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Skew Semigroup Rings

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Abstract. We investigate properties of skew semigroup rings. Specifically, we give necessary and sufficient conditions which ensure that these rings are unital finite normalizing extensions of the scalars. We then present a large class of examples of such skew semigroup rings in situations more general than groups.

Suppose R is a ring graded by the finite semigroup S. It is shown in [1] and [2] that, in many naturally occurring situations, the category $R - gr_*$ of those S-graded left R-modules having zero z-component is equivalent to the category $R\#S^* - mod$, the full module category over the smash product ring $R\#S^*$. In this article we investigate another role played by the ring $R\#S^*$ within the context of semigroup-graded rings. Specifically, we show that the semigroup S naturally induces endomorphisms of $R\#S^*$, which are in general not automorphisms of $R\#S^*$, in such a way that we may form the skew semigroup rings $(R\#S^*) * S^*$ and $S^* * (R\#S^*)$. As we will show in Section 4, the rings $(R\#S^*) * S^*$ and $S^* * (R\#S^*)$ actually possess some very nice ring-theoretic properties, even in this more general "non-automorphism" case.

The goal of this article is to undertake a general discussion of skew semigroup rings which arise from actions as endomorphisms. As one consequence we are able to more fully appreciate the structure of the aforementioned "smash-skew" rings. This course of investigation can be viewed as the logical continuation of some ideas presented in Okniński's book [9, especially Chapters 4 and 6]. In particular, we show that such skew semigroup rings are often unital rings which contain both the underlying ring and the semigroup, in such a way that the

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resulting ring is a finite normalizing extension of the scalars. Section 1 of this article is devoted to a general discussion of this task. In Section 2 we focus on special types of semigroups (called *semigroups with local identities*), and characterize which skew semigroup rings are unital in this setting. In Section 3 we investigate relationships between actions and reversing actions, especially some naturally occurring ones. In the final section we describe the motivating "smash-skew" examples, which sometimes (but not always) yield nontrivial normalizing extensions.

As a second consequence of our investigation we are able to provide larger contexts for the two "duality" theorems of Cohen & Montgomery [6, Theorems 3.2 and 3.5], and the "realization" theorems of Albu & Năstăsescu [5, Theorem 3.6]. This is done in [3] and [4], respectively.

Throughout this article S will denote a semigroup, possibly with zero z. The set of nonzero elements of S will be denoted by S^*. Unless otherwise indicated, the letter A will always denote an associative unital ring; $E(A)$ will denote the semigroup of ring endomorphisms of A (under composition), while $Aut(A)$ will denote the group of ring automorphisms of A. A ring endomorphism of a unital ring is **not** assumed to take the identity element of the ring to itself. Unless otherwise indicated, all functions will be written on the right, so that $f \circ g = fg$ means "first f, then g". If S is a semigroup and R is an associative ring, we say that R is *graded by* S in case $R = \bigoplus_{s\in S} R_s$ as abelian groups, and for each $s, t \in S$ we have $R_s R_t \subseteq R_{st}$.

If A is a unital subring of the unital ring B, then we say that B is a *(finite) normalizing extension* of A in case there exists a (finite) set I and a subset $\{b_i\}_{i\in I}$ of B such that $\sum_{i\in I} Ab_i = B$, and $Ab_i = b_iA$ for each $i \in I$. In this situation we call the set $\{b_i\}_{i\in I}$ a *(finite) set of normalizing elements* for B over A. The importance of normalizing extensions stems from the fact that if B is a normalizing extension of A, then many properties of A can be passed to B, and vice versa. For a more complete description, see e.g. [8, Chapter 10].

1. Skew semigroup rings

In this section we give the definition of a skew semigroup ring. We then record necessary and sufficient conditions which ensure that such rings have the desired properties mentioned in the introduction. Specifically, our investigation in this section is broken into three questions: When is the resulting ring unital? When does it contain copies of the semigroup and the scalars? And when is it a normalizing extension of the scalars?

Definitions 1.1 *Let A be an associative ring, and let S be a finite semigroup. Suppose $\sigma : S^* \to E(A)$ (resp. $\sigma : S^* \to Aut(A)$) has the property that for any pair g, h in S^* with $gh \neq z, (gh)\sigma = (g)\sigma \circ (h)\sigma$. In this case we say that σ is an* action of S^* as endomorphisms (resp. automorphisms) on A. *For $a \in A$ and $h \in S^*$ we denote $(a)(h)\sigma$ by $a^{(h)\sigma}$.*

We denote the abelian group $\bigoplus_{s\in S^*} A_s$ by $\langle S^* *_\sigma A\rangle$, where each $A_s = A$. For $s \in S^*$ and $a \in A$ we denote the element of $\langle S^* *_\sigma A\rangle$ which is a in the s-component and zero elsewhere by $s[a]$, or simply by sa. We define multiplication in $\langle S^* *_\sigma A\rangle$ by setting $ga \cdot hb = gh[a^{(h)\sigma}b]$ for each pair $g, h \in S$ having $gh \neq z$ and each pair $a, b \in A$, setting $ga \cdot hb = 0$ whenever $gh = z$, and extending linearly to all of $\langle S^* *_\sigma A\rangle$. $\langle S^* *_\sigma A\rangle$ is thus an associative ring.

If in addition A is unital, we denote the subgroup $\bigoplus_{s\in S^*} 1^{(s)\sigma}\cdot A_s$ of $\langle S^* *_\sigma A\rangle$ by $S^* *_\sigma A$. $S^* *_\sigma A$ is a subring of $\langle S^* *_\sigma A\rangle$, and these two rings are equal whenever $1^{(s)\sigma} = 1$ for all $s\in S^*$ (e.g. in case σ is an action as automorphisms on A).

We call any ring of the form $\langle S^* *_\sigma A\rangle$ or $S^* *_\sigma A$ a skew semigroup ring *of S^* with coefficients in A.*

Definitions 1.2 *Let A be an associative ring, and let S be a finite semigroup. Suppose $\gamma : S^* \to E(A)$ (resp. $\gamma : S^* \to Aut(A)$) has the property that for any pair g,h in S^* with $gh \neq z$, $(gh)\gamma = (h)\gamma \circ (g)\gamma$. In this case we say that γ is a* reversing action of S^* as endomorphisms (resp. automorphisms) on A. *For $a\in A$ and $h \in S^*$ we denote $(a)(h)\gamma$ by $a^{(h)\gamma}$.*

We denote the abelian group $\bigoplus_{s\in S^*} A_s$ by $\langle A *_\gamma S^*\rangle$, where each $A_s = A$. For $s\in S^*$ and $a \in A$ we denote the element of $\langle A *_\gamma S^*\rangle$ which is a in the s-component and zero elsewhere by $[a]s$, or simply by as. We define multiplication in $\langle A *_\gamma S^*\rangle$ by setting $ag\cdot bh = [ab^{(g)\gamma}]gh$ for each pair $g,h\in S$ having $gh\neq z$ and each pair $a,b\in A$, setting $ag\cdot bh = 0$ whenever $gh = z$, and extending linearly to all of $\langle A *_\gamma S^*\rangle$. $\langle A *_\gamma S^*\rangle$ is thus an associative ring.

If in addition A is unital, we denote the subgroup $\bigoplus_{s\in S^*} A_s \cdot 1^{(s)\gamma}$ of $\langle A *_\gamma S^*\rangle$ by $A *_\gamma S^*$. $A *_\gamma S^*$ is a subring of $\langle A *_\gamma S^*\rangle$, and these two rings are equal whenever $1^{(s)\gamma} = 1$ for all $s\in S^*$ (e.g. in case γ is a reversing action as automorphisms on A).

We call any ring of the form $\langle A *_\gamma S^*\rangle$ or $A *_\gamma S^*$ a skew semigroup ring *of S^* with coefficients in A.*

It is easy to show that $S^* *_\sigma A$ and $A *_\gamma S^*$ are indeed rings; for instance, the multiplicative closure of $S^* *_\sigma A$ follows from the computation

$$\begin{aligned}(g[1^{(g)\sigma}a])\cdot(h[1^{(h)\sigma}b]) &= gh([1^{(g)\sigma}a]^{(h)\sigma}\cdot[1^{(h)\sigma}b]) \\ &= gh[1^{(g)\sigma(h)\sigma}a^{(h)\sigma}1^{(h)\sigma}b] = gh[1^{(gh)\sigma}(a\cdot 1)^{(h)\sigma}b] \\ &= gh[1^{(gh)\sigma}a^{(h)\sigma}b] \in S^* *_\sigma A\ .\end{aligned}$$

We note that for each $ga \in \langle S^* *_\sigma A\rangle$ we have $ga\cdot h1^{(h)\sigma}b = gha^{(h)\sigma}1^{(h)\sigma}b = gha^{(h)\sigma}b = ga\cdot hb$. By extending linearly, this computation shows that for each $r\in\langle S^* *_\sigma A\rangle$ we have $r\cdot h1^{(h)\sigma}b = r\cdot hb$. But in any ring $R\neq\{0\}$, the existence of elements $x\neq y$ in R having $rx = ry$ for all $r\in R$ implies that R is nonunital. Therefore, in situations where $S^* *_\sigma A$ is a proper subring of $\langle S^* *_\sigma A\rangle$ (resp. $A *_\gamma S^*$ is a proper subring of $\langle A *_\gamma S^*\rangle$), only $S^* *_\sigma A$ (resp. $A *_\gamma S^*$) can possibly be unital. For this reason, and because these smaller skew semigroup rings arise quite naturally in other contexts (see e.g. [4]), we focus on skew semigroup rings of the form $S^* *_\sigma A$ and $A *_\gamma S^*$ for the remainder of this article.

Of course, certain statements about rings of the form $S^* *_\sigma A$ will yield analogous "dual" statements about rings of the form $A *_\gamma S^*$. In order to produce a reasonably compact article, in situations where not more than minimal confusion can occur we will often state and prove statements only about rings of the form $S^* *_\sigma A$, and let the reader invent the corresponding statements for rings of the second type.

For any semigroup S and any unital ring A, we define $\iota : S^* \to Aut(A)$ by setting $(g)\iota = 1_{E(A)}$ for each g in S^*. Then for g,h in S^*, ι clearly satisfies $(h)\iota(g)\iota = (gh)\iota = (g)\iota(h)\iota$, so that we may form the rings $S^* *_\iota A$ and $A *_\iota S^*$. We denote these simply by S^*A (resp. AS^*); these are the usual (contracted) semigroup rings of S with coefficients in A.

Analogous to the situation for groups, rings of the form $S^* *_\sigma A$ provide the canonical examples of semigroup-graded rings; here

$$(S^* *_\sigma A)_s = \left\{ s[1^{(s)\sigma} a] \mid a \in A \right\} \text{ for } s \in S^*,$$

and $(S^* *_\sigma A)_z = \{0\}$.

Example 1.3 Let A denote the matrix ring

$$A = \begin{pmatrix} \mathbf{Z} & 0 & 0 \\ 0 & \mathbf{Z} & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} \end{pmatrix}$$

where $\mathbf{Z}$ is the ring of integers, and $\mathbf{Q}$ is the field of rationals. Let S denote the semigroup which arises from the directed graph $\;_1\cdot \rightarrow \cdot_2\;$; so $S = \{1, \alpha, 2, z\}$ with 1 and 2 idempotent, $1\alpha = \alpha 2 = \alpha$, and all other products equal to z. Let $ae_{11} + be_{22} + ce_{23} + de_{33}$ denote an arbitrary element of A. We define $\sigma : S^* \rightarrow E(A)$ by setting:

$$\begin{aligned} (ae_{11} + be_{22} + ce_{23} + de_{33})^{(1)\sigma} &= ae_{11} \\ (ae_{11} + be_{22} + ce_{23} + de_{33})^{(\alpha)\sigma} &= ae_{22} \\ (ae_{11} + be_{22} + ce_{23} + de_{33})^{(2)\sigma} &= be_{22} + ce_{23} + de_{33}. \end{aligned}$$

A tedious check verifies that $(s)\sigma \in E(A)$ for each $s \in S^*$, and that σ is indeed an action of S^* as endomorphisms on A. Since $(1_A)^{(1)\sigma} = e_{11}$, $(1_A)^{(\alpha)\sigma} = e_{22}$, and $(1_A)^{(2)\sigma} = e_{22} + e_{33}$, an easy computation verifies that the elements of $S^* *_\sigma A$ can be written in the form

$$1\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & b' & c' \\ 0 & 0 & d' \end{pmatrix}$$

where $a, b, b' \in \mathbf{Z}$ and $c, c', d' \in \mathbf{Q}$. Finally, another tedious computation verifies that $S^* *_\sigma A$ is isomorphic to the ring of matrices of the form

$$\begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Z} & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} \end{pmatrix}$$

by the map which takes

$$1\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & b' & c' \\ 0 & 0 & d' \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & d' \end{pmatrix}.$$

As one consequence of Example 1.3 we see that, unlike the situation for groups, the collections of scalars which appear as the coefficients of the underlying semigroup elements need not be isomorphic to each other. Thus these skew semigroup rings do indeed describe a larger class of rings than skew group rings.

There are many additional examples of semigroups acting as endomorphisms on unital rings. For instance, in [4] we show that if R is a ring graded by the semigroup S and ${}_RN$ is any S-graded left R-module, then there are natural actions and reversing actions of S^* on the ring $End_{R-gr}(U(N))$. (This is the ring of graded endomorphisms of $U(N)$, the graded module produced by taking the direct sum of all shift modules of N.) As another example, if η is a fixed endomorphism of the unital ring A then the map $\sigma : (\mathbf{Z}^+, +) \to End(A)$ given by $(m)\sigma = \eta^m$ for each $m \in \mathbf{Z}^+$ is an action of $\mathbf{Z}^+$ as endomorphisms on A, and the skew semigroup ring $\mathbf{Z}^+ *_\sigma A$ is just $A[X, \eta]$, the usual skew polynomial ring over A. A nice overview of some general classes of skew semigroup rings (which include rings of the form $A[X, \eta]$) can be found in [7].

For any semigroup S and unital ring A, if ζ is a map from S^* to $E(A)$, we may also view ζ as a map from $(S^{op})^*$ to $E(A)$. It is easy to show that $\zeta : S^* \to E(A)$ is an action (resp. reversing action) of S^* as endomorphisms on A if and only if $\zeta : (S^{op})^* \to E(A)$ is a reversing action (resp. action) of $(S^{op})^*$ as endomorphisms on A. In particular, we may form $A *_\sigma (S^{op})^*$ (resp. $(S^{op})^* *_\gamma A$) whenever we can form $S^* *_\sigma A$ (resp. $A *_\gamma S^*$). These will turn out to be nontrivial observations, since the rings $S^* *_\sigma A$ and $A *_\sigma (S^{op})^*$ need not be isomorphic. (This happens, for instance, if k is a field, S is the semigroup arising from any poset which is not isomorphic to its own dual poset, and $\sigma = \iota$.)

For notational convenience, we will denote rings of the form $A *_\sigma (S^{op})^*$ by $A *_\sigma S^{o*}$. For general semigroups, there is no natural ring homomorphism from $S^* *_\sigma A$ to $A *_\sigma S^{o*}$ or vice versa. In fact, in general the only ring "related" to $S^* *_\sigma A$ to which there is a nontrivial homomorphism from $S^* *_\sigma A$ is $(A^{op} *_\sigma S^{o*})^{op}$; these two rings are isomorphic by the map $g[1^{(g)\sigma}a] \mapsto [a1^{(g)\sigma}]g$.

Example 1.4 Let A, S, and σ be the ring, semigroup, and action of Example 1.3. It is easily shown that S is isomorphic to S^{op} (by the map which interchanges 1 and 2). We view σ as a reversing action of $(S^{op})^*$ on A as described above. Using computations analogous to those given in the previous Example, we see that the elements of the skew semigroup ring $A *_\sigma S^{o*}$ are of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b' & c' \\ 0 & 0 & d' \end{pmatrix} 2$$

where $a, b, b' \in \mathbf{Z}$ and $c', d' \in \mathbf{Q}$. Finally, another tedious computation verifies that $A *_\sigma S^{o*}$ is isomorphic to the ring of matrices of the form

$$\begin{pmatrix} \mathbf{Z} & 0 & 0 \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} \end{pmatrix}$$

by the map which takes

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b' & c' \\ 0 & 0 & d' \end{pmatrix} 2 \quad \text{to} \quad \begin{pmatrix} a & 0 & 0 \\ b & b' & c' \\ 0 & 0 & d' \end{pmatrix}.$$

In particular, the above two Examples demonstrate that the rings $S^* *_\sigma A$ and $A *_\sigma S^{o*}$ need not be isomorphic even when S and S^{op} are isomorphic semigroups.

We now record necessary and sufficient conditions which answer the first of the three questions given above; namely: When do rings of the form $S^* *_\sigma A$ contain a multiplicative identity? The proof is absolutely straightforward, but long and tedious, and is therefore omitted.

Proposition 1.5 *Let S be a semigroup, and let σ be an action of S^* as endomorphisms on the ring A. Then $S^* *_\sigma A$ has a multiplicative identity if and only if there exist a finite subset $H = \{h_i \mid 1 \le i \le n\}$ of S^* and a subset $B = \{b_i \mid 1 \le i \le n\}$ of A such that for every $s \in S^*$ and every $a \in A$,*

(1) $$\sum_{\substack{h_i \in H \\ \text{with } sh_i = s}} 1^{(s)\sigma} a^{(h_i)\sigma} b_i = 1^{(s)\sigma} a$$

(2) *For each $g \neq s$ in S^*,* $$\sum_{\substack{h_j \in H \\ \text{with } sh_j = g}} 1^{(g)\sigma} a^{(h_j)\sigma} b_j = 0$$

(3) $$\sum_{\substack{h_k \in H \\ \text{with } h_k s = s}} {b_k}^{(s)\sigma} a = 1^{(s)\sigma} a$$

(4) *For each $g \neq s$ in S^*,* $$\sum_{\substack{h_l \in H \\ \text{with } h_l s = g}} 1^{(g)\sigma} b_l^{(s)\sigma} a = 0$$

Moreover, when these conditions are satisfied, then $1_{S^* *_\sigma A} = \sum_{i=1}^n h_i[1^{(h_i)\sigma} b_i]$.

In case G is a group with identity e, and σ acts as automorphisms on A, then $H = \{e\}$ and $B = \{1\}$ clearly satisfy the above conditions. The ring $S^* *_\sigma A$ of Example 1.3 is unital, with $H = \{1, 2\}$ and $B = \{e_{11}, e_{22} + e_{33}\}$. The expected dual result to the above Proposition can also be verified for reversing actions, and yields that the ring $A *_\sigma S^{o*}$ of Example 1.4 is unital, with the same sets H and B.

We cannot hope to get a "nicer" description than that given in Proposition 1.5 of those semigroups S, unital rings A, and actions σ of S on A which yield unital skew semigroup rings. In fact, such a classification is unknown even in the semigroup case (i.e. when $\sigma = \iota$). We will consider a special case of this issue in Section 2, where some additional conditions will be imposed on S. We now give necessary and sufficient conditions which answer the second question, namely: When do both the semigroup S and unital ring of scalars A embed naturally inside the skew semigroup ring $S^* *_\sigma A$?

Proposition 1.6 *Suppose that $S^* *_\sigma A$ is a unital skew semigroup ring, with $1_{S^* *_\sigma A} = \sum_{i=1}^n h_i 1^{(h_i)\sigma} b_i$ as in Proposition 1.5.*
(1) *The function $\tau : A \to S^* *_\sigma A$ defined by setting $(a)\tau = \sum_{i=1}^n h_i[1^{(h_i)\sigma} b_i a]$ for each $a \in A$ is a ring homomorphism. Moreover,*

$$Ker(\tau) = \left\{ a \in A \mid 1^{(h_i)\sigma} b_i a = 0 \text{ for all } 1 \le i \le n \right\},$$

so that τ is injective if and only if whenever $a \in A$ has $1^{(h_i)\sigma} b_i a = 0$ for all $1 \le i \le n$, then $a = 0$.

(2) *The function* $\upsilon : S \to S^* *_\sigma A$ *defined by setting* $(z)\upsilon = 0$ *(in case* S *contains a zero* z*) and* $(s)\upsilon = s[1^{(s)\sigma}]$ *for each* $s \in S^*$ *is a semigroup homomorphism. This homomorphism is injective if and only if* $1^{(s)\sigma} \neq 0$ *for each* $s \in S^*$.

Proof. (1) For $a, a' \in A$ we have

$$\begin{aligned}
&(a)\tau \cdot (a')\tau \\
&\quad = \textstyle\sum_{t=1}^n h_t[1^{(h_t)\sigma} b_t a] \cdot \sum_{i=1}^n h_i[1^{(h_i)\sigma} b_i a'] \\
&\quad = \textstyle\sum_{t=1}^n \sum_{i=1}^n h_t h_i [1^{(h_t)\sigma} b_t a]^{(h_i)\sigma} [1^{(h_i)\sigma} b_i a'] \text{ (multiplication in } S^*) \\
&\quad = \textstyle\sum_{t=1}^n \sum_{i=1}^n h_t h_i [1^{(h_t h_i)\sigma} (b_t a)^{(h_i)\sigma} b_i \cdot a'] \text{ (multiplication in } A) \\
&\quad = \textstyle\sum_{t=1}^n \Big\{ h_t [\sum_{\substack{h_i \in H \\ \text{with } h_t h_i = h_t}} 1^{(h_t)\sigma} (b_t a)^{(h_i)\sigma} b_i \cdot a'] \\
&\qquad\qquad + \textstyle\sum_{\substack{g \in H \\ \text{with } g \neq h_t}} g[\sum_{\substack{h_i \in h \\ \text{with } h_t h_i = g}} 1^{(g)\sigma} (b_t a)^{(h_i)\sigma} b_i \cdot a'] \Big\} \\
&\qquad\qquad \text{(reorder the sum)} \\
&\quad = \textstyle\sum_{t=1}^n \Big\{ h_t [1^{(h_t)\sigma} b_t a \cdot a'] + \sum_{\substack{g \in H \\ \text{with } g \neq h_t}} g[0 \cdot a'] \Big\} \text{ (using Proposition 1.5(1,2))} \\
&\quad = \textstyle\sum_{t=1}^n h_t [1^{(h_t)\sigma} b_t a a'] = (aa')\tau.
\end{aligned}$$

(2) As $s[1^{(s)\sigma}] \cdot t[1^{(t)\sigma}] = st[1^{(st)\sigma}]$ for all $s, t \in S^*$, υ is easily seen to be a semigroup homomorphism. The only way that two elements of the form $s[1^{(s)\sigma}]$ and $t[1^{(t)\sigma}]$ can be equal is if the corresponding coefficient is 0, which gives the second statement. □

As a side remark, suppose in the notation of 1.6 that $a \in Ker(\tau)$; so $1^{(h_i)\sigma} b_i a = 0$ for all $1 \leq i \leq n$. Then for every $s \in S^*$ we have $\sum_{\substack{h_k \in H \\ \text{with } h_k s = s}} 1^{(h_k)\sigma} b_k a = 0$, so that by applying $(s)\sigma$ we get $\sum_{\substack{h_k \in H \\ \text{with } h_k s = s}} 1^{(s)\sigma} b_k^{(s)\sigma} a^{(s)\sigma} = 0$. But by property (3) in Proposition 1.5 this last equation becomes $1^{(s)\sigma} a^{(s)\sigma} = 0$, so that $a^{(s)\sigma} = 0$. Thus $Ker(\tau) \subseteq \Big\{ a \in A \mid \ a^{(s)\sigma} = 0 \text{ for all } s \in S^* \Big\}$.

Example 1.7 Using the notation of Examples 1.3 and 1.4 and Proposition 1.6, it is straightforward to show that both the function $\tau : A \to S^* *_\sigma A$ and the analogous function $\tau : A \to A *_\sigma S^{o*}$ are injective, and that the image of each of these functions corresponds to matrices of the form

$$\begin{pmatrix} \mathbf{Z} & 0 & 0 \\ 0 & \mathbf{Z} & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} \end{pmatrix}.$$

Again using the matrix version of $S^* *_\sigma A$, the function $\upsilon : S \to S^* *_\sigma A$ has $(1)\upsilon = e_{11}$, $(\alpha)\upsilon = e_{12}$, and $(2)\upsilon = e_{22} + e_{33}$. Similarly, the function $\upsilon : S^{op} \to A *_\sigma S^{o*}$ has $(1)\upsilon = e_{11}$, $(\alpha)\upsilon = e_{21}$, and $(2)\upsilon = e_{22} + e_{23}$. Each is clearly injective.

We now answer the last of the three questions: When is the skew semigroup ring $S^* *_\sigma A$ a normalizing extension of A by the semigroup S under the embeddings described above?

Proposition 1.8 *Let S be a (finite) semigroup, and let σ (resp. γ) be an action (resp. reversing action) of S^* as endomorphisms on the unital ring A. Suppose A and S embed in the unital ring $S^* *_\sigma A$ (resp. $A *_\gamma S^*$) as described in Proposition 1.6, with $1_{S^* *_\sigma A} = \sum_{i=1}^n h_i[1^{(h_i)\sigma} b_i]$ (resp. $1_{A *_\gamma S^*} = \sum_{i=1}^n [b_i 1^{(b_i)\gamma}] h_i$) as given in Proposition 1.5.*
(1) The set $Im(v) = \left\{ s[1^{(s)\sigma}] \mid s \in S^* \right\}$ is a (finite) set of normalizing elements for $S^* *_\sigma A$ over $Im(\tau) = \left\{ \sum_{i=1}^n h_i[1^{(h_i)\sigma} b_i a] \mid a \in A \right\}$ if and only if $A^{(s)\sigma} = 1^{(s)\sigma} \cdot A$ for each $s \in S^*$.
(2) The set $Im(v) = \left\{ [1^{(s)\gamma}]s \mid s \in S^* \right\}$ is a (finite) set of normalizing elements for $A *_\gamma S^*$ over $Im(\tau) = \left\{ \sum_{i=1}^n [a b_i 1^{(h_i)\gamma}] h_i \mid a \in A \right\}$ if and only if $A^{(s)\gamma} = A \cdot 1^{(s)\gamma}$ for each $s \in S^*$.

Proof. (1) By definition, we must show that $S^* *_\sigma A = \sum_{s \in S^*} s[1^{(s)\sigma}] \cdot Im(\tau)$, and that $Im(\tau) \cdot s[1^{(s)\sigma}] = s[1^{(s)\sigma}] \cdot Im(\tau)$ for each $s \in S^*$. For future reference we note that for any $s, t, u \in S^*$ having $st = u$ one can easily check that $1^{(u)\sigma} = 1^{(u)\sigma} 1^{(t)\sigma}$. Using this fact, together with statements (1) and (2) of Proposition 1.5, we get

$$\begin{aligned}
& s[1^{(s)\sigma}] \cdot (\textstyle\sum_{i=1}^n h_i[1^{(h_i)\sigma} b_i a']) \\
&= \textstyle\sum_{i=1}^n s h_i [1^{(s h_i)\sigma} b_i a'] \\
&= \sum_{\substack{h_i \in H \\ \text{with } s h_i = s}} s[1^{(s)\sigma} b_i a'] + \sum_{\substack{g \in S^* \\ \text{with } g \neq s}} \Big(\sum_{\substack{h_j \in H \\ \text{with } s h_j = g}} g[1^{(g)\sigma} b_j a'] \Big) \\
&= s[(\sum_{\substack{h_i \in H \\ \text{with } s h_i = s}} 1^{(s)\sigma} 1^{(h_i)\sigma} b_i) a'] + \sum_{\substack{g \in S^* \\ \text{with } g \neq s}} g[(\sum_{\substack{h_j \in H \\ \text{with } s h_j = g}} 1^{(g)\sigma} 1^{(h_j)\sigma} b_j) a'] \\
&= s[(1^{(s)\sigma} 1^{(h_i)\sigma}) a'] + \sum_{\substack{g \in S^* \\ \text{with } g \neq s}} g[0 \cdot a'] \\
&= s[1^{(s)\sigma} a'].
\end{aligned}$$

This equation yields immediately that $S^* *_\sigma A = \sum_{s \in S^*} s[1^{(s)\sigma}] \cdot Im(\tau)$. We now use statements (3) and (4) of Proposition 1.5 to verify directly that for each $a, a' \in A$ and $s \in S^*$,

$$(\sum_{i=1}^n h_i[1^{(h_i)\sigma} b_i a]) \cdot s[1^{(s)\sigma}] = s[a^{(s)\sigma}] \ .$$

Hence any element of $Im(\tau) \cdot s[1^{(s)\sigma}]$ is of the form $s[a^{(s)\sigma}] = s[1^{(s)\sigma} a']$ (where $a' = a^{(s)\sigma}$), which in turn is an element of $s[1^{(s)\sigma}] \cdot Im(\tau)$ by the first equation. Using the same idea, we see that the inclusion $s[1^{(s)\sigma}] \cdot Im(\tau) \subseteq Im(\tau) \cdot s[1^{(s)\sigma}]$ holds if and only if every element in A of the form $1^{(s)\sigma} a'$ can be written as $a^{(s)\sigma}$ for some $a \in A$. The result is now immediate.

The proof of (2) is similar. □

Of course the condition $A^{(s)\sigma} = 1^{(s)\sigma} \cdot A$ is satisfied whenever σ is an action as automorphisms of S on A. Thus Proposition 1.8 generalizes the well-known result that any skew group ring is a normalizing extension of the appropriate scalar ring.

Example 1.9 We again refer to the notation of Examples 1.3 and 1.4. It is easy to show that $A^{(\alpha)\sigma} = \mathbf{Z} e_{22}$, while $1^{(\alpha)\sigma} \cdot A = e_{22} A = \mathbf{Z} e_{22} + \mathbf{Q} e_{23}$. Thus by Proposition 1.8(1) we conclude that $S^* *_\sigma A$ is not a normalizing extension of A by the elements of S.

On the other hand, it is straightforward to show that $A^{(s)\sigma} = A \cdot 1^{(s)\sigma}$ for each $s \in S^*$, so that by Proposition 1.8(2) we conclude that $A *_\sigma S^{o*}$ is in fact a normalizing extension of A by the elements of S^{op}.

We now show that, in fact, $S^* *_\sigma A$ is not a normalizing extension of A for *any* choice of elements in $S^* *_\sigma A$. Let

$$t = \begin{pmatrix} m_{11} & m_{12} & q_{13} \\ 0 & m_{22} & q_{23} \\ 0 & 0 & q_{33} \end{pmatrix}$$

be a fixed element of $S^* *_\sigma A$ (as given by the isomorphism of Example 1.3). If we denote by b the matrix

$$b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in A, \quad \text{then} \quad tb = \begin{pmatrix} 0 & 0 & m_{12} \\ 0 & 0 & m_{22} \\ 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, an arbitrary element of A can be written as

$$a = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix},$$

so that an arbitrary element of At must be of the form

$$\begin{pmatrix} x_{11}m_{11} & x_{11}m_{12} & x_{11}q_{13} \\ 0 & x_{22}m_{22} & x_{22}q_{23} + r_{23}q_{33} \\ 0 & 0 & r_{33}q_{33} \end{pmatrix}.$$

Just suppose that $tA = At$; in particular, $tb \in At$. Then by the general form of elements of At given above we must have $x_{11}m_{12} = 0$ and $x_{11}q_{13} = m_{12}$. But this is clearly impossible for $m_{12} \neq 0$. Thus we have shown that any normalizing element has $m_{12} = 0$. But then

$$\sum \{At \mid t \in S^* *_\sigma A \text{ with } At = tA\} \neq S^* *_\sigma A.$$

We conclude that T is not a normalizing extension of A.

2. Semigroups with local identities

We show in this section that for certain types of frequently occurring semigroups, there are "natural candidates" for the sets H and B of Proposition 1.5. As a partial motivation for considering such semigroups, we note that it is easy to show (using the same ideas discussed subsequent to Definitions 1.2) that a necessary condition on S for $S^* *_\sigma A$ to be unital is: if $g, g' \in S^*$ have $gh = g'h$ for all $h \in S^*$, then $g = g'$. One way of ensuring this "separation of points" condition is to require the existence of elements described in the next definition.

Definition 2.1 *We call the semigroup S a* semigroup with local identities *(or simply an* l.i. semigroup*) in case there exists a subset E of S^* consisting of (nonzero) orthogonal idempotents with the property that for each $g \in S^*$, there exist $e, e' \in E$ with $g = eg$ and $g = ge'$. We call E a set of local identities for S.*

The condition given above is clearly equivalent to the condition that for each $g \in S^*$ there exist e, e' in E with $g = ege'$. If $g \in S^*$ and $e \in E$ has $eg = g$, then such e is unique in E for g; we often denote e by e_g. Similarly, if $ge' = g$ we denote e' by e'_g. The semigroups arising from groups, preordered sets, and directed graphs are easily seen to be l.i. semigroups; in particular, the semigroup S given in Example 1.3 is l.i. (We note that each of these can be viewed as a semigroup of matrices, specifically a subsemigroup of the Brandt semigroup $\mathcal{M}(1, X, X, id)$ for an appropriate set X.) Furthermore, quotients of such semigroups by ideals are also l.i.. As a particular example, let $T = \{1, \alpha, 2, \beta, 3, z\}$, with $\{1, 2, 3\}$ as idempotents, $\alpha = 1\alpha 2$, $\beta = 2\beta 3$, and $\alpha\beta = z$. Then T may be viewed as $\Gamma / < \alpha\beta >$, where Γ is the semigroup arising from the graph $\cdot_1 \to \cdot_2 \to \cdot_3$.

When S is finite, the l.i. condition is stronger than the *Property (I)* condition given in [10, Definition 1.3]:

There exists a finite subset U of S such that for
each $s \in S$ there exist $e, f \in U$ with $es = sf = s$.

Specifically, Ponizovskiĭ does not require the subset U to consist of orthogonal idempotents. However, he shows [10, Lemma 1.2 and Theorem 1.4] that if the semigroup ring RS is unital, then S has *Property (I)* and the set U can be taken to consist of idempotents.

Lemma 2.2 *Let S be an l.i. semigroup.*
(1) *If E is a set of local identities for S, then for each $s \in S^*$ and $e \in E$, either $se = s$ or $se = z$. Similarly, either $es = s$ or $es = z$.*
(2) *The set of local identities in S is unique.*

Proof. (1) By definition there exists $f \in E$ with $sf = s$. If $se \neq z$, then $(sf)e \neq z$, so by the orthogonality of E we have $f = e$. Thus $s = sf = se$. The proof of the second statement is similar.

(2) Suppose E and F are sets of local identities for S. Let $e \in E$. Then there exists $f' \in F$ with $ef' = e$. In particular, $ef' \neq z$, so by the second statement of the part (1) (with $s = f'$) we have $ef' = f'$. So $e = ef' = f'$, and $E \subseteq F$. That $F \subseteq E$ follows similarly. □

The goal of this section is to present conditions which ensure that $S^* *_\sigma A$ is unital. We show in Proposition 2.4 below that the unicity of $S^* *_\sigma A$ is equivalent to a property which arises in the more general arena of rings graded by semigroups, of which skew semigroup rings are a specific example (see the comments prior to Example 1.3). We may view this Proposition as a result analogous to [10, Theorem 1.4(ii)] for skew semigroup rings.

Definition 2.3 *Let S be an l.i. semigroup (possibly with zero z) having a set of local identities E. If R is a ring graded by S we call R* locally unital *in case for each $e \in E$ there exists $a_e \in R_e$ such that for each pair $e, e' \in E$, each $g \in S^*$ with $g = ege'$, and each $r \in R_g$ we have $a_e r = r = r a_{e'}$.*

Since each e in E is idempotent, the above condition implies in particular that a_e is a multiplicative identity for the ring R_e.

It is easy to show that if S is any l.i. semigroup having a finite set of local identities E, then any locally unital S-graded ring R is unital; specifically, $1_R = \sum_{e \in E} a_e$. We show in the next result that for skew semigroup rings corresponding to this type of semigroup, the converse holds as well.

Proposition 2.4 *Let S be an l.i. semigroup with a finite set of local identities E, and let $\sigma : S^* \to E(A)$ be an action of S^* as endomorphisms on the unital ring A. The following are equivalent.*
(1) *$S^* *_\sigma A$ is unital. (In this case we get $1_{S^* *_\sigma A} = \sum_{e\in E} e[1^{(e)\sigma}]$.)*
(2) *$S^* *_\sigma A$ is locally unital, with $a_e = e[1^{(e)\sigma}]$ for each $e \in E$.*
(3) *For each $e \in E$ and $a \in A$ we have $a^{(e)\sigma} = 1^{(e)\sigma} a$.*
(4) *For each $e \in E$, $A^{(e)\sigma} = 1^{(e)\sigma} \cdot A$.*

Proof. It will be useful to note that for $g \in S^*$ and $a \in A$, if $e = e_g$, then $e[1^{(e)\sigma}] \cdot g[1^{(g)\sigma} a] = g[1^{(g)\sigma} a]$ in $S^* *_\sigma A$; similarly, if $e = e'_g$ then $g[1^{(g)\sigma} a] \cdot e[1^{(e)\sigma}] = g[a^{(e)\sigma}]$. This yields in particular that the element $\sum_{e\in E} e[1^{(e)\sigma}]$ is always a left identity for $S^* *_\sigma A$.

(1)$\Leftrightarrow$(2) For $\Rightarrow$, the previous paragraph indicates that the element $e[1^{(e)\sigma}]$ acts correctly when multiplied on the left. But as $S^* *_\sigma A$ is unital, the left identity $\sum_{e\in E} e[1^{(e)\sigma}]$ of $S^* *_\sigma A$ must in fact be the identity of $S^* *_\sigma A$. In particular, $g[1^{(g)\sigma} a] = g[1^{(g)\sigma} a] \cdot \sum_{e\in E} e[1^{(e)\sigma}] = g[1^{(g)\sigma} a] \cdot e'_g[1^{(e'_g)\sigma}] + \sum_{\substack{e\in E\\ e\neq e'_g}} 0 = g[1^{(g)\sigma} a] \cdot e'_g[1^{(e'_g)\sigma}]$, so that the multiplication on the right side behaves correctly as well. As noted above, the converse holds for all locally unital S-graded rings.

(2)$\Rightarrow$(3) In particular we have $e[1^{(e)\sigma} a] = e[1^{(e)\sigma} a] \cdot e[1^{(e)\sigma}] = e[(1^{(e)\sigma} a)^{(e)\sigma} \cdot 1^{(e)\sigma}] = e[a^{(e)\sigma}]$, and the result follows on equating coefficients of e.

(3)$\Rightarrow$(4) Trivial.

(4)$\Rightarrow$(1) As noted above, the element $\sum_{e\in E} e[1^{(e)\sigma}]$ is always a left identity, so it suffices to show that $\sum_{e\in E} e[1^{(e)\sigma}]$ is a right identity. It is easy to show that we always have $1^{(g)\sigma} \cdot 1^{(e'_g)\sigma} = 1^{(g)\sigma}$. By hypothesis we have that for each $a \in A$ there exists $c \in A$ with $1^{(e)\sigma} a = c^{(e)\sigma}$; upon applying $(e)\sigma$ to these expressions we get $(1^{(e)\sigma} a)^{(e)\sigma} = (c^{(e)\sigma})^{(e)\sigma} = c^{(e\cdot e)\sigma} = c^{(e)\sigma} = 1^{(e)\sigma} a$. Similarly one gets that $1^{(e)\sigma} a = 1^{(e)\sigma} a 1^{(e)\sigma}$ for all $e \in E$. Using these facts, we get

$$\begin{aligned} & g[1^{(g)\sigma} a] \cdot \textstyle\sum_{e\in E} e[1^{(e)\sigma}] \\ &\quad = g[1^{(g)\sigma} a] \cdot e'_g[1^{(e'_g)\sigma}] + 0 = g[1^{(g)\sigma} \cdot 1^{(e'_g)\sigma} a] \cdot e'_g[1^{(e'_g)\sigma}] \\ &\quad = g[(1^{(g)\sigma} \cdot 1^{(e'_g)\sigma} a)^{(e'_g)\sigma} \cdot 1^{(e'_g)\sigma}] = g[1^{(g)\sigma} \cdot 1^{(e'_g)\sigma} a \cdot 1^{(e'_g)\sigma}] = g[1^{(g)\sigma} a]. \end{aligned}$$

The result follows by extending linearly. $\square$

When σ satisfies the property that $a^{(e)\sigma} = 1^{(e)\sigma} a$ for each $e \in E$ and $a \in A$, we will for obvious reasons call σ a *locally unital* action. For locally unital actions we see that the hypotheses of Proposition 1.5 are satisfied, where if $E = \{e_i \mid 1 \le i \le n\}$, then we set $h_i = e_i$ and $b_i = 1^{(e_i)\sigma}$ for each i. It is easy to show that, in the context of Examples 1.3 and 1.4, σ is a locally unital action (resp. reversing action) of S^* (resp. $(S^{op})^*$) on A.

We conclude this section with a series of notes. First, statements (2), (3), and (4) of the previous Proposition are equivalent whether or not E is finite, and statements (3) and (4) are equivalent for any idempotents in S, not just elements of E. In addition, the conditions given above are easily shown to be equivalent to the condition that $a^{(e)\sigma} = 1^{(e)\sigma} a 1^{(e)\sigma}$ for each $e \in E$ and $a \in A$, and to the condition that $A^{(e)\sigma} = 1^{(e)\sigma} \cdot A \cdot 1^{(e)\sigma}$ for each $e \in E$.

Second, the unicity of a skew semigroup ring $S^* *_\sigma A$ does not necessarily imply that S is l.i. For instance, we let S be the semigroup described in [2, Example 3.3]; so $S = \{a, b, c, z\}$ with $aa = a$, $ba = ca = cb = b$, $cc = c$, and all other products equal z. For any unital ring T we let $R = S^*T$ be the usual contracted semigroup ring. Then it is easily shown that R is unital, with $1_R = a + b[-1_T] + c$. However, S is not l.i. (We note that in fact S has *Property (I)*, but that idempotents in S do not commute. In addition, there are non-idempotents in the support of 1_R. Cf. [10, Theorem 1.4(ii)].)

Third, we cannot extend the previous Proposition to arbitrary semigroup-graded rings. Specifically, if R is a ring graded by the finite l.i. semigroup S, then R may be unital without being locally unital. For example, let $S = \{z, a, b\}$ with $a^2 = ab = ba = a$, and $b^2 = b$. Then S is l.i. with $E = \{b\}$. For any unital ring T the ring direct sum $W = T \bigoplus T$ is graded by S by setting $W_a = T \bigoplus \{0\}$, $W_b = \{0\} \bigoplus T$, and $W_z = \{0\}$. Clearly W is unital. But W cannot be locally unital, as the graded components of W annihilate each other.

Fourth, we give an example of an action of an l.i. semigroup S on a unital ring A which does not satisfy the conditions of the Proposition. Specifically, let $S = \{e\}$, a one-element group, and let A be the 2×2 upper triangular matrices over the field k; so $A = \{k_{11}e_{11} + k_{12}e_{12} + k_{22}e_{22} \mid k_{ij} \in k\}$. We define the action $\sigma : S \to E(A)$ by setting $(k_{11}e_{11} + k_{12}e_{12} + k_{22}e_{22})^{(e)\sigma} = k_{11}e_{11}$. Then $1_A^{(e)\sigma} = e_{11}$, so that $1_A^{(e)\sigma} \cdot A = \{k_{11}e_{11} + k_{12}e_{12} \mid k_{ij} \in k\} \neq \{k_{11}e_{11} \mid k_{11} \in k\} = A^{(e)\sigma}$.

Finally, with Proposition 1.8 in mind, we note that we may view statement (4) of Proposition 2.4 as a "precondition" for a $S^* *_\sigma A$ to be a normalizing extension of A.

3. "Inverse" actions

If σ is an action of the group G as automorphisms on the ring A, then since automorphisms are invertible we may define $\gamma : G \to Aut(A)$ by setting $(g)\gamma = ((g)\sigma)^{-1}$ for each $g \in G$. As such, γ is a reversing action, so that we may form the ring $A *_\gamma G$. In this section we discuss analogs of this inverse action for actions as endomorphisms on more general semigroups. In fact, our motivating examples possess such "inverses".

Proposition 3.1 *Let A be a unital ring, let S be any semigroup, let $\sigma : S^* \to E(A)$ be an action of S^* as endomorphisms on A, and let $\gamma : S^* \to E(A)$ be a reversing action of S^* as endomorphisms on A. Suppose that for each $h \in S^*$ and each $a \in A$ we have*

$$(a^{(h)\gamma})^{(h)\sigma} = 1^{(h)\sigma} a. \qquad (*)$$

Then the map $\Psi : A *_\gamma S^* \to S^* *_\sigma A$ defined by the linear extension of

$$\Psi : \; [a1^{(h)\gamma}]h \longmapsto h[a^{(h)\sigma}]$$

is a surjective ring homomorphism.

Proof. We note first that $h[a^{(h)\sigma}] = h[(1a)^{(h)\sigma}] = h[1^{(h)\sigma}a^{(h)\sigma}]$, so that Ψ indeed maps into $S^* *_\sigma A$. To show that Ψ is well-defined, it suffices to show that $(a1^{(h)\gamma})^{(h)\sigma} = a^{(h)\sigma}$. But $(a1^{(h)\gamma})^{(h)\sigma} = a^{(h)\sigma}(1^{(h)\gamma})^{(h)\sigma} = a^{(h)\sigma}1^{(h)\sigma}1$ (by $(*)$) $= (a1)^{(h)\sigma} = a^{(h)\sigma}$.

To show that Ψ is a ring homomorphism, we let $h, g \in S^*$ and compute

$$
\begin{aligned}
&([a1^{(h)\gamma}]h \cdot [b1^{(g)\gamma}]g)\Psi \\
&\quad = ([ab^{(h)\gamma}1^{(hg)\gamma}]hg)\Psi \quad \text{(definition of multiplication in } A *_\gamma S^*) \\
&\quad = hg[(ab^{(h)\gamma})^{(hg)\sigma}] = hg[a^{(hg)\sigma}(b^{(h)\gamma})^{(h)\sigma\circ(g)\sigma}] \\
&\quad = hg[a^{(hg)\sigma}(1^{(h)\sigma}b)^{(g)\sigma}] \quad \text{(by } (*)) \\
&\quad = hg[a^{(hg)\sigma}1^{(hg)\sigma}b^{(g)\sigma}] = hg[a^{(hg)\sigma}b^{(g)\sigma}] = hg[(a^{(h)\sigma})^{(g)\sigma}b^{(g)\sigma}] \\
&\quad = ha^{(h)\sigma} \cdot gb^{(g)\sigma} \quad \text{(definition of multiplication in } S^* *_\sigma A) \\
&\quad = ([a1^{(h)\gamma}]h)\Psi \cdot ([b1^{(g)\gamma}]g)\Psi.
\end{aligned}
$$

Finally, to show that Ψ is surjective we note that for $h \in S^*$ and $c \in A$ we have $h[1^{(h)\sigma}c] = h[(c^{(h)\gamma})^{(h)\sigma}]$ (by $(*)$) $= ([c^{(h)\gamma}1^{(h)\gamma}]h)\Psi$. □

In a manner analogous to that used in the above Proposition, we also obtain

Proposition 3.2 *Let A be a unital ring, let S be any semigroup, let $\sigma : S^* \to E(A)$ be an action of S^* as endomorphisms on A, and let $\gamma : S^* \to E(A)$ be a reversing action of S^* as endomorphisms on A. Suppose that for each $h \in S^*$ and each $a \in A$ we have*

$$(a^{(h)\sigma})^{(h)\gamma} = a1^{(h)\gamma}. \tag{$**$}$$

Then the map $\Phi : S^* *_\sigma A \to A *_\gamma S^*$ defined by the linear extension of

$$\Phi : \; h[1^{(h)\sigma}a] \longmapsto [a^{(h)\gamma}]h$$

is a surjective ring homomorphism.

Of course, if σ is an action of the group G as automorphisms on A, then $\gamma : G \to Aut(A)$ defined by setting $(g)\gamma = ((g)\sigma)^{-1}$ for each $g \in G$ satisfies conditions $(*)$ and $(**)$.

Example 3.3 We again let A denote the matrix ring

$$A = \begin{pmatrix} \mathbf{Z} & 0 & 0 \\ 0 & \mathbf{Z} & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} \end{pmatrix}$$

and we let S denote the semigroup $\{1, \alpha, 2, z\}$ as in Examples 1.3 and 1.4. Let $ae_{11} + be_{22} + ce_{23} + de_{33}$ denote an arbitrary element of A. We define $\gamma : S^* \to E(A)$ by setting:

$$
\begin{aligned}
(ae_{11} + be_{22} + ce_{23} + de_{33})^{(1)\gamma} &= ae_{11} \\
(ae_{11} + be_{22} + ce_{23} + de_{33})^{(\alpha)\gamma} &= be_{11} \\
(ae_{11} + be_{22} + ce_{23} + de_{33})^{(2)\gamma} &= be_{22} + ce_{23} + de_{33}.
\end{aligned}
$$

A tedious check verifies that $(s)\gamma \in E(A)$ for each $s \in S^*$, and that γ is indeed a reversing action as endomorphisms on A. Since $(1_A)^{(1)\gamma} = e_{11}$, $(1_A)^{(\alpha)\gamma} = e_{11}$, and $(1_A)^{(2)\gamma} = e_{22} + e_{33}$, an easy computation verifies that the elements of $A *_\gamma S^*$ can be written in the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 1 + \begin{pmatrix} a' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & d \end{pmatrix} 2$$

where $a, a', b \in \mathbf{Z}$ and $c, d \in \mathbf{Q}$. Finally, another tedious computation verifies that $A *_\gamma S^*$ is isomorphic to the factor ring of matrices of the form

$$\begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Z} & \mathbf{Q} \\ 0 & 0 & \mathbf{Q} \end{pmatrix} / \begin{pmatrix} 0 & 0 & \mathbf{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by the map which takes

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 1 + \begin{pmatrix} a' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \alpha + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & d \end{pmatrix} 2 \quad \text{to} \quad \begin{pmatrix} a & a' & 0 \\ 0 & b & c \\ 0 & 0 & d \end{pmatrix} + \begin{pmatrix} 0 & 0 & \mathbf{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now let σ be the action described in Example 1.3. Then with γ as given in the above Example,

$$(ae_{11} + be_{22} + ce_{23} + de_{33})^{(\alpha)\gamma(\alpha)\sigma} = be_{22} \neq be_{22} + ce_{23} = 1^{(\alpha)\sigma} \cdot (ae_{11} + be_{22} + ce_{23} + de_{33}),$$

so that γ does not satisfy $(*)$ with σ. (In fact, as a consequence of Proposition 3.5 below, we show that NO reversing action satisfies $(*)$ with σ.) However, it is straightforward to compute that

$$(ae_{11} + be_{22} + ce_{23} + de_{33})^{(s)\sigma(s)\gamma} = (ae_{11} + be_{22} + ce_{23} + de_{33}) \cdot 1^{(s)\gamma}$$

for each $s \in S^*$, so that γ does in fact satisfy $(**)$ with σ. Thus by Proposition 3.2 there is a surjection $S^* *_\sigma A \to A *_\gamma S^*$; using the identifications with matrix rings made in the appropriate Examples, this surjection is simply the canonical homomorphism on the factor ring.

Conditions $(*)$ and $(**)$ will be important in the context of our motivating "smash-skew" examples. As we now show in Propositions 3.4 and 3.5, these conditions can be used to show (respectively) that certain pairs of skew semigroup rings are isomorphic, and that certain types of skew semigroup rings are normalizing extensions.

Proposition 3.4 *Let $\sigma : S^* \to E(A)$ be an action of S^* as endomorphisms on the unital ring A, and let $\gamma : S^* \to E(A)$ be a reversing action of S^* as endomorphisms on A. Suppose that conditions $(*)$ and $(**)$ are satisfied for each $h \in S^*$ and each $a \in A$. Then (using the notation of Proposition 3.1 and 3.2) $\Psi : A *_\gamma S^* \to S^* *_\sigma A$ is a ring isomorphism, and $\Phi = \Psi^{-1}$.*

Proof. As $(*)$ holds we have that Ψ is surjective by Proposition 3.1; hence it suffices to show that $\Psi \circ \Phi$ is the identity on $A *_\gamma S$. But for $h \in S^*$ and $a \in A$ we have$([a1^{(h)\gamma}]h)\Psi\Phi = (h[a^{(h)\sigma}])\Phi = [(a^{(h)\sigma})^{(h)\gamma}]h = [a1^{(h)\gamma}]h$ (by $(**)$). □

Proposition 3.5 *Let S be a (finite) l.i. semigroup with finite set of local identities E.*

(1) *Suppose $\sigma : S^* \to E(A)$ is a locally unital action of S^* as endomorphisms on A with the properties*

(a) If $a \in A$ has $1^{(e)\sigma}a = 0$ for all $e \in E$ then $a = 0$.

(b) $1^{(s)\sigma} \neq 0$ *for all* $s \in S^*$.

Suppose also there exists a reversing action $\gamma : S^* \to E(A)$ *of* S *on* A *such that condition* $(*)$ *holds. Then* $S^* *_\sigma A$ *is a (finite) normalizing extension of* A *by* S.

(2) *Suppose* $\gamma : S^* \to E(A)$ *is a locally unital reversing action of* S^* *as endomorphisms on* A *with the properties*

(a) If $a \in A$ *has* $a1^{(e)\gamma} = 0$ *for all* $e \in E$ *then* $a = 0$.

(b) $1^{(s)\gamma} \neq 0$ *for all* $s \in S^*$.

Suppose also there exists an action $\sigma : S^* \to E(A)$ *of* S^* *on* A *such that condition* $(**)$ *holds. Then* $A *_\gamma S^*$ *is a (finite) normalizing extension of* A *by* S.

Proof. (1) For each $s \in S^*$ the containment $A^{(s)\sigma} \subseteq 1^{(s)\sigma} \cdot A$ always holds. But by $(*)$, for each $a \in A$ we have $1^{(s)\sigma}a = (a^{(s)\gamma})^{(s)\sigma} \in A^{(s)\sigma}$. Thus $A^{(s)\sigma} = 1^{(s)\sigma} \cdot A$ for each $s \in S^*$. As conditions (a) and (b) imply the hypotheses of Proposition 1.6, Proposition 1.8 yields the result. The proof of (2) is similar. □

As one consequence of this Proposition we conclude that the action σ of Example 1.3 cannot have $(**)$ with any reversing action, since by Example 1.9 $S^* *_\sigma A$ is not a normalizing extension of A by S.

The final two results of this section focus on situations in which S acts as automorphisms on the ring A. We start by giving another consequence of Proposition 3.5.

Corollary 3.6 *Let* S *be a (finite) semigroup.*
(1) *Let* $\sigma : S^* \to Aut(A)$ *be an action of* S^* *as automorphisms on* A. *We define* $\gamma : S^* \to Aut(A)$ *by setting* $(g)\gamma = ((g)\sigma)^{-1}$ *for each* $g \in S^*$. *Then* γ *is a reversing action of* S^* *as automorphisms on* A, *and* $S^* *_\sigma A \cong A *_\gamma S^*$. *Moreover, if* S *is l.i. then* $S^* *_\sigma A$ *is a (finite) normalizing extension of* A *by* S.
(2) *Let* $\gamma : S^* \to Aut(A)$ *be a reversing action of* S^* *as automorphisms on* A. *We define* $\sigma : S^* \to A$ *by setting* $(g)\sigma = ((g)\gamma)^{-1}$ *for each* $g \in S^*$. *Then* σ *is an action of* S^* *as automorphisms on* A, *and* $A *_\gamma S^* \cong S^* *_\sigma A$. *Moreover, if* S *is l.i. then* $A *_\gamma S^*$ *is a (finite) normalizing extension of* A *by* S.

Proof. We prove (1); the proof of (2) is nearly identical. We note that for any ring automorphism $\Lambda \in Aut(A)$ we have $(1_A)\Lambda = 1_A$; thus σ and γ satisfy both $(*)$ and $(**)$, so that the isomorphism follows from Proposition 3.4. Now assume that S is l.i. As each $e \in E$ is idempotent we have that $(e)\sigma$ is an idempotent element of the group $Aut(A)$, so that $(e)\sigma$ is the identity on A. Therefore σ is automatically a locally unital action, and the last statement follows directly from Proposition 3.5. □

As a trivial consequence of this Corollary we have that the semigroup rings AS^* and S^*A are isomorphic for any ring A and any semigroup S. To put the following result in context, we remind the reader that in general, the three rings $S^* *_\sigma A$, $A *_\sigma S^{o*}$, and $A *_\gamma S^*$ may indeed be pairwise nonisomorphic.

Corollary 3.7 *Let* G *be a group,* A *any unital ring, and* $\sigma : G \to Aut(A)$ *an action of* G *as automorphisms on* A. *For each* $g \in G$ *we set* $(g)\gamma = ((g)\sigma)^{-1}$. *Then the four skew group rings* $G *_\sigma A$, $A *_\sigma G^{op}$, $A *_\gamma G$, *and* $G^{op} *_\gamma A$ *are isomorphic.*

Proof. As G is a group and σ is a group homomorphism we have $(g)\gamma = (g^{-1})\sigma$ for each $g \in G$. By Corollary 3.6 and symmetry it suffices to show that $A *_\gamma G$ is isomorphic to $A *_\sigma G^{op}$. As $1^{(g)\sigma} = 1$ for each $g \in G$, we can define $\varphi : A *_\gamma G \to A *_\sigma G^{op}$ to be the linear extension of $(ag)\varphi = ag^{-1}$. It is straightforward to show that φ is the desired isomorphism. □

4. Actions and reversing actions of semigroups on smash products

If R is a ring graded by the semigroup S, then the ring $R\Delta S^*$ has been defined and investigated in [1] and [2]. This ring is important in the study of the category $R - gr_*$ of those S-graded R-modules having zero z-component, since by [2, Corollary 2.27] there is a category equivalence between $R - gr_*$ and a full subcategory of $R\Delta S^* - mod$. When S is a finite semigroup then the ring $R\Delta S^*$ is denoted by $R\#S^*$, and is called the *smash product of R with S^**; in this case there is a category equivalence between $R - gr_*$ and $R\#S^* - mod$. If in addition S is l.i. and R is locally unital, then $R\#S^*$ is the collection of S^*-square matrices of the form

$$\sum_{\substack{f,h\in S^* \\ fh\neq z}} r_f e_{fh,h}$$

(where $r_f \in R_f$, and $e_{fh,h}$ denotes the standard matrix unit which is 1 in the (fh, h) coordinate and 0 elsewhere) under the usual matrix operations.

The goal of this last section is to describe both a natural action and a natural reversing action of S^* as endomorphisms on $R\#S^*$. In turn, we will study the resulting "smash-skew" rings by using the general results of the three previous sections.

Example 4.1 Let R denote the ring

$$R = \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{pmatrix}$$

where $\mathbf{Z}$ and $\mathbf{Q}$ are the integers and rationals, respectively. Then R is graded by the semigroup $S = \{1, \alpha, 2, z\}$ of Example 1.3, where $R_1 = \mathbf{Z}e_{11}$, $R_\alpha = \mathbf{Q}e_{12}$, $R_2 = \mathbf{Q}e_{22}$, and $R_z = \{0\}$. We note that there are four nonzero products in S: $1 \cdot 1 = 1$, $1 \cdot \alpha = \alpha$, $\alpha \cdot 2 = \alpha$, and $2 \cdot 2 = 2$. This yields that the elements of $R\#S^*$ can possibly be nonzero only in the $(1, 1)$, (α, α), $(\alpha, 2)$, and $(2, 2)$ entries. Indeed, it is easy to check that the ring $R\#S^*$ is precisely the ring A described in Example 1.3.

The semigroup S is called *right $*$-cancellative* in case whenever $f, g, h \in S^*$ have $fh = gh \neq z$ then $f = g$. For instance, the semigroups arising from partially ordered sets and directed graphs are right $*$-cancellative.

Definitions 4.2 *Let S be a finite right $*$-cancellative l.i. semigroup, and let R be a locally unital ring graded by S.*

(1) *We define an action of S^* as endomorphisms on $R\#S^*$ as follows:*

$$\rho : S^* \to E(R\#S^*)$$

where for each $h \in S$, $(h)\rho \in E(R\#S^*)$ is the linear extension of the function

$$(h)\rho: \quad r_f e_{fg,g} \longmapsto \begin{cases} r_f e_{fgh,gh} & \text{if } fgh \neq z \\ 0 & \text{otherwise} \end{cases}.$$

We show below (using the right $*$-cancellativity of S) that $(h)\rho$ is a ring homomorphism for each $h \in S^*$, and that ρ is indeed an action of S^* as endomorphisms on $R\#S^*$. In particular we may also view ρ as a reversing action of $(S^{op})^*$ on $R\#S^*$. Thus we may form the rings

$$S^* *_\rho [R\#S^*] \quad \text{and} \quad [R\#S^*] *_\rho S^{o*}.$$

(2) *We define a reversing action of S^* as endomorphisms on $R\#S^*$ as follows:*

$$\lambda : S^* \to E(R\#S^*)$$

where for each $h \in S^*$, $(h)\lambda \in E(R\#S^*)$ is the linear extension of the function

$$(h)\lambda: \quad r_f e_{fg,g} \longmapsto \begin{cases} r_f e_{fk,k} & \text{if } \exists k \in S^* \text{ with } kh = g \\ 0 & \text{otherwise} \end{cases}.$$

By the right $*$-cancellativity of S, if such a k exists it is necessarily unique. As above, it is straightforward to check that $(h)\lambda$ is a ring homomorphism for each $h \in S^*$, and that λ is a reversing action of S^* as endomorphisms on $R\#S^*$. In particular we may also view λ as an action of $(S^{op})^*$ on $R\#S^*$. Thus we may form the rings

$$[R\#S^*] *_\lambda S^* \quad \text{and} \quad S^{o*} *_\lambda [R\#S^*].$$

We verify here that $(h)\rho$ defined above is indeed a ring homomorphism for all $h \in S^*$. For $r_f e_{fg,g}$ and $r_{f'} e_{f'g',g'} \in R\#S^*$ we have

$$\begin{aligned}
&(r_f e_{fg,g} \cdot r_{f'} e_{f'g',g'})^{(h)\rho} \\
&\quad = \begin{cases} (r_f r_{f'} e_{fg,g'})^{(h)\rho} & \text{if } g = f'g' \\ 0 & \text{otherwise} \end{cases} \\
&\quad = \begin{cases} r_f r_{f'} e_{fgh,g'h} & \text{if } g = f'g' \text{ and } fgh \neq z \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(r_f e_{fg,g})^{(h)\rho} \cdot (r_{f'} e_{f'g',g'})^{(h)\rho} \\
&\quad = \begin{cases} r_f e_{fgh,gh} & \text{if } fgh \neq z \\ 0 & \text{otherwise} \end{cases} \cdot \begin{cases} r_{f'} e_{f'g'h,g'h} & \text{if } f'g'h \neq z \\ 0 & \text{otherwise} \end{cases} \\
&\quad = \begin{cases} r_f r_{f'} e_{fgh,g'h} & \text{if } fgh \neq z \text{ and } f'g'h \neq z \text{ and } gh = f'g'h \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Thus the resulting expressions are equal, as long as the conditionals are equivalent. It is easy to show that the first set of conditionals always implies the second. Conversely, if $gh = f'g'h$ and both are nonzero, then by right $*$-cancellativity we get that $g = f'g'$, which gives the first set of conditionals.

For the remainder of this article we will assume that S is a finite right $*$-cancellative l.i. semigroup, and that R is a locally unital S-graded ring. Then in particular $R\#S^*$ is unital, with $1_{R\#S^*} = \sum_{l \in S^*} 1_{e_l} e_{l,l}$.

Example 4.3 Let R be the ring of Example 4.1. Then $R\#S^*$ equals the ring A of Example 1.3. It is straightforward to show that the action σ described in the previous Examples is precisely the action ρ given in Definition 4.2 in this specific setting. Similarly, the reversing action γ given in Example 3.3 is precisely the action λ given in Definition 4.2 in this setting.

The following side remark will point out one of the distinctions between the behaviour of the maps ρ and λ given in Definitions 4.2 versus more general actions and reversing actions. Suppose S is a semigroup with zero z; then we may define a new semigroup S_{new} with zero z_{new} by setting $S_{new} = S \cup \{z_{new}\}$ and defining multiplication in the obvious way. Clearly we may identify S^*_{new} with S; in particular, if A is any ring and $\sigma : S \to End(A)$ is a semigroup homomorphism, then we may view $\sigma : S^*_{new} \to End(A)$ as an action of S^*_{new} as endomorphisms on A. Thus the study of actions of semigroups as endomorphisms on rings (which by definition describe functions from the nonzero elements of a semigroup) actually includes the (seemingly more general) study of semigroup homomorphisms into $End(A)$. In fact, there are situations in which we have a semigroup T and an action of T^* on a ring A which cannot be extended to a semigroup homomorphism from T to $End(A)$ (see e.g. [3, Section 2]). In contrast, it is easy to check that the function ρ (resp. λ) given in this smash product setting may in fact be extended to a semigroup homomorphism from S (resp. S^{op}) to $E(R\#S^*)$ by setting $(z)\rho = 0$ (resp. $(z)\lambda = 0$).

Using the fact that $\sum_{l\in S^*} 1_{e_l} e_{l,l}$ is the multiplicative identity of $R\#S^*$, it is easily verified that for each $h \in S^*$ we have

$$(1_{R\#S^*})^{(h)\rho} = \sum \{1_{e_l} e_{lh,lh} \mid l \in S^*, lh \neq z\} \text{ and}$$

$$(1_{R\#S^*})^{(h)\lambda} = \sum \{1_{e_l} e_{k_l,k_l} \mid l \in S^* \text{ for which } \exists\ k_l \in S^* \text{ with } k_l h = l\}.$$

Moreover, we claim that $(1_{R\#S^*})^{(h)\rho} \neq 0$ and $(1_{R\#S^*})^{(h)\lambda} \neq 0$. For if $e \in E$ has $eh = h$, then by the previous displayed equation the term $1_e e_{eh,eh} = 1_e e_{h,h}$ appears in the summation expression for $(1_{R\#S^*})^{(h)\rho}$. In addition, the right $*$-cancellativity of S implies that this is the only term in the expression for $(1_{R\#S^*})^{(h)\rho}$ which has an entry in the (h,h) component of $R\#S^*$. Thus no cancellation can occur in this coordinate, and we conclude that $(1_{R\#S^*})^{(h)\rho}$ $\neq 0$. The proof for $(1_{R\#S^*})^{(h)\lambda}$ is similar.

Using these observations, it is straightforward to check the following explicit descriptions of the elements of the four smash-skew rings:

$$S^* *_\rho [R\#S^*] = \textstyle\sum_{h\in S^*} \{h[r_f e_{fg,g}] \mid fg \neq z,\ r_f \in R_f, \text{ and } fg = lh \text{ for some } l \in S^*\},$$

$$[R\#S^*] *_\rho S^{o*} = \textstyle\sum_{h\in S^*} \{[r_f e_{fg,g}]h \mid fg \neq z,\ r_f \in R_f, \text{ and } g = lh \text{ for some } l \in S^*\},$$

$$[R\#S^*] *_\lambda S^* = \textstyle\sum_{h\in S^*} \{[r_f e_{fg,g}]h \mid fg \neq z,\ r_f \in R_f, \text{ and } gh \neq z\}, \text{ and}$$

$$S^{o*} *_\lambda [R\#S^*] = \textstyle\sum_{h\in S^*} \{h[r_f e_{fg,g}] \mid fg \neq z,\ r_f \in R_f, \text{ and } fgh \neq z\}.$$

Lemma 4.4 *The action ρ and reversing action λ described in Definitions 4.2 are locally unital. In particular, by Proposition 2.4, the four rings $S^* *_\rho [R\#S^*]$, $[R\#S^*] *_\rho S^{o*}$, $[R\#S^*] *_\lambda S^*$, and $S^{o*} *_\lambda [R\#S^*]$ are unital.*

Proof. We show that the action $\rho : S^* \to E(R\#S^*)$ is locally unital; the others are similar. By definition we need to show that $(r_t e_{tf,f})^{(e)\rho} = 1^{(e)\rho} \cdot r_t e_{tf,f}$ for each $e \in E$, each $r_t \in R_t$, and each pair $t, f \in S^*$ with $tf \neq z$. But by Lemma 2.2(1), for $l \in S^*$ and $e \in E$ we have $le \neq z$ if and only if $le = l$. So by the above description of $1^{(e)\rho}$ we have

$$1^{(e)\rho} \cdot r_t e_{tf,f} = (\sum_{\substack{l \in S^* \\ le \neq z}} 1_{e_l} e_{le,le}) \cdot r_t e_{tf,f} = \begin{cases} r_t e_{tf,f} & \text{if } tfe \neq z \\ 0 & \text{otherwise} \end{cases},$$

which is precisely the definition of $(r_t e_{tf,f})^{(e)\rho}$. □

We now apply the general results about skew semigroup rings which were developed in Sections 1 through 3 to this smash-skew setting. We begin by describing the analog of Proposition 1.6. Recall that for $g \in S^*$, if $e' \in E$ with $ge' = g$ then we denote e' by e'_g.

Lemma 4.5 *The linear extension of each of the following maps is an injective ring homomorphism from $R\#S^*$ to the indicated smash-skew ring. In fact, each of these maps is the appropriate analog of the map τ given in Proposition 1.6.*
(1) *The map $\tau : R\#S^* \to S^* *_\rho [R\#S^*]$ defined by* $\tau : r_f e_{fg,g} \longmapsto e'_g[r_f e_{fg,g}]$.
(2) *The map $\tau : R\#S^* \to [R\#S^*] *_\rho S^{o*}$ defined by* $\tau : r_f e_{fg,g} \longmapsto [r_f e_{fg,g}]e'_g$.
(3) *The map $\tau : R\#S^* \to [R\#S^*] *_\lambda S^*$ defined by* $\tau : r_f e_{fg,g} \longmapsto [r_f e_{fg,g}]e'_g$.
(4) *The map $\tau : R\#S^* \to S^{o*} *_\lambda [R\#S^*]$ defined by* $\tau : r_f e_{fg,g} \longmapsto e'_g[r_f e_{fg,g}]$.

Proof. It is easy to verify that for $g \in S^*$ and $e \neq e'_g \in E$ we have $e_{fg,g}1^{(e)\lambda} = 0 = 1^{(e)\sigma}e_{fg,g}$. The results follow using this fact and a long tedious check. As a representative example, for the map given in (2) we have $\sum_{e \in E}[r_f e_{fg,g}(1_{R\#S})^{(e)\rho}]e = [r_f e_{fg,g}]e'_g = (r_f e_{fg,g})\tau$. □

We now give the first of two main structure results about smash-skew rings.

Proposition 4.6 *Let ρ and λ denote the action and reversing action (resp.) of S^* on $R\#S^*$ given above.*
(1) *There is an isomorphism of rings*

$$[R\#S^*] *_\rho S^{o*} \cong S^{o*} *_\lambda [R\#S^*].$$

(2) *Using the appropriate injections described in the previous Lemma, each of the rings $[R\#S^*] *_\rho S^{o*}$ and $S^{o*} *_\lambda [R\#S^*]$ is a normalizing extension of $R\#S^*$ by $(S^{op})^*$.*

Proof. We will show that Propositions 3.4 and 3.5 may be applied in this setting, where the semigroup under consideration is S^{op}, $A = R\#S^*$, $\gamma = \rho$, and $\sigma = \lambda$. We note that $\gamma = \rho$ is in fact an *action* (not a reversing action) of S^{op} on A. We first show that conditions $(*)$ and $(**)$ are satisfied here. For $(*)$, it suffices to show that $(r_f e_{fg,g})^{(h)\rho\circ(h)\lambda} = 1^{(h)\lambda} r_f e_{fg,g}$ for all elements f, g, h in S^* with $fg \neq z$, and each $r_f \in R_f$. But a straightforward computation verifies this equation, since each of these expressions can easily be shown to equal

$$\begin{cases} r_f e_{fg,g} & \text{if } fgh \neq z \\ 0 & \text{otherwise} \end{cases}.$$

For $(**)$, it suffices to show that $(r_f e_{fg,g})^{(h)\lambda\circ(h)\rho} = r_f e_{fg,g} 1^{(h)\rho}$ for all elements f, g, h in S^* with $fg \neq z$, and each $r_f \in R_f$. But a computation analogous to the one above yields this equation, as each of these expressions equals

$$\begin{cases} r_f e_{fg,g} & \text{if } \exists\, l \in S^* \text{ with } g = lh \\ 0 & \text{otherwise} \end{cases} .$$

We now apply Proposition 3.4 to obtain the isomorphism of statement (1). For completeness, we note that in this setting the isomorphism

$$\Phi = \Psi^{-1} : S^{o*} *_\lambda [R\#S^*] \to [R\#S^*] *_\rho S^{o*}$$

may be explicitly described as the linear extension of the map which takes, for each $h[r_f e_{fg,g}]$ in $S^{o*} *_\lambda [R\#S^*]$, $h[r_f e_{fg,g}] \longmapsto [r_f e_{fgh,gh}]h$. (That this assignment indeed gives a function from $S^{o*} *_\lambda [R\#S^*]$ to $[R\#S^*] *_\rho S^{o*}$ follows from the descriptions of the elements of these two rings given above.)

For statement (2) we again note (as in the proof of 4.5) that the map τ defined here is precisely the map τ of Proposition 1.6 in this particular setting. In addition, the previous Lemma yields that τ is injective. Statement (2) now follows immediately from Proposition 3.5 together with the observations made prior to Lemma 4.4 that $(1_{R\#S^*})^{(h)\rho} \neq 0$ and $(1_{R\#S^*})^{(h)\lambda} \neq 0$ for all $h \in S^*$. □

In light of the above result, one might hope that the other two smash-skew rings are isomorphic, and are normalizing extensions of the smash. However, Examples 1.3 and 3.3, combined with the observation made in Example 4.3, indicate that this is in fact not the case in general. We now show that if the underlying semigroup has somewhat more structure than has been previously assumed, then these two additional skew semigroup rings are in fact connected by a surjective ring homomorphism. Additionally, in this setting one of these two rings is a normalizing extension of the smash.

The l.i. semigroup S is called a *category* in case whenever $f, g, h \in S^*$ have $fg \neq z$ and $gh \neq z$, then $fgh \neq z$.

Proposition 4.7 *Suppose S is a right $*$-cancellative category, and let R be a locally unital ring graded by S.*
(1) *There is a surjection of rings*

$$\Phi : S^* *_\rho [R\#S^*] \to [R\#S^*] *_\lambda S^*.$$

(2) *$[R\#S^*] *_\lambda S^*$ is a normalizing extension of $R\#S^*$ by S via the embedding τ of Lemma 4.5(3).*

Proof. We show that property $(**)$ of Proposition 3.2 holds in the situation where the semigroup under consideration is S, $A = R\#S^*$, $\sigma = \rho$, and $\gamma = \lambda$. So we must show that $(r_f e_{fg,g})^{(h)\rho\circ(h)\lambda} = r_f e_{fg,g} 1^{(h)\lambda}$ for all $f, g, h \in S^*$ with $fg \neq z$ and $r_f \in R_f$. But

$$(r_f e_{fg,g})^{(h)\rho\circ(h)\lambda} = \begin{cases} r_f e_{fg,g} & \text{if } fgh \neq z \\ 0 & \text{otherwise} \end{cases} , \qquad \text{while}$$

$$r_f e_{fg,g} 1^{(h)\lambda} = \begin{cases} r_f e_{fg,g} & \text{if } gh \neq z \\ 0 & \text{otherwise} \end{cases}.$$

As we are assuming that $fg \neq z$, we conclude that the two conditionals are equal for all $f, g, h \in S^*$ precisely when S is a category.

Statement (1) now follows directly from Proposition 3.2, while statement (2) follows from Proposition 3.5(2). □

We note here that condition $(*)$ of Proposition 3.1 rarely holds in this smash-skew setting. Specifically, straightforward computations yield

$$(r_f e_{fg,g})^{(h)\lambda \circ (h)\rho} = \begin{cases} r_f e_{fg,g} & \text{if } \exists\, k \in S^* \text{ with } g = kh \\ 0 & \text{otherwise} \end{cases}, \qquad \text{while}$$

$$1^{(h)\rho} r_f e_{fg,g} = \begin{cases} r_f e_{fg,g} & \text{if } \exists\, l \in S^* \text{ with } fg = lh \\ 0 & \text{otherwise} \end{cases}.$$

Of course both of these conditionals are true if S is a group. But the equivalence of the conditionals implies that for $f \in S^*$ and $e' \in E$ with $fe' = f$, there exists $k \in S^*$ with $e' = kf$. This is obviously a very restrictive condition on S.

Next, the observations made in Examples 1.9 and 4.3 imply that, in contrast to statement (2) of the previous Proposition, the ring $S^* *_\rho [R\#S^*]$ need not be a normalizing extension of $R\#S^*$, even when S is a category.

Finally, we note that condition $(**)$ of Proposition 3.2 need not hold in the setting of Proposition 4.7 in case S is not a category. For example, let T be the semigroup which arises from the directed graph ${}_1\cdot \rightarrow_2 \cdot \rightarrow_3 \cdot \rightarrow_4$, where the arrows are labelled f, g, h respectively, and let S be the semigroup $T/ < fgh >$; so S is the semigroup gotten from T by identifying the product fgh with z. It is clear that S is l.i. and right $*$-cancellative, and that S is not a category (as $fg \neq z$ and $gh \neq z$ but $fgh = z$). Let A be any unital ring, let R denote the (contracted) semigroup ring AS^*, and let r_f denote the element $1_A f$ of R. We show that the element $r_f e_{fg,g} = r_f e_{fg,g} 1^{(h)\lambda}$ is not in $B^{(h)\lambda}$ for $B = R\#S^*$. To this end, assume that $b \in B$ with $b^{(h)\lambda} = r_f e_{fg,g}$. Write $b = \sum_{\substack{s,t \in S^* \\ st \neq z}} b_s e_{st,t}$; then from $b^{(h)\lambda} = r_f e_{fg,g}$ we get $\sum_{\substack{s,k \\ kh = t \neq z}} b_s e_{sk,k} = r_f e_{fg,g}$. But this is a contradiction as $t = kh$ and $sk = fg$ imply $st = skh = fgh = z$.

Apart from being of interest in their own right, the smash-skew rings described in this last section may also be viewed in relation to the theorems of Cohen & Montgomery [6, Theorems 3.2 and 3.5] and of Albu & Năstăsescu [5, Theorem 3.6]. We establish such connections in [3] and [4], respectively.

References

[1] Abrams, G.; Menini, C.: *Categorical equivalences and realization theorems.* J. Pure App. Alg., to appear.

[2] Abrams, G.; Menini, C.; del Río, A.: *Realization theorems for categories of graded modules over semigroup-graded rings.* Comm. Alg. **22**(13) (1994), 5343–5388.

[3] Abrams, G.; Menini, C.: *Duality theorems for finite semigroups.* J. Pure App. Alg. **109** (1996), 1–22.

[4] Abrams, G.; Menini, C.: *Rings of endomorphisms of semigroup-graded modules.* Rocky Mtn. J. Math. **26**(2) (1996), 375–406.

[5] Albu, T.; Năstăsescu, C.: *Infinite group-graded rings, rings of endomorphisms, and localization.* J. Pure App. Alg. **59** (1989), 125–150.

[6] Cohen, M.; Montgomery, S.: *Group-graded rings, smash products, and group actions.* Trans. A.M.S. **282**(1) (1984), 237–258.

[7] Dauns, J.: *Generalized skew polynomial rings.* Trans. A.M.S. **271**(2) (1982), 575–586.

[8] McConnell, J. C.; Robson, J. C.: *Noncommutative Noetherian Rings.* John Wiley & Sons, Wiley-Interscience Series of Texts, Monographs, and Tracts, 1987.

[9] Okniński, J.: *Semigroup Algebras.* Marcel Dekker, Monographs and Books in Pure and Applied Mathematics (Volume 138), 1991.

[10] Ponizovskiĭ, J. S.: *Semigroup rings.* Semigroup Forum **36** (1987), 1–46.

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