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Jacobi Maps Between Riemannian Manifolds

Martin Belger[1], **Velichka Milousheva**[2], **Grozio Stanilov**[2]

[1] *Fachbereich Mathematik, Universität Leipzig*
Augustusplatz 10, D-04109 Leipzig, Deutschland

[2] *Faculty of Mathematics and Informatics, Sofia University*
blvd. G.Boucher 5, Sofia, 1126, Bulgaria

Abstract. In the present paper we define the notion of Jacobi map between two Riemannian manifolds as a diffeomorphism, preserving the Jacobi operator. The main results are: non-trivial Jacobi maps exist only between locally conformal flat Riemannian manifolds; the Jacobi class of a Riemannian manifold consists of all locally conformal flat Riemannian manifolds, for which the Jacobi map commutes with the corresponding Ricci operators. Remarks about the invariance of the eigenvalues and eigenvectors of Jacobi operators are given.

1. Jacobi maps

Let (M, g) be an n-dimensional Riemannian manifold and R be the curvature operator. For each unit vector X of the tangential space T_pM at $p \in M$ we consider the Jacobi operator

$$R_X : T_pM \to T_pM,$$

defined by

$$R_X(u) = R(u, X, X), \tag{1}$$

where $u \in T_pM$. R_X is a self-adjoint operator. If $X \in T_pM$ is an arbitrary non-zero vector, we consider the operator

$$R_X(u) = \frac{R(u, X, X)}{g(X, X)}. \tag{1'}$$

Thus, the one-dimensional subspace $E_1 = \{v \in T_pM : v = \alpha X, \alpha \in \mathbf{R}\}$ determines the self-adjoint operator R_{E_1} with

$$R_{E_1}(u) = \frac{R(u, X, X)}{g(X, X)}. \tag{1''}$$

Let $(\overline{M}, \overline{g})$ be another n-dimensional Riemannian manifold with a curvature operator $\overline{R}$. Let

$$f : M \to \overline{M}$$

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be a diffeomorphism and

$$f_{*p} : T_pM \to T_{\overline{p}}\overline{M}$$

be the tangent map, induced by f, where $\overline{p} = f(p)$. Thus E_1 is transformed by f_{*p} into $\overline{E}_1 = f_{*p}E_1 = \{f_{*p}v, v \in E_1\}$.

Definition. We call the diffeomorphism f a *Jacobi map* (*J-map*), if for each point $p \in M$ and each one-dimensional subspace $E_1 \subset T_pM$ the tangent map f_{*p} and the Jacobi operator R_{E_1} commute in the following way:

$$f_{*p} \circ R_{E_1} = \overline{R}_{\overline{E}_1} \circ f_{*p}. \tag{2}$$

2. Jacobi invariance of the eigenvalues and the eigenvectors of the Jacobi operator

Let $f : M \to \overline{M}$ be a Jacobi map. From (2) and the eigenvalue equation

$$R_{E_1}(u) = c\,(p, E_1)\,u \tag{3}$$

with respect to each p and E_1 we get

$$\overline{R}_{\overline{E}_1} \circ f_{*p}(u) = f_{*p} \circ R_{E_1}(u) = c\,(p, E_1)\,f_{*p}u. \tag{4}$$

So, we have immediately

Proposition 1. *If u is an eigenvector of R_{E_1} with a corresponding eigenvalue $c\,(p, E_1)$, then $f_{*p}u$ is an eigenvector to the same eigenvalue $c\left(\overline{p}, \overline{E}_1\right) = c\,(p, E_1)$of the operator $\overline{R}_{\overline{E}_1}$, which means that the eigenvalues and in a natural sense also the eigenvectors of the Jacobi operator are invariants of the Jacobi maps.*

3. Relations between Jacobi maps, isometries and curvature-preserving maps

It is easy to prove the following proposition, which lights up the relations between these maps.

Proposition 2. *Each two of the following conditions imply the third one:*

$$\overline{R}\,(f_{*p}X, f_{*p}Y, f_{*p}Z) = f_{*p}R(X, Y, Z), \quad X, Y, Z \in T_pM, \tag{5}$$

i.e. f is a curvature-preserving map;

$$\overline{g}\,(f_{*p}X, f_{*p}Y) = g(X, Y), \quad X, Y \in T_pM, \tag{6}$$

i.e. f is an isometry;

$$\frac{\overline{R}\,(f_{*p}u, f_{*p}X, f_{*p}X)}{\overline{g}\,(f_{*p}X, f_{*p}X)} = f_{*p}\frac{R(u, X, X)}{g(X, X)}, \quad u, X \in T_pM, \tag{7}$$

i.e. f is a Jacobi map.

4. A Jacobi class and a Jacobi group of a Riemannian manifold

Let $\left(\overline{\overline{M}}, \overline{\overline{g}}\right)$ be also a n-dimensional Riemannian manifold and $h : \overline{M} \to \overline{\overline{M}}$ be a J-map. Then the composition $h \circ f$, where $(h \circ f)(p) = h(f(p))$, is also a J-map. Indeed, if

$$\overline{\overline{p}} = h(\overline{p}) = h(f(p)) = (h \circ f)(p)$$

and

$$\overline{\overline{E}}_1 = h_{*\overline{p}}(E_1) = (h \circ f)_{*p} E_1,$$

then

$$\begin{aligned}\overline{\overline{R}}_{\overline{\overline{E}}_1} \circ (h \circ f)_{*p} &= \overline{\overline{R}}_{\overline{\overline{E}}_1} \circ (h_{*\overline{p}} \circ f_{*p}) = \left(\overline{\overline{R}}_{\overline{\overline{E}}_1} \circ h_{*\overline{p}}\right) \circ f_{*p} = \left(h_{*\overline{p}} \circ \overline{R}_{\overline{E}_1}\right) \circ f_{*p} \\ &= h_{*\overline{p}} \circ \left(\overline{R}_{\overline{E}_1} \circ f_{*p}\right) = h_{*\overline{p}} \circ f_{*p} \circ R_{E_1} = (h \circ f)_{*p} \circ R_{E_1}.\end{aligned}$$

If f is a J-map, then f^{-1} is also a J-map. Indeed, multiplying (2) by $\left(f^{-1}\right)_{*\overline{p}} = (f_{*p})^{-1}$ from the left and from the right, we get

$$R_{E_1} \circ \left(f^{-1}\right)_{*\overline{p}} = \left(f^{-1}\right)_{*\overline{p}} \circ \overline{R}_{\overline{E}_1},$$

which shows that f^{-1} is a J-map.

So, if we consider all J-maps from M onto M, we can formulate the following

Proposition 3. *The set of all J-maps of a Riemannian manifold (M, g) onto itself is a group.*

We call it the *Jacobi group of* (M, g).

Corollary. *The eigenvalues of any Jacobi operator, defined on a Riemannian manifold (M, g) are invariants of the Jacobi group of the manifold.*

We shall call two Riemannian manifolds *Jacobi-equivalent*, if there is a J-map between them. The set of all Riemannian manifolds, which are Jacobi-equivalent to (M, g), we call a *Jacobi class* of (M, g).

5. Relations between J-maps and conformal maps

Let $f : (M, g) \to \left(\overline{M}, \overline{g}\right)$ be a J-map. We shall express the curvature operator $\overline{R}$ by R. Using the definition of a J-map and denoting $X^* = f_* X$, $u^* = f_* u$, and yet, for shorteness, $f_* = f_{*p}$, we have

$$\frac{\overline{R}(u^*, X^*, X^*)}{\overline{g}(X^*, X^*)} = f_* \frac{R(u, X, X)}{g(X, X)}.$$

Replacing X by $X+Y$ in the last equation and using it for the couple of vectors u, X and u, Y, we get

$$\begin{aligned}\overline{R}(u^*,X^*,Y^*)+\overline{R}(u^*,Y^*,X^*) = &\left[\frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}(X^*,X^*)}{g(X,X)}\right] f_*R(u,X,X)\\ &+\left[\frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}(Y^*,Y^*)}{g(Y,Y)}\right] f_*R(u,Y,Y)\\ &+\frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)}\left[f_*R(u,X,Y)+f_*R(u,Y,X)\right].\end{aligned}$$

Now, using the properties of the curvature operator, we obtain the following expression for $\overline{R}$:

$$\begin{aligned}\overline{R}(U^*,X^*,Y^*) = &\frac{1}{3}\left[\frac{\overline{g}(X^*+U^*,X^*+U^*)}{g(X+U,X+U)} + 2\frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)}\right] f_*R(Y,X,U)\\ &+\frac{1}{3}\left[4\frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}(X^*+U^*,X^*+U^*)}{g(X+U,X+U)}\right] f_*R(U,Y,X)\\ &+\frac{2}{3}\left[\frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}(X^*,X^*)}{g(X,X)}\right] f_*R(U,X,X)\\ &+\frac{2}{3}\left[\frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}(Y^*,Y^*)}{g(Y,Y)}\right] f_*R(U,Y,Y)\\ &+\frac{2}{3}\left[\frac{\overline{g}(X^*+U^*,X^*+U^*)}{g(X+U,X+U)} - \frac{\overline{g}(X^*,X^*)}{g(X,X)}\right] f_*R(Y,X,X)\\ &+\frac{1}{3}\left[\frac{\overline{g}(X^*+U^*,X^*+U^*)}{g(X+U,X+U)} - \frac{\overline{g}(U^*,U^*)}{g(U,U)}\right] f_*R(Y,U,U),\end{aligned} \tag{8}$$

where $U, X, Y \in T_pM$. Thus we have proved the following

Proposition 4. *If $f : (M,g) \to (\overline{M},\overline{g})$ is a J-map, then the curvature operator $\overline{R}$ can be expressed by the curvature operator R by the formula (8).*

Let $M, \overline{M}$ be J-equivalent. Then, if we take into consideration that

$$\overline{R}(X^*,X^*,Y^*) = 0$$

for each $X, Y \in T_pM$, from (8) we get the equation

$$A(X,Y)f_*R(X,Y,Y) - B(X,Y)f_*R(Y,X,X) = 0, \tag{9}$$

where

$$\begin{aligned}A(X,Y) &= \frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}(Y^*,Y^*)}{g(Y,Y)},\\ B(X,Y) &= \frac{\overline{g}(X^*+Y^*,X^*+Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}(X^*,X^*)}{g(X,X)}.\end{aligned}$$

Since $f_* = f_{*p}$ is an isomorphism at every point $p \in M$, (9) implies

$$A(X,Y)R(X,Y,Y) - B(X,Y)R(Y,X,X) = 0, \tag{10}$$

which holds for any arbitrary vectors X, Yat any point of M.

Let $X \wedge Y \subset T_pM$ be an arbitrary section of finite sectional curvature $\mathcal{K}$(p;X,Y). Multiplying (10) by X and Y, we get

$$\begin{aligned} A(X,Y)\mathcal{K}(p;X,Y) = 0, \\ B(X,Y)\mathcal{K}(p;X,Y) = 0. \end{aligned} \tag{11}$$

If

$$\mathcal{K}(q;U,V) \neq 0$$

at a point $q \in M$ and for a section $U \wedge V \subset T_qM$, so there exist neighbourhoods Ω of q and Λ of the section $U \wedge V$ such that

$$\mathcal{K}(p;X,Y) \neq 0 \qquad (g(X,X)g(Y,Y) \neq g(X,Y)^2)$$

for each $p \in \Omega$ and each $X \wedge Y \in \Lambda$. Then (11) imply

$$\begin{aligned} \frac{\overline{g}\,(X^* + Y^*, X^* + Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}\,(Y^*,Y^*)}{g(Y,Y)} = 0, \\ \frac{\overline{g}\,(X^* + Y^*, X^* + Y^*)}{g(X+Y,X+Y)} - \frac{\overline{g}\,(X^*,X^*)}{g(X,X)} = 0, \end{aligned}$$

i.e.

$$\frac{\overline{g}\,(X^*,X^*)}{g(X,X)} = \frac{\overline{g}\,(Y^*,Y^*)}{g(Y,Y)}, \tag{12}$$

which holds for $X \wedge Y \in \Lambda$, $p \in \Omega$. Hence, the left hand side is a function only of p and therefore

$$\overline{g}\,(X^*,X^*) = e^{2\sigma} g(X,X), \tag{13}$$

where σ is a function of p. In other words, f is a conformal map. Using (13) and (8) we get

$$\overline{R}\,(f_*U, f_*X, f_*Y) = e^{2\sigma} f_* R(U,X,Y), \tag{14}$$

where U, X, Y are arbitrary vectors of T_pM. Thus we have proved the main result in our paper

Theorem 1. *Let (M,g) be an n-dimensional $(n \geq 2)$ Riemannian manifold with non-zero sectional curvature. Then any Jacobi map $f : (M,g) \to \left(\overline{M}, \overline{g}\right)$ is a conformal map and the corresponding curvature operators are related by (14). The converse is also true.*

For a conformal map f we have ([1])

$$\begin{aligned} &\overline{R}\,(X^*,Y^*,Z^*) = \\ &f_*\left[R(X,Y,Z) + M(X,Z)Y - M(Y,Z)X + g(X,Z)N(Y) - g(Y,Z)N(X)\right], \end{aligned} \tag{15}$$

where $N : T_pM \to T_pM$ is defined by $N(X) = -(X\sigma)\nabla\sigma$ and $M(X,Y) = M(Y,X)$ for any $X, Y \in T_pM$ because the Hesse form to σ and metric g are symmetric forms. Using (14) and (15) we get

$$\begin{aligned}&\left(e^{2\sigma}-1\right)R(X,Y,Z) = \\ &M(X,Z)Y - M(Y,Z)X + g(X,Z)M(Y) - g(Y,Z)M(X).\end{aligned} \tag{16}$$

If $\{e_i\}_{i=1}^n$ is an orthonormal base of T_pM, from (16) we get

$$\begin{aligned}&\left(e^{2\sigma}-1\right)\sum_{i=1}^n R\left(X,e_i,e_i\right) = \sum_{i=1}^n M\left(X,e_i\right)e_i - \sum_{i=1}^n M\left(e_i,e_i\right)X \\ &+\sum_{i=1}^n g\left(X,e_i\right)M\left(e_i\right) - \sum_{i=1}^n g\left(e_i,e_i\right)M(X),\end{aligned}$$

i.e.

$$\left(e^{2\sigma}-1\right)\rho(X) = \sum_{i=1}^n M\left(X,e_i\right)e_i - MX + \sum_{i=1}^n g\left(X,e_i\right)M\left(e_i\right) - nM(X), \tag{17}$$

where ρ is the Ricci operator and $M = \sum\limits_{i=1}^n M\left(e_i,e_i\right)$. Multiplying (17) by Y and using the equations

$$M(X,Y) = \sum_{i=1}^n M\left(X,e_i\right)g\left(Y,e_i\right) = \sum_{i=1}^n g\left(X,e_i\right)M\left(Y,e_i\right)$$

we get

$$\left(e^{2\sigma-1}\right)\rho(X,Y) = (2-n)M(X,Y) - Mg(X,Y). \tag{18}$$

Hence

$$\left(e^{2\sigma}-1\right)\tau = (2-n)M - nM,$$

i.e.

$$M = \left(e^{2\sigma}-1\right)\frac{\tau}{2(1-n)},$$

where τ is the scalar curvature of (M,g). Thus we obtain

$$M(X,Y) = \left(e^{2\sigma}-1\right)\left[\frac{\rho(X,Y)}{2-n} + \frac{\tau}{2(2-n)(1-n)}g(X,Y)\right], \tag{19}$$

or

$$M(X) = \left(e^{2\sigma}-1\right)\left[\frac{\rho(X)}{2-n} + \frac{\tau}{2(2-n)(1-n)}X\right]. \tag{20}$$

So, the equation (16) takes the form

$$\begin{aligned}&\Big\{R(X,Y,Z) + \frac{1}{n-2}\left[\rho(X,Z)Y - \rho(Y,Z)X + g(X,Z)\rho(Y) - g(Y,Z)\rho(X)\right] \\ &-\frac{\tau}{(n-1)(n-2)}\left[g(X,Z)Y - g(Y,Z)X\right]\Big\}\left(e^{2\sigma}-1\right) = 0.\end{aligned} \tag{21}$$

There are two cases possible:

(i) $\sigma(p) = 0$ for each $p \in M$, i.e. f is an isometry. We call such a map a trivial Jacobi map.

(ii) $\sigma(q) \neq 0$ for a point $q \in M$. Then there exists a neighbourhood Ω of q, such that $\sigma(p) \neq 0$ for each $p \in \Omega$. Hence

$$\begin{aligned} R(X,Y,Z) = {} & \frac{1}{n-2}\left[\rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)\rho(X) - g(X,Z)\rho(Y)\right] \\ & - \frac{\tau}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right] \end{aligned} \tag{22}$$

for each $p \in \Omega$, $X, Y, Z \in T_pM$. The last equation shows, that in case $n > 3$ (M, g) is a locally conformal flat manifold ([1]). Thus, as a consequence of Theorem 1, we have

Corollary. *Any n-dimensional ($n > 3$) Riemannian manifold (M, g) with non-zero sectional curvature, which admits a non-trivial Jacobi map on a Riemannian manifold $\left(\overline{M}, \overline{g}\right)$ is locally conformal flat.*

It's obvious, that in this case $\left(\overline{M}, \overline{g}\right)$ is also a locally conformal flat manifold.

6. The Jacobi class of a Riemannian manifold with non-zero sectional curvature

From the equation

$$\frac{\overline{R}\,(X^*, Y^*, Y^*)}{\overline{g}\,(Y^*, Y^*)} = f_* \frac{R(X,Y,Y)}{g(Y,Y)}, \quad X, Y \in T_pM$$

follows

$$\overline{\rho}\,(f_*X) = f_*\rho(X), \tag{23}$$

which mean that f_* commutes with the Ricci operators ρ and $\overline{\rho}$. If f is conformal map, then (23) implies

$$\overline{\rho}\,(f_*X, f_*Y) = e^{2\sigma}\rho(X,Y) \tag{24}$$

and

$$\overline{\tau} = \tau. \tag{25}$$

So, for a conformal J-map (23), (24) and (25) are fulfilled. It's easy to prove also, that a conformal J-map f, for which $f_* \circ \rho = \overline{\rho} \circ f_*$ is a J-map. Thus we can formulate the following

Theorem 2. *The Jacobi class of any n-dimensional ($n \geq 3$) Riemannian manifold (M, g) with non-zero sectional curvature consists of all locally conformal flat Riemannian manifolds, for which the differential f_* of a J-map f commutes with the corresponding Ricci operators.*

As a consequence we get the following result ([2]): If (M, g) is a Riemannian manifolds with constant sectional curvature c, then the Jacobi class of (M, g) consists of all Riemannian manifolds with the same constant curvature c and the manifolds are conformally equivalent.

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References

[1] Gromoll, D., Klingenberg, W., Meyer, E.: *Riemannsche Geometrie im Grossen.* Springer-Verlag, Berlin, 1968.

[2] Milousheva, V.: *Jacobi Maps Between Riemannian Manifolds with Constant Curvature.* Mathematics and Education of Mathematics, Proceedings of Twenty Third Spring Conference of the Union of Bulgarian Mathematicians, 1994 (to appear).

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