{stkl} = 0. \tag{15}$$

We note that the identities (13) are satisfied by the *curvature tensor* of the *Levi-Civita connection* of g.

The trace decomposition of R has the form $R^i_{klm} = B^i_{klm} + \delta^i_l D_{km} + \delta^i_m E_{kl}$, where

$$D_{kl} = -E_{kl} = \frac{1}{n-1} R_{kl}, \quad B^i_{klm} = R^i_{klm} - \frac{1}{n-1}\left(\delta^i_l R_{km} - \delta^i_m R_{kl}\right).$$

The second formula defines the *Weyl projective curvature tensor* of the Levi-Civita connection.

4.5. The Weyl conformal curvature tensor

As in Sec. 4.4, we assume that X is endowed with a regular, symmetric tensor field g of type (0,2). Notation and formulas (11)–(15) remain valid. Recall that the *scalar curvature* of g is defined by $r = g^{kl} g^{st} R_{sktl}$.

Let R be a tensor (10) whose associated tensor R' (11) satisfies (13). We wish to apply the trace decomposition formula of Theorem 6 to the tensor

$$R^{ij}_{kl} = g^{js} R^i_{skl} = g^{ir} g^{js} g_{rp} R^p_{skl} = g^{ir} g^{js} R_{rskl}. \tag{16}$$

We get $R^{ij}_{kl} = B^{ij}_{kl} + \delta^i_k C^j_l + \delta^i_l D^j_k + \delta^j_k E^i_l + \delta^j_l F^i_k + \delta^i_k \delta^j_l G + \delta^i_l \delta^j_k H$, where

$$\begin{aligned} G &= \frac{1}{n(n-1)} r, \quad H = -\frac{1}{n(n-1)} r, \\ C^j_l &= F^j_l = -D^j_l = -E^j_l = \frac{1}{n(n-2)}\left(n g^{jm} R_{ml} - \delta^j_l r\right), \end{aligned} \tag{17}$$

and

$$\begin{aligned} B^{ij}_{kl} = R^{ij}_{kl} &- \frac{1}{n-2}\left(\delta^i_k g^{jm} R_{ml} - \delta^i_l g^{jm} R_{mk} - \delta^j_k g^{im} R_{ml} + \delta^j_l g^{im} R_{mk}\right) \\ &+ \frac{1}{(n-1)(n-2)}\left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right) r. \end{aligned} \tag{18}$$

Formulas (17) and (18) represent the desired trace decomposition of the tensor (16).

If we use these formulas in the identity $R_{pqkl} = g_{ip} g_{jq} R^{ij}_{kl}$ and define $B_{pqkl} = g_{ip} g_{jq} B^{ij}_{kl}$, we get the *Weyl conformal curvature tensor*

$$\begin{aligned} B_{pqkl} = R_{pqkl} &- \frac{1}{n-2}\left(g_{kp} R_{ql} - g_{lp} R_{qk} - g_{kq} R_{pl} + g_{lq} R_{pk}\right) \\ &+ \frac{1}{(n-1)(n-2)}\left(g_{kp} g_{kq} - g_{lp} g_{kq}\right) r. \end{aligned}$$

This formula shows us that the Weyl conformal curvature tensor is precisely the traceless component of the curvature tensor of a metric, considered as a tensor of type (2,2).

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