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On the Derivation of the Non-Euclidean Angle of Parallelism Function

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Abstract. This paper investigates the role that elementary area theory can play in deriving the formula for the non-Euclidean angle of parallelism function. It is shown how the pivotal feature of area theory, the connection between area and angle defect of a polygon, extends to a quantitative relation among the entities associated with a given angle of parallelism. The sought-after formula then is easily determined. The whole investigation is carried out in the framework laid out by Oskar Perron.

1. Introduction

The non-Euclidean angle of parallelism formula has been derived in quite a variety of ways, on the one hand because of its great importance for non-Euclidean geometry, and on the other hand because no single way of deducing the function has emerged as the most natural one. In 1944 Oskar Perron published a particularly elementary deduction of the angle of parallelism function in the Mathematische Annalen; in his approach the function is considered together with and calculated right after the introduction of the trigonometric functions (s. [5]). This procedure (which has also been integrated in Perron's textbook of non-Euclidean geometry [6]) suffers however from the fact that long-winded and non-intuitive analytic passages make up a considerable part of it.
As will be shown in this paper, it is possible to derive the angle of parallelism function in a direct, geometric way, and to do that still in the context of the trigonometric functions, and still in an entirely elementary way. The decisive connection in determining the sought-after formula will be that between area measure and angular defect of (proper and improper) polygons. In section 2 a few preliminary facts will be compiled, and in section 3 the actual derivation of the angle of parallelism function will be presented.

2. Abbreviations, Preliminaries

Let A, B be two distinct points. Then AB denotes the *segment* determined by them, $|AB|$ the *length* of the segment, $g(AB)$ the *straight line* through A and B, $\overrightarrow{(A)B}$ the *halfline*

bounded by A, through B, and $\overrightarrow{AB} = \{A\} \cup \overrightarrow{(A)B}$ the related *ray*. By $|\sphericalangle AOB|$ we mean the *size of* the *angle* $\sphericalangle AOB$ and by $|\Delta ABC|$ the *area of* the *triangle* ΔABC.
The letters $\omega, \tau, \ldots$ are used for *ends*, improper points of concurrence of boundary parallel lines. The end symbols appear prominently in the abbreviations $\Delta AB\omega, \ldots$ for *trilaterals*, i.e. for simply asymptotic triangles with two proper vertices and an improper one. Our deduction of the angle of parallelism function depends especially on the following facts.

(i) For $\sphericalangle AOB$ a fixed angle of size α, Y a running point on the halfline $\overrightarrow{(O)B}$ and X its normal projection on $\overrightarrow{(O)A}$ (s. Fig. 1),

$$\lim_{|OY| \to 0} \frac{|XY|}{|OY|} = \sin\alpha, \qquad \lim_{|OY| \to 0} \frac{|OX|}{|OY|} = \cos\alpha.$$

(This result has been deduced by Perron; s. [5], p. 247ff., [6], p. 51ff.)

Fig. 1

Fig. 2

(ii) For a polygon the defect ($n\pi - 2\pi -$ angle sum in the case of an n-gon) constitutes a suitable area measure in the following sense. Any decomposition of the polygon into polygonal parts has the property that the sum of the defects of the parts is equal to the defect of the whole polygon. (For details see any of the books listed in the Bibliography.) Also for a trilateral the defect ($\pi -$ sum of the proper angles) qualifies as a suitable area measure.

(iii) A trilateral has an area of less than ε if only its (finite) inner side is sufficiently small. In order to show that we construct out of a right trilateral $\Delta AB\omega$ another right trilateral $\Delta A_1B_1\omega$ of less than half the area (s. Fig. 2). Repeated arbitrarily this construction leads to a right trilateral $\Delta UV\omega$ of area $< \varepsilon/2$.
The area of the equiangular trilateral $\Delta UU^*\omega$ with symmetry line $g(V\omega)$ then measures $< \varepsilon$ (s. Fig. 3), and the same is true for any other trilateral whose inner side is not larger than UU^*.

Fig. 3

Fig. 4

(iv) Let again, as in (i), $\sphericalangle AOB$ be a fixed angle of size α, Y a variable point on $\overrightarrow{(O)B}$ and X its normal projection on $\overrightarrow{(O)A}$. Then

$$\lim_{|OX|\to 0} \frac{|\Delta OXY|}{|OX|} = 0 \quad \text{and} \quad \lim_{|XY|\to 0} \frac{|\Delta OXY|}{|XY|} = 0.$$

Proof. Given ΔOX_0Y_0, ΔOX_1Y_1, where $|OX_1| = 1/2|OX_0|$, we construct a point Z_0 such that ΔOX_1Y_1, $\Delta X_0X_1Y_1$ and $\Delta Y_1Z_0X_0$ form three congruent triangles with mutually disjoint interior, and such that $|\sphericalangle Y_1X_0Z_0| = |\sphericalangle OY_1X_1| = \beta_1$ (s. Fig. 4). Since $|\sphericalangle OX_0Z_0| = |\sphericalangle X_1Y_1Z_0| = \alpha + \beta_1$ and $\alpha + \beta_1 < \pi/2$ (see ΔOX_1Y_1) it follows that $\sphericalangle OX_0Z_0 < \sphericalangle OX_0Y_0$ and $\sphericalangle X_1Y_1Z_0 < \sphericalangle X_1Y_1Y_0$. As a result Z_0 lies in the interior of ΔOX_0Y_0, and the three triangles are part of a polygonal decomposition of ΔOX_0Y_0. Consequently $|\Delta OX_1Y_1| < 1/3|\Delta OX_0Y_0|$ and

$$\frac{|\Delta OX_1Y_1|}{|OX_1|} < \frac{\frac{1}{3}|\Delta OX_0Y_0|}{\frac{1}{2}|OX_0|} = \frac{2}{3}\frac{|\Delta OX_0Y_0|}{|OX_0|}.$$

Introducing additional triangles ΔOX_2Y_2, $\Delta OX_3Y_3, \ldots$ which one after another fulfill $|OX_{i+1}| = 1/2|OX_i|$ we can expand our result to

$$\frac{|\Delta OX_nY_n|}{|OX_n|} < \left(\frac{2}{3}\right)^n \frac{|\Delta OX_0Y_0|}{|OX_0|}. \qquad (n = 1, 2, 3, \ldots)$$

Because $|\Delta OXY|$, and for $|OX| > 0$ also $|\Delta OXY|/|OX|$ are continuous functions of $|OX|$ this provides the proof for the first formula of our statement. By means of

$$\lim_{|OX|\to 0} \frac{|OX|}{|XY|} = \cot\alpha$$

the second formula follows from the first one at once.

3. The Derivation of the Angle of Parallelism Function

Let $\sphericalangle\omega O\tau$ be a right angle, defined by the proper point O and by the ends ω and τ, and let $X, Y, \ldots$ be arbitrary points on $\overrightarrow{(O)\tau}$. The rays $\overrightarrow{X\omega}$, $\overrightarrow{Y\omega}, \ldots$ are boundary parallel to $\overrightarrow{O\omega}$ and the angles $\sphericalangle OX\omega$, $\sphericalangle OY\omega, \ldots$ are *the angles of parallelism* associated with the segments $OX, OY, \ldots$ (s. Fig. 5). The function which relates the measures of these segments and angles is *the angle of parallelism function* $\varphi = \Pi(x)$. If $|OX| < |OY|$, $\sphericalangle OX\omega$ is an exterior angle of $\Delta XY\omega$, and consequently $|\sphericalangle OX\omega| > |\sphericalangle XY\omega| = |\sphericalangle OY\omega|$. This shows $\Pi(x)$ is a decreasing function. Also, $|\sphericalangle OX\omega| - |\sphericalangle OY\omega| = \pi - |\sphericalangle YX\omega| - |\sphericalangle XY\omega|$, i.e. the difference of the angles of parallelism related to OX, OY is identical with the defect, and thus with the area measure of $\Delta XY\omega$. Because $|\Delta XY\omega| \to 0$ for $|XY| \to 0$ (see 2., (iii)) the function $\Pi(x)$ must be continuous.

Fig. 5

We now consider three points A, B, C in this order on $\overrightarrow{(O)\tau}$ and assume B is the midpoint of AC. From these points we draw the rays $\overrightarrow{A\omega}$, $\overrightarrow{B\omega}$, $\overrightarrow{C\omega}$, as well as the rays $\overrightarrow{A\omega_A}$ and $\overrightarrow{C\omega_C}$ which are defined by the equation $|\sphericalangle OA\omega_A| = |\sphericalangle OC\omega_C| = |\sphericalangle OB\omega| = \varphi$, and by the stipulation that ω_A and ω_C lie on the same side of line $g(O\tau)$ as ω (s. Fig. 6). We complete our figure by normally projecting the points A, C onto the points A', C' of $g(B\omega)$. The congruence of the ensuing triangles $\Delta AA'B$, $\Delta CC'B$ entails that the segments $A'A$, $C'C$ have equal length, and as a consequence that the related angles of parallelism $\sphericalangle A'A\omega$, $\sphericalangle C'C\omega$ satisfy $|\sphericalangle A'A\omega| = |\sphericalangle C'C\omega| = \alpha$. Note also $|\sphericalangle A'AB| = |\sphericalangle C'CB| = \beta$. Finally, defining $|\sphericalangle\omega_A A\omega| = \delta_A, |\sphericalangle\omega_C C\omega| = \delta_C$, we have that $\Pi(OA) = \varphi + \delta_A$, $\Pi(OB) = \varphi$ and $\Pi(OC) = \varphi - \delta_C$, in agreement with the decreasing character of the angle of parallelism function. In addition, the relations

$$\begin{aligned} \varphi + \beta &< \frac{\pi}{2} && (\text{in } \Delta AA'B\,), \\ \varphi + \delta_A + \alpha + \beta &= \pi && (\text{at the point } A\,) \quad \text{and} \\ \delta_C + \alpha &= \varphi + \beta && (\text{at the point } C\,) \end{aligned}$$

lead to $\delta_A + \alpha > \frac{\pi}{2}$, $\delta_C + \alpha < \frac{\pi}{2}$ and thus to $\delta_C < \delta_A$. This means that the continuous, decreasing function $\varphi = \Pi(x)$ is convex downwards at $x = |OB|$ and has negative left-hand side and right-hand side derivatives there. These one-sided derivatives φ'_-, φ'_+ satisfy (s. Fig. 6)

$$|\varphi'_-| = \lim_{|AB|\to 0} \frac{|(\varphi + \delta_A) - \varphi|}{|AB|} = \lim_{|AB|\to 0} \frac{\delta_A}{|AB|}$$

and

$$|\varphi'_+| = \lim_{|CB|\to 0} \frac{|(\varphi - \delta_C) - \varphi|}{|CB|} = \lim_{|CB|\to 0} \frac{\delta_C}{|CB|}\,.$$

Fig. 6

At this point one has to take the important fact into account that $\delta_A = \pi - (\alpha + \beta + \varphi)$ is the defect and thus the area measure of the trilateral $\Delta AB\omega$, and $\delta_C = \beta + \varphi - \alpha = \pi - [(\alpha - \beta) + (\pi - \varphi)]$ is the defect and thus the area measure of the trilateral $\Delta CB\omega$. Consequently

$$|\varphi'_-| = \lim_{|AB|\to 0} \frac{|\Delta AB\omega|}{|AB|} \quad \text{and} \quad |\varphi'_+| = \lim_{|CB|\to 0} \frac{|\Delta CB\omega|}{|CB|}.$$

From the expansion

$$|\varphi'_-| = \lim_{|AB|\to 0} \frac{|\Delta AB\omega|}{|AB|} = \lim_{|AB|\to 0} \frac{|\Delta AB\omega|}{|AA'|} \lim_{|AB|\to 0} \frac{|AA'|}{|AB|} = \lim_{|AA'|\to 0} \frac{|\Delta AB\omega|}{|AA'|} \sin\varphi$$

of the left-hand side derivative follows that the limit which appears last in the chain of equations exists and is positive. Because of $\Delta AB\omega = \Delta AA'\omega + \Delta AA'B$ it satisfies the equation

$$\lim_{|AA'|\to 0} \frac{|\Delta AB\omega|}{|AA'|} = \lim_{|AA'|\to 0} \frac{|\Delta AA'\omega|}{|AA'|} + \lim_{|AA'|\to 0} \frac{|\Delta AA'B|}{|AA'|}$$

in which the second limit on the right-hand side is 0 (s. 2., (iv)). The remaining first limit on the right-hand side applies to the areas of right trilaterals as a function of, and divided by their inner sides; it is independent of $x = |OB|$ resp. φ and represents a general non-Euclidean constant κ. Thus for every argument $x > 0$

$$|\varphi'_-| = \kappa \sin\varphi.$$

The right-hand side derivative is calculated analogously (except that $\Delta CB\omega = \Delta CC'\omega - \Delta CC'B$), and the result is likewise

$$|\varphi'_+| = \kappa \sin\varphi.$$

This shows that φ has an all-over derivative which is determined by the differential equation

$$\varphi' = -\kappa \sin\varphi.$$

Its solution, the angle of parallelism function, has the formula

$$\varphi = \Pi(x) = 2\tan^{-1}\left(e^{-\kappa x}\right).$$

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