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Metric Properties of the Stephanos Bijection

Dedicated to Professor G. Geise on his 65^{th} birthday

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Abstract. The three-dimensional projective image space P^3 of the Stephanos bijection becomes a hyperbolic space $P^3_{|2}$ with index 2, if the ruled quadric Q^2_{42} which corresponds to the singular collineations of the projective line P^1 is regarded as an absolute quadric. We study properties of the Stephanos bijection which are connected with the Cayley/Klein metric defined in $P^3_{|2}$.

1. Introduction

Stephanos' bijection of the collineations of the real projective line P^1 onto the points of the real three-dimensional projective space P^3 has been often investigated (v. Horniaček [5], Medek [6],[7],[8], Geise [2] and the references in [2]), but all these articles are restricted to projective properties of the Stephanos bijection. If the ruled quadric Q^2_{42} which corresponds to the singular collineations is regarded as an absolute quadric, the P^3 becomes a hyperbolic space $P^3_{|2}$ with index 2 (cf. Giering [3], pp. 234-235). This paper examines for the first time properties of the Stephanos bijection which are connected with the Cayley/Klein metric defined in $P^3_{|2}$. The locus of the Stephanos image points P_A of all elliptic or all directly hyperbolic projectivities Π_A with a given characteristic cross ratio c_A is a distance sphere with the Stephanos image point P_E of the identity as centre. For the Stephanos image points P_A and P_B of two hyperbolic projectivities Π_A and Π_B with altogether four pairwise distinct fixed points Q_A, R_A, Q_B and R_B, the angle of the lines $P_E + P_A$ and $P_E + P_B$ can be given as a function of the cross ratio $\delta_F = DV(Q_A, R_A, Q_B, R_B)$, if Q_A and Q_B are not separated by R_A and R_B on the projective line P^1. If Π_A and Π_B are directly hyperbolic projectivities, the first cosine law of hyperbolic trigonometry yields a formula which expresses the distance $d(P_A, P_B)$ by δ_F and the characteristic cross ratios c_A and c_B of Π_A and Π_B (Proposition 7). For three projectivities Π_A, Π_B and Π_C we represent the angle of the planes $\varepsilon_{AB} = P_E + P_A + P_B$ and $\varepsilon_{AC} = P_E + P_A + P_C$ (if it is defined) depending on the relative position of the six fixed points of the three projectivities.

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The Stephanos bijection is a kinematic mapping of the collineations of the projective line P^1. Rath [9] studies a kinematic mapping of the collineations of the n-dimensional projective spaces which includes the Stephanos bijection as the special case $n = 1$. Metric properties which are similar to the results obtained in this paper can be given especially for those kinematic mappings that have -as the Stephanos bijection- a Cayley/Klein space as image space (see for example Grünwald [4], Blaschke [1], pp. 177-258, Rehbock [10], Strubecker [12] and Röschel [11]).

2. Preliminaries

Let Π_A be a collineation of the real projective line P^1 with the transformation matrix

$$A = \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}. \tag{1}$$

If Π_A is a projectivity ($det\, A \neq 0$) and different from the identity $\Pi_E = id$, then Π_A has two conjugate complex or real (possibly coincident) fixed points Q_A and R_A with the appertaining eigenvalues λ_A and μ_A:

$$\begin{aligned} &\lambda_A = \frac{1}{2}(a_0 + a_3) - \frac{1}{2}\sqrt{D_A}\ , \ \ \mu_A = \frac{1}{2}(a_0 + a_3) + \frac{1}{2}\sqrt{D_A}\ ; \\ &D_A := (a_0 - a_3)^2 + 4a_1 a_2 . \end{aligned} \tag{2}$$

For the homogeneous coordinates q_A and r_A of Q_A and R_A we have:

$$\left.\begin{aligned} &q_A = \begin{pmatrix} a_1 \\ \lambda_A - a_0 \end{pmatrix},\ r_A = \begin{pmatrix} a_1 \\ \mu_A - a_0 \end{pmatrix}, \text{ if } a_1 \neq 0, \\ &q_A = \begin{pmatrix} \lambda_A - a_3 \\ a_2 \end{pmatrix},\ r_A = \begin{pmatrix} \mu_A - a_3 \\ a_2 \end{pmatrix}, \text{ if } a_2 \neq 0, \\ &q_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ r_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ or } q_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ r_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \text{ if } a_1 = a_2 = 0 . \end{aligned}\right\} \tag{3}$$

The projectivity $\Pi_A \neq id$ is called $\left\{\begin{matrix} \textit{elliptic} \\ \textit{hyperbolic} \\ \textit{parabolic} \end{matrix}\right\}$, if the fixed points Q_A and R_A are $\left\{\begin{matrix} \text{conjugate complex} \\ \text{real and distinct} \\ \text{coincident} \end{matrix}\right\}$. The *characteristic cross ratio* c_A of Π_A is defined as

$$c_A := \frac{\lambda_A}{\mu_A} = DV(Q_A, R_A, X, \Pi_A X)\ \ \forall X \in P^1 . \tag{4}$$

A parabolic projectivity has the characteristic cross ratio $c_A = 1$. For an elliptic projectivity, c_A has the form

$$c_A = e^{i\cdot\varphi_A}\,;\ \ 0 < \varphi_A < 2\pi. \tag{5}$$

A hyperbolic projectivity is called $\left\{ \begin{array}{c} \textit{directly} \\ \textit{indirectly} \end{array} \right\}$ hyperbolic, if $\left\{ \begin{array}{c} c_A > 0 \\ c_A < 0 \end{array} \right\}$. The involutions are characterized by the characteristic cross ratio $c_A = -1$:

$$\Pi_A^2 = id \iff c_A = -1\,. \tag{6}$$

Two non-parabolic projectivities Π_A and Π_B with identical fixed points $Q_A = Q_B$ and $R_A = R_B$ have the same characteristic cross ratio, if and only if they are inverse to each other:

$$Q_A = Q_B \neq R_A = R_B\, :\;\; c_A = c_B \iff \Pi_A = \Pi_B^{-1}\,. \tag{7}$$

3. The Stephanos Bijection

The Stephanos bijection maps the collineations of the real projective line P^1 onto the points of the real three-dimensional projective space P^3. A collineation Π_A with the transformation matrix A (A as in (1)) is mapped onto its *Stephanos image point*

$$P_A = [(a_0, a_1, a_2, a_3)^\top] \in P^3\,. \tag{8}$$

The singular collineations are mapped onto the points of a ruled quadric $Q_{42}^2 \subset P^3$:

$$Q_{42}^2 = \{X \in P^3 : \; x_0x_3 - x_1x_2 = 0\}. \tag{9}$$

If the quadric Q_{42}^2 is regarded as an absolute quadric, the projective space P^3 becomes a *hyperbolic space* $P_{|2}^3$ *with index* 2 (see [3], pp. 234-235). The points $X \in P^3 \backslash Q_{42}^2$ are called the *finite* points, the points $X \in Q_{42}^2$ are called the *infinite* points of $P_{|2}^3$. The Stephanos image point P_E of the identity $\Pi_E = id$ is

$$P_E = [(1,0,0,1)^\top]\,. \tag{10}$$

The polar plane Γ_E of P_E with respect to Q_{42}^2 has the equation

$$\Gamma_E = \{X \in P^3 : \;\; x_0 + x_3 = 0\}. \tag{11}$$

Proposition 1. *The Stephanos bijection maps the projectivities of P^1 with a given (real or complex) fixed point Q onto the finite points of a (real or complex) tangent plane of the absolute quadric Q_{42}^2 which contains the Stephanos image point P_E of the identity.*

Proof: If λ is the eigenvalue of a fixed point $Q = [(q_0, q_1)^\top]$ of a collineation Π_A (A as in (1), then we get from

$$a_0q_0 + a_1q_1 = \lambda q_0\,, \;\; a_2q_0 + a_3q_1 = \lambda q_1 \tag{12}$$

by elimination of λ:

$$q_0q_1(a_0 - a_3) + q_1^2a_1 - q_0^2a_2 = 0\,. \tag{13}$$

If q_0 and q_1 are given (homogeneous) coordinates of the fixed point Q, (13) is the equation of a plane containing P_E (see (10)). The pole of this plane with respect to Q^2_{42} has the homogeneous coordinates $(-q_0q_1, q_0^2, -q_1^2, q_0q_1)^\top$ and is therefore a point of the absolute quadric Q^2_{42}. Hence the plane is a tangent plane of Q^2_{42}. □

Corollary 2. *The Stephanos bijection maps the projectivities of P^1 with two given conjugate complex or real (possibly coincident) fixed points Q and R onto the finite points of a (real) line which contains the Stephanos image point P_E of the identity.*

Proof: If the fixed points Q and R are distinct, the assertion follows from Proposition 1. If Q and R are coincident, we get from (2) for the appertaining (double) eigenvalue λ:

$$\lambda = \frac{1}{2}(a_0 + a_3)\,. \tag{14}$$

From (12) and (14) follows:

$$-q_0a_0 + 2q_1a_1 + q_0a_3 = 0\,, \quad -q_1a_0 + 2q_0a_2 + q_1a_3 = 0\,. \tag{15}$$

If q_0 and q_1 are given (homogeneous) fixed point coordinates, (15) describes a line which contains the point P_E (see (10)). □

Remark R1. Proposition 1 yields a result which was already given by Geise in [2], pp. 374-375: If $Q^2_{31} \subset P^3_{|2}$ denotes the tangent cone from P_E to Q^2_{42}, then the Stephanos image point P_A of a $\left\{\begin{array}{c}\text{elliptic}\\ \text{hyperbolic}\\ \text{parabolic}\end{array}\right\}$ projectivity $\Pi_A \neq id$ is a point of $\left\{\begin{array}{c}\text{the interior domain of } Q^2_{31}\\ \text{the exterior domain of } Q^2_{31}\\ Q^2_{31}\end{array}\right\}$.

Proposition 3. *The Stephanos bijection maps the projectivities of P^1 with a given characteristic cross ratio c_A onto the finite points of a quadric $Q^2(c_A) \subset P^3_{|2}$ which belongs to the pencil of quadrics which is spanned by Q^2_{42} and the doubly counted polar plane Γ_E of the Stephanos image point P_E of the identity with respect to Q^2_{42}.*

Proof: For the eigenvalues λ_A and μ_A and the characteristic cross ratio c_A of a projectivity Π_A we get with (4):

$$c_A(\lambda_A + \mu_A)^2 = (1 + c_A)^2 \cdot \lambda_A\mu_A\,. \tag{16}$$

With $\lambda_A + \mu_A = trA = a_0 + a_3$ and $\lambda_A\mu_A = detA = a_0a_3 - a_1a_2$ (see (1)) follows

$$c_A(a_0 + a_3)^2 = (1 + c_A)^2(a_0a_3 - a_1a_2)\,. \tag{17}$$

The statement of Proposition 3 follows with (9) and (11). □

Remark R2. The projectivities with the characteristic cross ratio $c_A = -1$ (the involutions) correspond to the finite points of the polar plane Γ_E. The projectivities with $c_A = 1$ (the parabolic projectivities) are mapped onto the finite points of the tangent cone

Q^2_{31} from P_E to Q^2_{42} according to Remark 1. All other quadrics of the pencil of quadrics considered in Proposition 3 are non-degenerate ruled quadrics.

Remark R3. The Stephanos image points P_A and $P_{A^{-1}}$ of the inverse projectivities Π_A and Π_A^{-1} with two given distinct fixed points Q_A and R_A and the given characteristic cross ratio $c_A \notin \{-1, 1\}$ (see equation (7)) are the intersection points of the line of $P^3_{|2}$ through P_E corresponding to Q_A and R_A (see Corollary 2) and the ruled quadric corresponding to c_A (see Proposition 3). Therefore the points P_A and $P_{A^{-1}}$ are mapped onto each other by the harmonic homology [1] with the centre P_E and the axis Γ_E. If the intersection point of the line $P_E + P_A = P_E + P_{A^{-1}}$ and the polar plane Γ_E is denoted by S_A, then the points P_A, $P_{A^{-1}}$, P_E and S_A are in harmonic position (see [3], p. 72, Satz 1):

$$c_A \notin \{1, -1\} \,:\; DV(P_A, P_{A^{-1}}, P_E, S_A) = -1\,;\;\; S_A := (P_E + P_A) \cap \Gamma_E\,. \tag{18}$$

4. Metric Properties of the Stephanos Bijection

Proposition 4. *Let $\Pi_A \neq id$ be a non-parabolic projectivity of P^1, P_A the Stephanos image point of Π_A and S_1 and S_2 the (real or complex) intersection points of the absolute quadric $Q^2_{42} \subset P^3_{|2}$ and the connecting line of P_A and the Stephanos image point P_E of the identity. If the points S_1 and S_2 are numbered suitably, then the cross ratio of the four points P_E, P_A, S_1 and S_2 is equal to the characteristic cross ratio c_A of Π_A:*

$$DV(P_E, P_A, S_1, S_2) = c_A\,;\;\; \{S_1, S_2\} = (P_E + P_A) \cap Q^2_{42}\,. \tag{19}$$

Proof: By using the set-up

$$S_i = [\alpha_i p_A + \beta_i p_E]\,;\;\; i = 1, 2\,, \tag{20}$$

we get the coefficients α_i and β_i on account of (8), (9) and (10) as homogeneous solutions of the quadratic equation

$$(\alpha_i a_0 + \beta_i)(\alpha_i a_3 + \beta_i) - \alpha_i a_1 \cdot \alpha_i a_2 = 0\,;\;\; i = 1, 2\,. \tag{21}$$

If we denote the eigenvalues appertaining to Π_A with λ_A and μ_A (see (2)), then from (21) follows:

$$\left(\frac{\beta_i}{\alpha_i}\right)^2 + (\lambda_A + \mu_A) \cdot \frac{\beta_i}{\alpha_i} + \lambda_A \mu_A = 0\,;\;\; i = 1, 2\,. \tag{22}$$

Equation (22) has the two solutions

$$\frac{\beta_1}{\alpha_1} = -\mu_A\,,\;\; \frac{\beta_2}{\alpha_2} = -\lambda_A\,. \tag{23}$$

[1] See [3], pp. 40-41.

For the cross ratio considered in (19) we have because of (20):

$$DV(P_E, P_A, S_1, S_2) = \frac{\beta_2}{\alpha_2} : \frac{\beta_1}{\alpha_1}\,. \tag{24}$$

The assertion (19) follows from (23), (24) and (4). □

Remark R4. Let Π_A and Π_B ($\Pi_A \neq \Pi_B$) be two non-parabolic projectivities, being different from the identity, with common fixed points $Q_A = Q_B$ and $R_A = R_B$ and the characteristic cross ratios c_A and c_B. Let P_A and P_B be the Stephanos image points of Π_A and Π_B and S_1 and S_2 the intersection points of the absolute quadric Q^2_{42} of $P^3_{|2}$ and the line $P_A + P_B$ (which contains P_E). Then we have

$$DV(P_A, P_B, S_1, S_2) = \frac{DV(P_E, P_B, S_1, S_2)}{DV(P_E, P_A, S_1, S_2)} = \frac{c_B}{c_A}\,. \tag{25}$$

Remark R5. If Π_A is an elliptic projectivity, then the connecting line $P_E + P_A$ is a passant of the hyperbolic space $P^3_{|2}$ according to Remark R1, and on account of (19) the (directed) distance [2] $d(P_E, P_A)$ of P_E and P_A can be written as (φ_A as in (5)):

$$d(P_E, P_A) = \frac{1}{2i} \ln c_A = \frac{1}{2}\varphi_A\,. \tag{26}$$

If Π_A is a hyperbolic projectivity, then $P_E + P_A$ is a secant of $P^3_{|2}$ according to Remark R1. If furthermore Π_A is directly hyperbolic ($c_A > 0$), then P_E and P_A are not separated on $P_E + P_A$ by the intersection points S_1 and S_2 of $P_E + P_A$ and Q^2_{42} (see [2], p. 374, 3.). In this case we have:

$$d(P_E, P_A) = \frac{1}{2} \ln\, c_A\,. \tag{27}$$

According to (26) and (27) the locus of the Stephanos image points of all elliptic or all directly hyperbolic projectivities with a given characteristic cross ratio is a *distance sphere* of the hyperbolic space $P^3_{|2}$ with the centre P_E.

Proposition 5. *Let Π_A and Π_B be two projectivities of P^1, being different from the identity, with altogether four pairwise distinct (real or complex) fixed points Q_A, R_A, Q_B and R_B. Furthermore let ε_{AB} be the plane of the hyperbolic space $P^3_{|2}$ which is spanned by the Stephanos image points P_A, P_B and P_E of Π_A, Π_B and the identity and let g_1 and g_2 be the two (real or complex) tangents from P_E to Q^2_{42} which lie in ε_{AB}. Then the cross ratios*

$$\delta_F := DV(Q_A, R_A, Q_B, R_B)\,,\;\; \delta_G := DV(P_E + P_A, P_E + P_B, g_1, g_2) \tag{28}$$

are connected by

$$(1 - \delta_F)^2(1 - \delta_G)^2 = 16 \cdot \delta_F \delta_G\,. \tag{29}$$

[2] See [3], pp. 177-178, Satz 3.

Proof: Case 1: Let all four fixed points be complex. Without loss of generality, Q_A and R_A have the homogeneous coordinates

$$q_A = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad r_A = \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{30}$$

Then we get:

$$\delta_F = \frac{q_{B1} - iq_{B0}}{q_{B1} + iq_{B0}} : \frac{r_{B1} - ir_{B0}}{r_{B1} + ir_{B0}}. \tag{31}$$

If the entry b_1 of the transformation matrix

$$B = \begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix} \tag{32}$$

of Π_B were zero, the point $[(0,1)^\top]$ would be a fixed point in contradiction to the assumption. Therefore $b_1 \neq 0$ holds, and we get from (31) by application of (3) (λ_B and μ_B denote the eigenvalues of Π_B):

$$\delta_F = \frac{(\lambda_B - b_0) - ib_1}{(\lambda_B - b_0) + ib_1} : \frac{(\mu_B - b_0) - ib_1}{(\mu_B - b_0) + ib_1} \overset{(2)}{=} \frac{(b_1 - b_2) + i(\lambda_B - \mu_B)}{(b_1 - b_2) - i(\lambda_B - \mu_B)}. \tag{33}$$

Let $\tilde{P}_A$, $\tilde{P}_B$, G_1 and G_2 be the intersection points of the four lines $P_E + P_A$, $P_E + P_B$, g_1 and g_2 with the polar plane Γ_E of P_E with respect to Q^2_{42}:

$$\begin{gathered} \tilde{P}_A := (P_E + P_A) \cap \Gamma_E \, , \tilde{P}_B := (P_E + P_B) \cap \Gamma_E \, , \\ G_1 := g_1 \cap \Gamma_E \, , G_2 := g_2 \cap \Gamma_E \, . \end{gathered} \tag{34}$$

Then we have

$$\delta_G = DV(\tilde{P}_A, \tilde{P}_B, G_1, G_2). \tag{35}$$

We obtain the point $\tilde{P}_B$ with (10) and (11):

$$\tilde{P}_B = [(b_0 - b_3, 2b_1, 2b_2, b_3 - b_0)^\top]. \tag{36}$$

The point $\tilde{P}_A$ is the Stephanos image point of the involution $\Pi_{\tilde{A}}$ with the fixed points $Q_A = [(1,i)^\top]$ and $R_A = [(1,-i)^\top]$:

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{P}_A = P_{\tilde{A}} = [(0,1,-1,0)^\top]. \tag{37}$$

The points G_1 and G_2 are the intersection points of the line $\tilde{P}_A + \tilde{P}_B$ and the absolute quadric Q^2_{42}. We use the set-up

$$G_i = [\alpha_i \tilde{p}_A + \beta_i \tilde{p}_B]; \quad i = 1, 2. \tag{38}$$

The coefficients α_i and β_i are, on account of (36), (37) and (9), the homogeneous solutions of the quadratic equation

$$(\beta_i(b_0 - b_3))^2 - (\alpha_i + 2\beta_i b_1)(\alpha_i - 2\beta_i b_2) = 0\,;\; i = 1, 2\,. \tag{39}$$

Because of (2) the equation (39) is equivalent to

$$\left(\frac{\alpha_i}{\beta_i}\right)^2 + 2(b_1 - b_2)\cdot\frac{\alpha_i}{\beta_i} - (\lambda_B - \mu_B)^2 = 0\,;\; i = 1, 2\,. \tag{40}$$

Finally from (35), (38) and (40) follows: [3]

$$\delta_G = \frac{\alpha_2}{\beta_2} : \frac{\alpha_1}{\beta_1} = \frac{b_2 - b_1 - \nu\sqrt{(b_1 - b_2)^2 + (\lambda_B - \mu_B)^2}}{b_2 - b_1 + \nu\sqrt{(b_1 - b_2)^2 + (\lambda_B - \mu_B)^2}}\,;\; \nu \in \{-1, 1\}\,. \tag{41}$$

From (33) and (41) we get the assertion (29).

Case 2: Now let at least two of the fixed points of Π_A and Π_B - say Q_A and R_A - be real. Without loss of generality they have the homogeneous coordinates

$$q_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\; r_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{42}$$

It follows that

$$\delta_F = \frac{q_{B1}}{q_{B0}} : \frac{r_{B1}}{r_{B0}}\,. \tag{43}$$

The entry b_1 of the transformation matrix B of Π_B is not zero; otherwise the point $R_A = [(0,1)^\top]$ would be a fixed point of Π_B in contradiction to the assumption $Q_B \neq R_A \neq R_B$. Therefore we get from (43) by application of (3):

$$\delta_F = \frac{\lambda_B - b_0}{\mu_B - b_0}\,. \tag{44}$$

The points $\tilde{P}_A$, $\tilde{P}_B$, G_1 and G_2 are defined as in (34). The point $\tilde{P}_A$ is the Stephanos image point of the involution $\Pi_{\tilde{A}}$ with the fixed points $Q_A = [(1,0)^\top]$ and $R_A = [(0,1)^\top]$:

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \tilde{P}_A = P_{\tilde{A}} = [(1,0,0,-1)^\top]\,. \tag{45}$$

The coefficients α_i and β_i in (38) are, on account of (36), (45) and (9), the homogeneous solutions of the quadratic equation

$$(\alpha_i + \beta_i(b_0 - b_3))^2 + 2\beta_i b_1 \cdot 2\beta_i b_2 = 0\,;\; i = 1, 2\,. \tag{46}$$

[3] Changing the sign of the factor $\nu \in \{-1, 1\}$ corresponds to the exchange of the two (not distinguishable) lines g_1 and g_2.

Because of (2) the equation (46) is equivalent to

$$\left(\frac{\alpha_i}{\beta_i}\right)^2 + 2(b_0 - b_3)\cdot\frac{\alpha_i}{\beta_i} + (\lambda_B - \mu_B)^2 = 0\,;\ i = 1,2\,. \tag{47}$$

Finally we get:

$$\delta_G = \frac{b_3 - b_0 - 2\nu\sqrt{-b_1 b_2}}{b_3 - b_0 + 2\nu\sqrt{-b_1 b_2}}\,;\ \ \nu \in \{-1, 1\}\,. \tag{48}$$

From (44) and (48) follows again the assertion (29). □

Remark R6. If T_1 and T_2 are the intersection points of the line $P_A + P_B$ and the tangent cone Q^2_{31} from P_E to Q^2_{42}, then we have for δ_G:

$$\delta_G = DV(P_A, P_B, T_1, T_2)\,. \tag{49}$$

Remark R7. Since the cross ratios δ_F and δ_G are different from 1, (29) is equivalent to

$$\left(\frac{1+\delta_F}{1-\delta_F}\right)^2\cdot\left(\frac{1+\delta_G}{1-\delta_G}\right)^2 = \left(\frac{1+\delta_F}{1-\delta_F}\right)^2 + \left(\frac{1+\delta_G}{1-\delta_G}\right)^2\,. \tag{50}$$

Remark R8. From (50) we obtain the equivalence

$$\delta_F = -1 \iff \delta_G = -1\,. \tag{51}$$

Therefore the four fixed points Q_A,R_A,Q_B and R_B of Π_A and Π_B are in harmonic position, if and only if the two lines $P_E + P_A$ and $P_E + P_B$ are orthogonal. [4]

Remark R9. In the case $\delta_F \neq -1$, $\delta_G \neq -1$ (50) has the simpler form:

$$\left(\frac{1-\delta_F}{1+\delta_F}\right)^2 + \left(\frac{1-\delta_G}{1+\delta_G}\right)^2 = 1\,. \tag{52}$$

Lemma 6. *Consider the situation given in* Proposition 5. *If the conic*

$$Q^1_{31}(A,B) := \varepsilon_{AB} \cap Q^2_{42} \tag{53}$$

is regarded as an absolute conic, the plane ε_{AB} becomes a hyperbolic plane $P^2_{|1}(A,B)$, and the following statement holds:
The point P_E is a point of the interior domain of $Q^1_{31}(A,B)$, if and only if the four fixed points Q_A, R_A, Q_B and R_B of Π_A and Π_B are all real and the points Q_A and R_A are separated on the projective line P^1 by Q_B and R_B ($\delta_F < 0$).

[4] See [3], p. 72, Satz 1 and pp. 201-205, Abschnitt 9D.

Proof: Since the four fixed points of Π_A and Π_B are pairwise distinct, the plane $\varepsilon_{AB} = P_E + P_A + P_B$ is not a tangent plane of the absolute quadric Q^2_{42} (see Proposition 1). Hence the conic $Q^1_{31}(A,B)$ is non-degenerate. The point P_E lies in the interior domain of $Q^1_{31}(A,B)$, if and only if the plane ε_{AB} intersects the tangent cone Q^2_{31} from P_E to Q^2_{42} in two complex generators. According to the equations (41) and (48) for the cross ratio δ_G, which were given in the proof of Proposition 5, this is the case, if and only if the fixed points Q_A and R_A are real (without loss of generality let $Q_A = [(1,0)^\top]$ and $R_A = [(0,1)^\top]$) and furthermore $-b_1b_2 < 0$ holds. From $b_1b_2 > 0$ follows with (2), (3) and (44), that Q_B and R_B and also the cross ratio δ_F are real. For a real δ_F, however, we have

$$-b_1b_2 < 0 \;\overset{(2)}{\Longleftrightarrow}\; (\lambda_B - b_0)(\mu_B - b_0) < 0 \;\overset{(44)}{\Longleftrightarrow}\; \delta_F < 0\,. \tag{54}$$

If, the other way round, δ_F is real and negative, then we get $-b_1b_2 < 0$ directly from (54). Hence the assertion is shown in both directions. □

Proposition 7. *Let Π_A and Π_B be two directly hyperbolic projectivities of P^1 with altogether four pairwise distinct fixed points Q_A, R_A, Q_B and R_B and the characteristic cross ratios c_A and c_B ($c_A, c_B > 0$). Let the fixed points of Π_A be separated on P^1 by the fixed points of Π_B. (Thus the cross ratio $\delta_F = DV(Q_A, R_A, Q_B, R_B)$ is negative.) The connecting plane ε_{AB} of the Stephanos image points P_A, P_B and P_E of Π_A, Π_B and the identity becomes a hyperbolic plane $P^2_{|1}(A,B)$, if the (non-degenerate) conic $Q^1_{31}(A,B) = \varepsilon_{AB} \cap Q^2_{42}$ (see (53)) is regarded as an absolute conic. Then the following statement holds: The three points P_E, P_A and P_B of $P^2_{|1}(A,B)$ are points of the interior domain of the absolute conic $Q^1_{31}(A,B)$. The angle $\alpha = \sphericalangle(P_E + P_A, P_E + P_B)$ ($0 < \alpha \le \frac{\pi}{2}$) and the distance $d(P_A, P_B)$ are given by*

$$\cos\alpha = \left|\frac{1+\delta_F}{1-\delta_F}\right|\,, \tag{55}$$

$$\cosh d(P_A,P_B) = \frac{1}{4\sqrt{c_Ac_B}}\left((1+c_A)(1+c_B) - \cos\alpha\cdot(1-c_A)(1-c_B)\right)\,. \tag{56}$$

Proof: According to Lemma 6, the point P_E lies in the interior domain of $Q^1_{31}(A,B)$. Because of $c_A > 0$ and $c_B > 0$ neither P_E and P_A nor P_E and P_B are separated on their connecting line by the (real) intersection points of $P_E + P_A$ and $P_E + P_B$, respectively, with Q^2_{42} (see (19)). Hence the points P_A and P_B also lie in the interior domain of $Q^1_{31}(A,B)$. For the angle $\alpha = \sphericalangle(P_E + P_A, P_E + P_B)$ we get with [3], p. 177, Satz 3 (δ_G as in (28)):

$$\alpha = \frac{1}{2i}\ln\delta_G\,,\quad \delta_G = e^{2i\alpha} = (\cos^2\alpha - \sin^2\alpha) + i\cdot 2\sin\alpha\cos\alpha\,. \tag{57}$$

From (57) and (50) we get

$$\cos^2\alpha = \frac{(1+\delta_G)^2}{4\delta_G} = \left(\frac{1+\delta_F}{1-\delta_F}\right)^2\,. \tag{58}$$

Because of the restriction $0 < \alpha \leq \frac{\pi}{2}$ we have $\cos\alpha \geq 0$, and therefore the equation (55) holds. The first cosine law of hyperbolic trigonometry (see [3], p. 306, Satz 1) yields

$$\begin{aligned}\cosh d(P_A, P_B) = \cosh d(P_E, P_A) \cdot \cosh d(P_E, P_B) - \\ - \cos\alpha \cdot \sinh d(P_E, P_A) \cdot \sinh d(P_E, P_B)\,.\end{aligned} \tag{59}$$

The assertion (56) follows with equation (27). □

Proposition 8. *Let Π_A, Π_B and Π_C be three projectivities of P^1, all being different from the identity, with altogether six (real or complex) fixed points Q_A, R_A, Q_B, R_B, Q_C and R_C. Let δ_{AB}, δ_{AC} and δ_{BC} be the cross ratios*

$$\begin{aligned}&\delta_{AB} := DV(Q_A, R_A, Q_B, R_B)\,, \quad \delta_{AC} := DV(Q_A, R_A, Q_C, R_C)\,, \\ &\delta_{BC} := DV(Q_B, R_B, Q_C, R_C)\,.\end{aligned} \tag{60}$$

Furthermore let P_A, P_B, P_C and P_E be the Stephanos image points of Π_A, Π_B, Π_C and the identity, let

$$\varepsilon_{AB} := P_E + P_A + P_B\,, \quad \varepsilon_{AC} := P_E + P_A + P_C\,, \tag{61}$$

and let τ_1^A and τ_2^A be the (real or complex) tangent planes of the absolute quadric $Q_{42}^2 \subset P_{|2}^3$ which contain the line $P_E + P_A$ and δ_ε the cross ratio

$$\delta_\varepsilon := DV(\varepsilon_{AB}, \varepsilon_{AC}, \tau_1^A, \tau_2^A)\,. \tag{62}$$

Then the following equation holds:

$$\frac{(1+\delta_\varepsilon)^2}{\delta_\varepsilon} = \frac{1}{\delta_{AB}\delta_{AC}} \left(\frac{(\delta_{AB} + \delta_{AC}) - \delta_{BC}(1 + \delta_{AB}\delta_{AC})}{1 - \delta_{BC}} \right)^2 =: f(\delta_{AB}, \delta_{AC}, \delta_{BC})\,. \tag{63}$$

Proof: Without loss of generality, each of the fixed points Q_B ,R_B ,Q_C and R_C of Π_B and Π_C is different from the point $[(0,1)^\top]$. Then the entries b_1 and c_1 of the transformation matrices B and C of Π_B and Π_C (see (1)) are not zero, and the following quantities can be defined (λ_B, μ_B, λ_C and μ_C denote the eigenvalues of B and C):

$$\beta_\lambda := \frac{\lambda_B - b_0}{b_1}\,, \; \beta_\mu := \frac{\mu_B - b_0}{b_1}\,, \gamma_\lambda := \frac{\lambda_C - c_0}{c_1}\,, \; \gamma_\mu := \frac{\mu_C - c_0}{c_1}\,. \tag{64}$$

With (2) we get from (64):

$$\beta_\lambda\beta_\mu = -\frac{b_2}{b_1}\,, \;\; \gamma_\lambda\gamma_\mu = -\frac{c_2}{c_1}\,, \beta_\lambda + \beta_\mu = \frac{b_3 - b_0}{b_1}\,, \;\; \gamma_\lambda + \gamma_\mu = \frac{c_3 - c_0}{c_1}\,. \tag{65}$$

Because of (3) we have for the cross ratio δ_{BC}:

$$\delta_{BC} = \frac{\gamma_\lambda - \beta_\lambda}{\gamma_\lambda - \beta_\mu} : \frac{\gamma_\mu - \beta_\lambda}{\gamma_\mu - \beta_\mu}\,. \tag{66}$$

Case 1: Let the fixed points Q_A and R_A of Π_A be real. Then without loss of generality they have the homogeneous coordinates $q_A = (1,0)^\top$ and $r_A = (0,1)^\top$, and from (44) and (64) follows

$$\delta_{AB} = \frac{\beta_\lambda}{\beta_\mu}\,,\ \ \delta_{AC} = \frac{\gamma_\lambda}{\gamma_\mu}\,. \tag{67}$$

From (66) and (67) we get: [5]

$$f(\delta_{AB},\delta_{AC},\delta_{BC}) = \frac{(\beta_\lambda\beta_\mu + \gamma_\lambda\gamma_\mu)^2}{\beta_\lambda\beta_\mu\cdot\gamma_\lambda\gamma_\mu} \overset{(65)}{=} \frac{(b_1c_2+b_2c_1)^2}{b_1b_2\cdot c_1c_2}\,. \tag{68}$$

If $\tilde{P}_A$ denotes the intersection point of the line $P_E + P_A$ and the polar plane Γ_E of P_E with respect to Q^2_{42}, then the planes ε_{AB} and ε_{AC} defined in (61) can be written as:

$$\varepsilon_{AB} = P_E + \tilde{P}_A + P_B\,,\ \ \varepsilon_{AC} = P_E + \tilde{P}_A + P_C\,. \tag{69}$$

With (9), (10) and (45) we obtain the homogeneous coordinates of the poles H_{AB} and H_{AC} of ε_{AB} and ε_{AC} with respect to Q^2_{42}:

$$h_{AB} = (0,b_1,-b_2,0)^\top\,,\ \ h_{AC} = (0,c_1,-c_2,0)^\top\,. \tag{70}$$

If H_1 and H_2 denote the intersection points of the line $H_{AB} + H_{AC}$ and Q^2_{42}, then we have

$$\delta_\varepsilon = DV(H_{AB},H_{AC},H_1,H_2)\,. \tag{71}$$

By using the set-up

$$H_i = [\sigma_i h_{AB} + \tau_i h_{AC}]\,;\ \ i = 1,2\,, \tag{72}$$

we get the coefficients σ_i and τ_i because of (9) and (70) as homogeneous solutions of the quadratic equation

$$(\sigma_i b_1 + \tau_i c_1)(\sigma_i b_2 + \tau_i c_2) = 0\,. \tag{73}$$

Hence follows

$$\delta_\varepsilon = \frac{\sigma_2}{\tau_2} : \frac{\sigma_1}{\tau_1}\,, \tag{74}$$

and therefore

$$\frac{(1+\delta_\varepsilon)^2}{\delta_\varepsilon} = \frac{(\sigma_1\tau_2+\sigma_2\tau_1)^2}{\sigma_1\sigma_2\cdot\tau_1\tau_2} \overset{(73)}{=} \frac{(b_1c_2+b_2c_1)^2}{b_1b_2\cdot c_1c_2}\,. \tag{75}$$

According to (68) and (75) the assertion (63) holds.

Case 2: Let the fixed points Q_A and R_A of Π_A be conjugate complex. Without loss of generality their homogeneous coordinates are $q_A = (1,i)^\top$ and $r_A = (1,-i)^\top$. With (31) and (64) follows:

$$\delta_{AB} = \frac{\beta_\lambda - i}{\beta_\lambda + i} : \frac{\beta_\mu - i}{\beta_\mu + i}\,,\ \ \delta_{AC} = \frac{\gamma_\lambda - i}{\gamma_\lambda + i} : \frac{\gamma_\mu - i}{\gamma_\mu + i}\,. \tag{76}$$

[5] Since Q_B, R_B, Q_C and R_C are different from $Q_A = [(1,0)^\top]$ according to the supposition, b_2 and c_2 are not zero.

From (66) and (76) we get:

$$f(\delta_{AB},\delta_{AC},\delta_{BC}) = 4\cdot\frac{\left((1-\beta_\lambda\beta_\mu)(1-\gamma_\lambda\gamma_\mu)+(\beta_\lambda+\beta_\mu)(\gamma_\lambda+\gamma_\mu)\right)^2}{\left((1-\beta_\lambda\beta_\mu)^2+(\beta_\lambda+\beta_\mu)^2\right)\left((1-\gamma_\lambda\gamma_\mu)^2+(\gamma_\lambda+\gamma_\mu)^2\right)}\,. \tag{77}$$

With (9), (10) and (37) we obtain the homogeneous coordinates of the poles H_{AB} and H_{AC}:

$$\begin{aligned} h_{AB} &= (b_1+b_2, b_3-b_0, b_3-b_0, -(b_1+b_2))^\top\,,\\ h_{AC} &= (c_1+c_2, c_3-c_0, c_3-c_0, -(c_1+c_2))^\top\,. \end{aligned} \tag{78}$$

Because of (9) and (78) we get the coefficients σ_i and τ_i in (72) as homogeneous solutions of the quadratic equation

$$\left(\sigma_i(b_1+b_2)+\tau_i(c_1+c_2)\right)^2+\left(\sigma_i(b_3-b_0)+\tau_i(c_3-c_0)\right)^2 = 0\,;\ i=1,2\,. \tag{79}$$

From (74) and (79) follows:

$$\frac{(1+\delta_\varepsilon)^2}{\delta_\varepsilon} = 4\cdot\frac{\left((b_1+b_2)(c_1+c_2)+(b_3-b_0)(c_3-c_0)\right)^2}{\left((b_1+b_2)^2+(b_3-b_0)^2\right)\left((c_1+c_2)^2+(c_3-c_0)^2\right)}\,. \tag{80}$$

According to (65), (77) and (80) the assertion (63) holds. □

Remark R10. The planes ε_{AB} and ε_{AC} are identical, if and only if $\delta_\varepsilon = 1$. With (63) follows:

$$\begin{aligned} P_E, P_A, P_B, P_C \text{ coplanar} \iff &(\delta_{AB}\delta_{AC}\delta_{BC}-(\delta_{AB}+\delta_{AC}+\delta_{BC})+1)^2 = \\ &2\delta_{AB}\delta_{AC}\delta_{BC}+(1-2\delta_{AB})(1-2\delta_{AC})(1-2\delta_{BC})\,. \end{aligned} \tag{81}$$

Remark R11. According to equation (63) the planes ε_{AB} and ε_{AC} are orthogonal ($\delta_\varepsilon = -1$), if and only if $\delta_{AB}+\delta_{AC} = (1+\delta_{AB}\delta_{AC})\cdot\delta_{BC}$. Since the cross ratio of four pairwise distinct points is different from 1, we have $1+\delta_{AB}\delta_{AC}\neq 0$, and therefore

$$\varepsilon_{AB}\perp\varepsilon_{AC} \iff \delta_{BC} = \frac{\delta_{AB}+\delta_{AC}}{1+\delta_{AB}\delta_{AC}}\,. \tag{82}$$

Remark R12. If the fixed points Q_A and R_A of Π_A are complex, then the line P_E+P_A is a passant according to Remark R1, and we get for the angle $\beta := \sphericalangle(\varepsilon_{AB},\varepsilon_{AC})$ $(0\le\beta\le\frac{\pi}{2})$ (see [3], p. 177-178, Satz 3):

$$\beta = \frac{1}{2i}\cdot\ln\delta_\varepsilon\,,\ \ \cos^2\beta = \frac{(1+\delta_\varepsilon)^2}{4\delta_\varepsilon}\,. \tag{83}$$

If the fixed points Q_A and R_A are real, then $P_E + P_A$ is a secant. If furthermore the planes ε_{AB} and ε_{AC} are not separated by the (in this case real) tangent planes τ_1^A and τ_2^A ($\delta_\varepsilon > 0$), then we have for the angle β ($\beta \geq 0$):

$$\beta = \frac{1}{2} \cdot \ln \delta_\varepsilon\,, \quad \cosh^2 \beta = \frac{(1+\delta_\varepsilon)^2}{4\delta_\varepsilon}\,. \tag{84}$$

From (83), (84) and (63) we get a formula for $\cos\beta$ and $\cosh\beta$, respectively, depending on δ_{AB}, δ_{AC} and δ_{BC}.

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