Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 2, 399-414.

The Ohm Type Properties for Multiplication Ideals

Majid M. Ali
Al-Zaytoonah University, Dept. of Mathematics and Computer Science
P.O.Box 130, Amman 11733, JORDAN

Abstract. Let R be a commutative ring with identity. An ideal I in R is called a multiplication ideal if every ideal contained in I is a multiple of I. Ohm's properties for finitely generated ideals in Prüfer domains are investigated by Gilmer. These properties are generalized by Naoum and studied for finitely generated multiplication ideals. The purpose of this work is to generalize the results of Gilmer and Naoum to the case of multiplication ideals (not necessarily finitely generated ones).

Key words and phrases: Multiplication ideal, Ohm condition, weak-cancellation ideal, localization.

MSC 1991: Primary 13A15, Secondary 13B20.

0. Introduction

Let R be a ring. An ideal $aR + bR$ is said to satisfy property (k) if

$$(aR + bR)^k = a^k R + b^k R$$

for each positive integer k, see [6]. It is known that this property holds if $aR + bR$ is invertible [6]. This property is extended for finitely generated cancellation ideals [5], finitely generated projective ideals [10] and [11], and for finitely generated multiplication ideals [9]. It is also shown in [6] that if I_λ $(\lambda \in \Lambda)$ is a collection of ideals in a Prüfer domain R then

$$\left(\sum_{\lambda\in\Lambda} I_\lambda\right)^k = \sum_{\lambda\in\Lambda} I_\lambda^k$$

for each positive integer k. It follows easily from this fact that if $I = \sum_{\alpha\in\Lambda} x_\alpha R$ is a non-finitely generated ideal in a Prüfer domain R, then I satisfies property (k). In the first part of this paper we prove that if I_λ $(\lambda \in \Lambda)$ is a collection of ideals in a ring R such that $\sum_{\lambda\in\Lambda} I_\lambda$ is a multiplication ideal, then for each positive integer k

$$\left(\sum_{\lambda\in\Lambda} I_\lambda\right)^k = \sum_{\lambda\in\Lambda} I_\lambda^k$$

from which it follows easily that property (k) is satisfied for non-finitely generated multiplication ideals (see Theorem 2.2 and Corollary 2.3).

On the other hand, we say that the ideal $aR + bR$ satisfies property (k*) if

$$(aR \cap bR)^k = a^k R \cap b^k R$$

for each positive integer k. It is proved that $aR + bR$ has property (k*) if it is invertible [6], projective [10], and multiplication [9]. If R is a Prüfer domain and I_i $(1 \le i \le n)$ is a finite collection of ideals in R, then

$$\left(\bigcap_{l=1}^{n} I_l\right)^k = \bigcap_{l=1}^{n} I_l^k$$

for each positive integer k, [6]. A direct consequence of the above fact is the following: If R is a Prüfer domain and $I = \sum_{i=1}^{n} a_i R$ is a finitely generated (i.e. invertible) ideal in R, then I has property (k*). In the second part of our paper we prove that if I_i $(1 \le i \le n)$ is a finite collection of ideals in a ring R such that $\sum_{l=1}^{n} I_l$ is multiplication, then

$$\left(\bigcap_{l=1}^{n} I_l\right)^k = \bigcap_{l=1}^{n} I_l^k$$

for each positive integer k (Theorem 3.6). From this fact we show that property (k*) holds for finitely generated multiplication ideals (Corollary 3.8).

Unfortunatly, property (k*) is not valid for non-finitely generated multiplication ideals. However, we give sufficient conditions for non-finitely generated ideals to satisfy property (k*), see Theorem 3.7. Some examples which help to see that the converse of Theorems 2.2, 3.6 and 3.7 is not satisfied are given at the end of the second part of this paper. The properties defined above are refered to as Ohm type properties. In the last part of this paper, we study other Ohm type properties for multiplication ideals (not necessarily finitely generated ones). Finally, it should be mentioned that all rings considered in this paper are commutative with 1. Consult [4], [5], [6], [7] and [13] for the basic concepts used.

1. Preliminaries

Let R be a ring and I and J two ideals in R. Following [14, p.113], the conductor of J into I is the set of all elements $x \in R$ such that $xJ \subseteq I$. It is denoted by $[I : J]$. Obviously, $[I : J]$ is an ideal in R. The annihilator of I is denoted by $ann(I)$ and equals to $[O : I]$. For $a \in I$, the annihilator of a is denoted by $ann(a)$ and equals to $[O : aR]$. An ideal I in a ring R is called a multiplication ideal provided that for every ideal $J \subseteq I$, there exists an ideal K in R such that $J = KI$ [7]. Note that $K \subseteq [J : I]$. Therefore $J = KI \subseteq [J : I]I \subseteq J$ so that $J = [J : I]I$. It is useful to remark that if I is a multiplication ideal and J any ideal in R, then

$$I \cap J = \big[(I \cap J) : I\big]I = \big[J : I\big]I$$

(see [12, Lemma 3.1]).

We have the following results [2]:

Proposition 1.1. *Let R be a ring and I_λ $(\lambda \in \Lambda)$ a collection of ideals in R such that $\sum_{\lambda\in\Lambda} I_\lambda$ is a multiplication ideal. Then*

$$\sum_{\lambda\in\Lambda}\left[I_\lambda : \left(\sum_{\lambda\in\Lambda} I_\lambda\right)\right] + ann(a) = R \quad \textit{for each} \quad a \in \sum_{\lambda\in\Lambda} I_\lambda.$$

Proposition 1.2. *Let R be a ring and I_i $(1 \le i \le n)$ a finite collection of ideals such that $\sum_{k=1}^{n} I_k$ is a multiplication ideal. Then*

$$\sum_{l=1}^{n}\left[\left(\bigcap_{k=1}^{n} I_k\right) : \hat{I}_l\right] + ann(a) = R \quad \textit{for each} \quad a \in \sum_{k=1}^{n} I_k,$$

where $\hat{I}_l$ denotes the intersection of all I_i except I_l.

Lemma 1.3. *Let R be a ring and I, J two ideals in R such that $I + J$ is a multiplication ideal. Then*

$$[I_M : J_M] + [J_M : I_M] = R_M$$

for every maximal ideal M in R.

The following result is given in [1]:

Lemma 1.4. *Let R be a ring and I, J two ideals in R such that*

$$[I : J] + [J : I] = R,$$

then the following statements are true:

i) $IJ = (I + J)(I \cap J)$,

ii) $(I \cap J)K = IK \cap JK$ *for every ideal* K *in* R.

In particular, if $I + J$ *is a multiplication ideal in* R *then* (i) *and* (ii) *are satisfied.*

2. The equality $\left(\sum_{\lambda\in\Lambda} I_\lambda\right)^k = \sum_{\lambda\in\Lambda} I_\lambda^k$ for ideals

It is well-known that if $I = \sum_{i=1}^n a_iR$ is a multiplication ideal in a ring R, then

$$\left(\sum_{i=1}^{n} a_iR\right)^k = \sum_{i=1}^{n} a_i^kR$$

for each positive integer k. It is also proved that if I_i $(1 \leq i \leq n)$ is a finite collection of finitely generated ideals in an arithmetical ring R (i.e. a ring in which every finitely generated ideal is multiplication), then

$$\left(\sum_{i=1}^{n} I_i\right)^k = \sum_{i=1}^{n} I_i^k$$

for each positive integer k, [9]. Finally, it is shown that if I_λ $(\lambda \in \Lambda)$ is a collection of ideals in a Prüfer domain R, then

$$\left(\sum_{\lambda\in\Lambda} I_\lambda\right)^k = \sum_{\lambda\in\Lambda} I_\lambda^k$$

for each positive integer k, [6].

Now, we generalize the above results to a collection (not necessarily a finite one) of multiplication ideals (not necessarily finitely generated ideals).

We start by the following proposition.

Proposition 2.1. *Let* I *and* J *be ideals in a ring* R *such that* $[I : J] + [J : I] = R$. *Then*

$$(I + J)^k = I^k + J^k$$

for each positive integer k. *In particular, if* $I + J$ *is a multiplication ideal, then the result also holds.*

Proof. Since

$$[I : J] + [J : I] = R,$$

there exist $x \in [I : J]$, $y \in [J : I]$ with $x + y = 1$. Hence $xJ \subseteq I$ and $yI \subseteq J$. It follows that

$$IJ = xIJ + yIJ \subseteq I^2 + J^2.$$

Thus

$$(I+J)^2 = I^2 + J^2.$$

We prove the proposition by induction on k. The proposition is true for $k = 2$. Assume that $k > 2$ and the proposition is true for k, i.e.

$$(I+J)^k = I^k + J^k.$$

It follows that

$$IJ^{k-1},\ I^{k-1}J \subseteq I^k + J^k.$$

Next

$$\begin{aligned}(I+J)^{k+1} &= (I+J)^k(I+J) = (I^k+J^k)(I+J)\\ &= I^{k+1} + I^kJ + IJ^k + J^{k+1}.\end{aligned}$$

It is enough to show that

$$I^kJ\,,\ IJ^k \subseteq I^{k+1} + J^{k+1}.$$

As $x + y = 1$, we have

$$\begin{aligned}I^kJ = xI^kJ + yI^kJ &\subseteq I^{k+1} + I^{k-1}J(yI)\\ &\subseteq I^{k+1} + (I^k + J^k)(yI)\\ &\subseteq I^{k+1} + yI^{k+1} + J^{k+1}\\ &= I^{k+1} + J^{k+1}.\end{aligned}$$

Similarly,

$$IJ^k \subseteq I^{k+1} + J^{k+1}.$$

The proposition is thus concluded. The second assertion follows locally first by using Lemma 1.3 and hence globally.

The following result is a generalization of proposition 2.1.

Theorem 2.2. *Let R be a ring and I_λ $(\lambda \in \Lambda)$ a collection of ideals in R such that $\sum_{\lambda\in\Lambda} I_\lambda$ is a multiplication ideal. Then*

$$\left(\sum_{\lambda\in\Lambda} I_\lambda\right)^k = \sum_{\lambda\in\Lambda} I_\lambda^k \quad \textit{for each positive integer } k.$$

Proof. Since the inclusion relation $\sum_{\lambda\in\Lambda} I_\lambda^k \subseteq \left(\sum_{\lambda\in\Lambda} I_\lambda\right)^k$ always holds, we need only to prove that $\left(\sum_{\lambda\in\Lambda} I_\lambda\right)^k \subseteq \sum_{\lambda\in\Lambda} I_\lambda^k$. It is enough to prove it locally. Thus we may assume that R is local. As $\sum_{\lambda\in\Lambda} I_\lambda$ is a multiplication ideal, we have from Proposition 1.1 that

$$R = \sum_{\lambda\in\Lambda}\left[I_\lambda : \left(\sum_{\lambda\in\Lambda} I_\lambda\right)\right] + ann(a) \quad \text{for each} \quad a \in \sum_{\lambda\in\Lambda} I_\lambda.$$

If $\sum_{\lambda\in\Lambda} I_\lambda = O$, then $I_\lambda = O$ for each $\lambda \in \Lambda$. Therefore the statement is trivially true. Thus we may assume that $\sum_{\lambda\in\Lambda} I_\lambda \neq O$. It follows that there exists $0 \neq a \in \sum_{\lambda\in\Lambda} I_\lambda$, and hence $ann(a) \neq R$. As R is a local ring, we infer that

$$R = \left[I_{\lambda_0} : \Big(\sum_{\lambda\in\Lambda} I_\lambda \Big) \right] \quad \text{for some } \lambda_0 \in \Lambda,$$

and hence $I_{\lambda_0} = \sum_{\lambda\in\Lambda} I_\lambda$. It follows that

$$\left(\sum_{\lambda\in\Lambda} I_\lambda \right)^k = I_{\lambda_0}^k \subseteq \sum_{\lambda\in\Lambda} I_\lambda^k.$$

This proves the theorem.

The following two corollaries are direct conseqences of Theorem 2.2.

Corollary 2.3. *Let $I = \sum_{\alpha\in\Lambda} x_\alpha R$ be a multiplication ideal in a ring R. Then for each positive integer k,*

$$\left(\sum_{\alpha\in\Lambda} x_\alpha R \right)^k = \sum_{\alpha\in\Lambda} x_\alpha^k R\,.$$

Corollary 2.4. *If I is a multiplication ideal in a ring R, and if I contains no non-zero nilpotent element, then*

$$ann(I) = ann(I^k)$$

for each positive integer k.

The proof is essentially the same as that of [9, Corollary 5.3] in the case of finitely generated multiplication ideals. Hence it is omitted.

The previous corollaries are generalizations of [9, Theorem 5.1 and Corollary 5.3] to multiplication ideals (not necessarily finitely generated ones) respectively.

3. The equality $\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right)^k = \bigcap_{\lambda\in\Lambda} I_\lambda^k$ for ideals

If I and J are two finitely generated ideals in a Prüfer domain R, then

$$(I \cap J)^k = I^k \cap J^k$$

for each positive integer k ([6] and [13]). The localization argument is used in [9] to show that this property holds if $I + J$ is a finitely generated multiplication ideal in a ring R. Our next result is a generalization of these results.

Proposition 3.1. *Let I and J be ideals in a ring R such that $[I : J] + [J : I] = R$. Then*

$$(I \cap J)^k = I^k \cap J^k$$

for each positive integer k. In particular, if $I + J$ is a multiplication ideal, then the result is also satisfied.

To prove this proposition, we need several lemmas.

Lemma 3.2. *Let I and J be ideals in a ring R such that $[I : J] + [J : I] = R$. Then*

$$I^2 \cap J^2 \subseteq IJ.$$

Proof. By using Lemma 1.4 (i), we have

$$IJ = (I + J)(I \cap J) \supseteq (I + J)I \cap (I \cap J)J \supseteq I \cdot I \cap J \cdot J = I^2 \cap J^2.$$

Lemma 3.3. *If I, J, K and L are ideals in a ring R such that*

$$[I : J] + [J : I] = R \quad \textit{and} \quad [K : L] + [L : K] = R,$$

then

$$(I \cap J)(K \cap L) = IK \cap IL \cap JK \cap JL.$$

Proof. By using Lemma 1.4 (ii) twice, we have

$$(I \cap J)(K \cap L) = I(K \cap L) \cap J(K \cap L) = IK \cap IL \cap JK \cap JL.$$

Lemma 3.4. *Let I and J be ideals in a ring R such that $[I : J] + [J : I] = R$. Then*

$$(I \cap J)^2 \subseteq I^2 \cap J^2.$$

Proof. It follows from Lemma 3.3 that

$$(I \cap J)^2 = (I \cap J)(I \cap J) = I^2 \cap IJ \cap J^2.$$

Also, from Lemma 3.2 we have that

$$I^2 \cap J^2 \subseteq IJ.$$

Hence the result.

Lemma 3.5. *If I and J are ideals in a ring R such that $[I : J] + [J : I] = R$, then*

$$[I^k : J^k] + [J^k : I^k] = R$$

for each positive integer k.

Proof. By using the fact that $A \subseteq [A : B]$ for all ideals A and B in R, we have

$$R = [I : J] + [J : I] \subseteq \big[[I : J] : [J : I]\big] + \big[[J : I] : [I : J]\big], \quad \text{i.e.}$$
$$\big[[I : J] : [J : I]\big] + \big[[J : I] : [I : J]\big] = R.$$

From Proposition 2.1, it follows that

$$R = \big([I : J] + [J : I]\big)^k = [I : J]^k + [J : I]^k.$$

It finally implies that

$$R = [I : J]^k + [J : I]^k \subseteq [I^k : J^k] + [J^k : I^k].$$

Hence

$$[I^k : J^k] + [J^k : I^k] = R.$$

Proof of Proposition 3.1. The proof is by induction on k. The proposition is true for $k = 2$ (Lemma 3.4). Let $k > 2$ and we assume that the proposition is true for k, i.e.

$$(I \cap J)^k = I^k \cap J^k.$$

From Lemmas 3.3 and 3.5 it follows that

$$(I \cap J)^{k+1} = (I \cap J)^k (I \cap J) = \big(I^k \cap J^k\big)(I \cap J)$$
$$= I^{k+1} \cap I^k J \cap J^k I \cap J^{k+1}.$$

To finish the proof of the proposition, it is enough obviously to prove that

$$I^{k+1} \cap J^{k+1} = I^k J \cap J^k I.$$

It is known that

$$[I : J] + [J : I] = R.$$

Therefore by Lemma 3.5,

$$\big[I^k : J^k\big] + \big[J^k : I^k\big] = R.$$

Then $1 = u + v = x + y$ for some $u \in [I : J]$, $v \in [J : I]$, $x \in \big[I^k : J^k\big]$ and $y \in \big[J^k : I^k\big]$. Let $z \in I^{k+1} \cap J^{k+1}$. Therefore

$$z = z(x + y)(u + v) = zxu + zxv + zyu + zyv. \qquad (*)$$

Because of $z \in I^{k+1}$, $vz \in vI\,I^k \subseteq J\,I^k$ and $yz \in yI^k\,I \subseteq J^k\,I$. Hence $zxv, zyv \in I^kJ$ and $zyu, zyv \in J^kI$. Similarly, since $z \in J^{k+1}$, $zxu, zyu \in J^kI$ and $zxu, zxv \in I^kJ$. Therefore $zxu, zyv \in I^kJ \cap J^kI$. As

$$(I \cap J)^k = I^k \cap J^k,$$

we infer that

$$I^k \cap J^k \subseteq I^{k-1}J\,,\ J^{k-1}I.$$

Moreover, from Lemma 3.5, we have that

$$\left[I^k : J^k\right] + \left[J^k : I^k\right] = R.$$

It follows from Lemma 1.4 (ii) that

$$J\left(I^k \cap J^k\right) \subseteq J\,I^k \cap J^{k+1}.$$

This implies that

$$J\,I^k \cap J^{k+1} \subset J^kI,$$

and hence $zxv \in J^kI$. Similarly, $zyu \in I^kJ$. It follows that each term on the right hand side of the relation $(*)$ belongs to $I^kJ \cap J^kI$ and so z belongs to this intersection. The second assertion of the proposition is now obvious by using Lemmas 1.3 and 1.4.
This completes the proof of the proposition.

It is an apparent from the proof of [6, Theorem 2.2] that

$$\left(\bigcap_{i=1}^{n} I_i\right)^k = \bigcap_{i=1}^{n} I_i^k$$

for any collection I_i $(1 \le i \le n)$ of ideals in a Prüfer domain R and any positive integer k. The analogous equality for an arbitrary collection of ideals in a Prüfer domain is not valid. The following example is given by Gilmer.

Let $V = Q[\,[X]\,]$ and $M = XV$, where Q is the field of rational numbers. Then, if Z denotes the ring of integers, the domain $D = Z + M$ is Prüfer [6, p.561], but if $I_i = p_iD$, where $p_1 < p_2 < \ldots$ is the sequence of positive prime integers, then for any positive integer k,

$$\left(\bigcap_{i=1}^{\infty} I_i\right)^k = M^k \subset \bigcap_{i=1}^{\infty} I_i^k = M.$$

As a generalization of the above fact, we give the following theorem:

Theorem 3.6. *Let R be a ring and I_i $(1 \le i \le n)$ a finite collection of ideals in R such that $\sum_{l=1}^{n} I_l$ is a multiplication ideal. Then*

$$\left(\bigcap_{l=1}^{n} I_l\right)^k = \bigcap_{l=1}^{n} I_l^k \quad \textit{for each positive integer } k.$$

Proof. We prove the statement by induction on n. It is true for $n = 2$ by Proposition 3.1. Assume that $n > 2$ and let the statement be true for $n - 1$, i.e.

$$\left(\hat{I}_i\right)^k = I_1^k \cap I_2^k \cap \cdots \cap I_{i-1}^k \cap I_{i+1}^k \cap \cdots \cap I_n^k,$$

where $\hat{I}_i$, as we have said, the intersection of all I_j except I_i. We now prove the theorem for n. It is enough to show that locally. Thus we may assume that R is a local ring. As $\sum_{l=1}^n I_l$ is a multiplication ideal, we infer from Proposition 1.2 that

$$\sum_{i=1}^n \left[\left(\bigcap_{l=1}^n I_l\right) : \hat{I}_i\right] + ann(a) = R \quad \text{for each} \quad a \in \sum_{l=1}^n I_l.$$

Suppose that $\sum_{l=1}^n I_l = O$, then $I_l = O$ for all $l \in \{1, \cdots, n\}$ and, moreover $\bigcap_{l=1}^n I_l = O$. The result in this case is trivial. Hence we may assume that $\sum_{l=1}^n I_l \neq O$, and therefore there exists $0 \neq a \in \sum_{l=1}^n I_l$. It follows that $ann(a) \neq R$, and this yields that

$$\sum_{i=1}^n \left[\left(\bigcap_{l=1}^n I_l\right) : \hat{I}_i\right] = R.$$

Then there exists $i_0 \in \{1, \ldots, n\}$ such that

$$\left[\left(\bigcap_{l=1}^n I_l\right) : \hat{I}_{i_0}\right] = R$$

and hence $\bigcap_{l=1}^n I_l = \hat{I}_{i_0}$. This finally gives that

$$\bigcap_{l=1}^n I_l^k \subseteq I_1^k \cap I_2^k \cap \cdots \cap I_{i_0-1}^k \cap I_{i_0+1}^k \cap \cdots \cap I_n^k = \left(\hat{I}_{i_0}^k\right) = \left(\bigcap_{l=1}^n I_l\right)^k \subseteq \bigcap_{l=1}^n I_l^k,$$

$$\text{i.e.} \quad \left(\bigcap_{l=1}^n I_l\right)^k = \bigcap_{l=1}^n I_l^k \quad \text{for each positive integer } k.$$

This completes the proof of the theorem.

Now, we have two remarks about Theorem 3.6 which we wish to mention. The first one is a direct consequence of the theorem. It is the following:

Let $I = \sum_{i=1}^n a_i R$ be a finitely generated multiplication ideal in a ring R. Then

$$\left(\bigcap_{i=1}^n a_i R\right)^k = \bigcap_{i=1}^n a_i^k R$$

for each positive integer k. The second remark is the example of Gilmer which was pointed before Theorem 3.6. It is easy to see that this example shows that our theorem fails when we leave the case of a finite collection of ideals. This is because $\sum_{i=1}^{\infty} p_i D = D$ is a multiplication ideal but

$$\left(\bigcap_{i=1}^{\infty} p_i D\right)^k \neq \bigcap_{i=1}^{\infty} p_i^k D$$

for each positive integer k.

However, the next theorem generalizes Theorem 3.6 under certain conditions.

Theorem 3.7. *Let R be a ring and I_λ $(\lambda \in \Lambda)$ a collection of multiplication ideals such that*

(i) $\displaystyle\bigcap_{\lambda\in\Lambda} I_\lambda$ *is a multiplication ideal,*

(ii) $\displaystyle J\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) = \bigcap_{\lambda\in\Lambda} JI_\lambda$ *where* $\displaystyle J = \sum_{\lambda\in\Lambda}\left[\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) : I_\lambda\right].$

Then

$$\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right)^k = \bigcap_{\lambda\in\Lambda} I_\lambda^k$$

for each positive integer k.

Proof. It follows from [2, Theorem 2.1] that

$$\sum_{\lambda\in\Lambda}\left[\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) : I_\lambda\right] + ann(a) = R \quad \text{for each} \quad a \in \bigcap_{\lambda\in\Lambda} I_\lambda.$$

It is enough to prove the statement locally. Thus we may assume that R is local. If $\bigcap_{\lambda\in\Lambda} I_\lambda = O$, then the statement is trivial. Thus we may assume that $\bigcap_{\lambda\in\Lambda} I_\lambda \neq O$, and hence there exists $a \in \bigcap_{\lambda\in\Lambda} I_\lambda$ with $ann(a) \neq R$. This implies that

$$\sum_{\lambda\in\Lambda}\left[\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) : I_\lambda\right] = R,$$

and hence there exists $\lambda_0 \in \Lambda$ such that

$$\left[\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) : I_{\lambda_0}\right] = R.$$

It follows that $\bigcap_{\lambda\in\Lambda} I_\lambda = I_{\lambda_0}$, and hence

$$\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right)^k = I_{\lambda_0}^k \supseteq \bigcap_{\lambda\in\Lambda} I_\lambda^k \supseteq \left(\bigcap_{\lambda\in\Lambda} I_\lambda\right)^k .$$

This concludes the proof of the theorem.

A direct consequence of Theorem 3.7 is the following:

Corollary 3.8. *Let R be a ring and $I = \sum_{\alpha\in\Lambda} x_\alpha R$ an ideal in R such that $\bigcap_{\alpha\in\Lambda} x_\alpha R$ is multiplication and*

$$J\left(\bigcap_{\alpha\in\Lambda} x_\alpha R\right) = \bigcap_{\alpha\in\Lambda} J x_\alpha R,$$

where $J = \sum_{\alpha\in\Lambda}\left[\left(\bigcap_{\alpha\in\Lambda} x_\alpha R\right) : x_\alpha R\right]$. Then

$$\left(\bigcap_{\alpha\in\Lambda} x_\alpha R\right)^k = \bigcap_{\alpha\in\Lambda} x_\alpha^k R$$

for each positive integer k.

Some examples will help to see that the converse of Theorems 2.2, 3.6 and 3.7 may not always be true. Let R be a ring and M an R-module. Let $[\,R\,M\,]$ denote the ring consisting of all matrices

$$\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$$

where $r \in R$ and $m \in M$ with the usual definitions of matrix addition and multiplication. It is shown by Smith [15] that the module M is a multiplication module (an R-module M is called a multiplication module if for every submodule N of M there exists an ideal I in R such that $N = I\,M$) if and only if the ideal generated by

$$\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$$

where $m \in M$ is a multiplication ideal. Let now $R = [\,F\,V\,]$ where F is any field and V the two-dimensional vector space over F with basis $\{u, v\}$. Let $U = uF$ and $W = vF$. Suppose also that

$$A = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} R \quad \text{and} \quad B = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} R.$$

Then

$$(A \cap B)^k = A^k \cap B^k$$

for each positive integer k but $A+B=[\,O\,V\,]$ is not a multiplication ideal. Assume that $R=[\,Z\,Z_{p^\infty}\,]$ where Z_{p^∞} is the Prüfer p-group. Then there exist elements $b_n \in Z_{p^\infty}$ ($n \geq 1$) such that $pb_1 = 0$ and $pb_n = b_{n-1}$ ($n \geq 2$). For each $n \geq 1$, let

$$A_n = \begin{bmatrix} 0 & b_n \\ 0 & 0 \end{bmatrix} R.$$

Then

$$\left(\sum_{i=1}^{\infty} A_i\right)^k = O = \sum_{i=1}^{\infty} A_i^k$$

but $\sum_{i=1}^{\infty} A_i$ is not a multiplication ideal. On the other hand, for each positive integer t, let $C_t = [\,tZ\,Z_{p^\infty}\,]$. Then $\bigcap_{t\geq 1} C_t = [\,O\,Z_{p^\infty}\,]$ is not a multiplication ideal. However

$$\left(\bigcap_{t\geq 1} C_t\right)^k = O = \bigcap_{t\geq 1} C_t^k$$

for each positive integer k.

4. Other Ohm type properties for multiplication ideals

Let R be a ring. Recall that an ideal I in R is called a weak-cancellation ideal if for all ideals A and B in R such that $AI \subseteq BI$, then $A \subseteq B + ann(I)$. It is well-known that finitely generated multiplication ideals are weak-cancellation, see [3] and [8]. However, the following property holds for multiplication ideals in general and can be found in [2].

Lemma 4.1. *Let I be a multiplication ideal in a ring R. Then $AI \subseteq BI$ for any ideals A and B in R if and only if $A \subseteq B + ann(I)$ for each $a \in I$.*

Another useful lemma is the following:

Lemma 4.2. *Let R be a ring and M any maximal ideal in R. If I is a multiplication ideal in R which contains no non-zero nilpotent elements, then*

$$ann(I_M^k) \subseteq ann(a)_M$$

for each $a \in I$ and each positive integer k.

Proof. It is easy to check that I_M has also no non-zero nilpotent elements. By [3, Theorem 1], I_M is a principal ideal in R_M (hence multiplication). Finally, from Corollary 2.4 we obtain that

$$ann(I_M^k) = ann(I_M) \subseteq ann(a)_M$$

for each $a \in I$ and each positive integer k.

We can now draw the following two results which give other Ohm type properties for multiplication ideals (not necessarily finitely generated ones). The first one may be compared with [8, Theorem 2.4 and Remark 2.5] and the second one with [8, Proposition 3.1].

Proposition 4.3. *Let R be a ring. Let I and J be ideals in R such that $I+J$ is a multiplication ideal which contains no non-zero nilpotent elements. If $I^k = J^k$ for some positive integer k, then the following statements are satisfied:*

(i)
$$I + ann(a) = J + ann(a) \quad \textit{for each} \quad a \in I+J, \quad \textit{i.e.}$$
$$I = J \ (\,mod\ ann(a)\,) \quad \textit{for each} \quad a \in I+J;$$

(ii)
$$ann(I) = ann(J).$$

Proof. Since $I+J$ is a multiplication ideal, we have from Proposition 2.1 that

$$(I+J)^k = I^k + J^k.$$

It follows that

$$I^k = I^k + J^k = (I+J)^k,$$

and hence

$$J(I+J)^{k-1} \subseteq (I+J)^k = I^k.$$

On the other hand

$$I^k = I\,I^{k-1} \subseteq I(I+J)^{k-1}.$$

Therefore

$$J(I+J)^{k-1} \subseteq I(I+J)^{k-1}.$$

Suppose that M is a maximal ideal in R. Then

$$J_M\big(I+J\big)_M^{k-1} \subseteq I_M\big(I+J\big)_M^{k-1}.$$

Moreover, $(I+J)^{k-1}$ is a multiplication ideal [4], and hence $\big(I+J\big)_M^{k-1}$ is a principal ideal. Hence $\big(I+J\big)_M^{k-1}$ is a weak-cancellation ideal, and therefore

$$J_M \subseteq I_M + ann\left(\big(I+J\big)_M^{k-1}\right).$$

Next, from Lemma 4.2 we have that

$$ann\left(\big(I+J\big)_M^{k-1}\right) \subseteq ann(a)_M \quad \text{for each } a \in I+J.$$

It follows that

$$J_M \subseteq I_M + ann(a)_M \quad \text{for each } a \in I+J,$$

and hence

$$J \subseteq I + ann(a) \quad \text{for each } a \in I + J.$$

It follows now immediately that

$$J + ann(a) \subseteq I + ann(a) \quad \text{for each } a \in I + J.$$

Similarly

$$I + ann(a) \subseteq J + ann(a) \quad \text{for each } a \in I + J.$$

Thus

$$I + ann(a) = J + ann(a) \quad \text{for each } a \in I + J.$$

This finishes the proof of the first part. For the second one, we infer from Proposition 2.1 and Corollary 2.4 that

$$ann(I^k) = ann(I^k + J^k) = ann(I + J)^k = ann(I + J) \subseteq ann(I) \subseteq ann(I^k).$$

Similarly, $ann(J) = ann(J^k)$. Thus $ann(I) = ann(J)$, as required.

Proposition 4.4. *Let R be a ring and I, J two ideals in R such that J and $I + J$ are multiplication ideals. Then for each positive integer k,*

$$[I : J]^k + ann(a) = [I^k : J^k] + ann(a) \quad \textit{for each} \quad a \in J^k, \quad \textit{i.e.}$$
$$[I : J]^k = [I^k : J^k]\,(mod\, ann(a)\,) \quad \textit{for each} \quad a \in J^k.$$

Moreover, if J has no non-zero nilpotent elements, then

$$[I : J]^k = [I^k : J^k]\,(\,mod\, ann(a)\,) \quad \textit{for each} \quad a \in J.$$

The proof of the first assertion is essentially similar to that given for [8, Proposition 3.1] by Lemma 4.1 and that of the second one follows from Lemma 4.2.

Note. A part of this work appeared in the Ph.D. thesis written by the author under the supervision of Prof. Lüneburg. It was submitted to the university of Kaiserslautern in 1993. I wish to thank Prof. Lüneburg and Prof. Naoum for their suggestions.

References

[1] Ali, Majid M.: *The equality $[A : B] + [B : A] = R$ for modules.* To appear.

[2] Ali, Majid M.: *Some remarks on Smith's theorem.* To appear.

[3] Anderson, D. D.: *Multiplication ideals, multiplication rings, and the ring $R(X)$.* Canad. J. Math. **28** (1976), 760–768.

[4] Anderson, D. D.: *Some remarks on multiplication ideals.* Math. Japonica **25** (1980), 463–469.

[5] Gilmer, R.: *On a condition of J. Ohm for integral domains.* Canad. J. Math. **20** (1968), 970–983.

[6] Gilmer, R.; Grams, A.: *The equality* $(A \cap B)^n = A^n \cap B^n$ *for ideals.* Canad. J. Math. **24** (1972), 792–798.

[7] Larsen, M. D.; McCarthy, P. J.: *Multiplicative Theory of Ideals.* New York 1971.

[8] Naoum, A. G.: *The Ohm type properties for finitely generated multiplication ideals.* Period. Math. Hungar. **18** (1987), 287–293.

[9] Naoum, A. G.; Balboul, Majid M.: *On finitely generated multiplication ideals in commutative rings.* Beiträge zur Algebra und Geom. **19** (1985), 75–82.

[10] Naoum, A. G.; Hasan, M. A. K.: *On finitely generated projective ideals and Ohm condition.* Period. Math. Hungar. **16** (1985), 51–60.

[11] Naoum, A. G.; Hasan, M. A. K.: *On the sum and intersection of finitely generated projective ideals.* Arab. J. Math. **4** (1983), 61–75.

[12] Naoum, A. G.; Hasan, M. A. K.: *The residual of finitely generated multiplication modules.* Arch. Math. **46** (1986), 225–230.

[13] Ohm, J.: *Integral closure and* $(x, y)^n = (x^n, y^n)$. Monatsh. Math. **71** (1967), 32–39.

[14] Ribenboim, P.: *Algebraic Numbers.* New York 1972.

[15] Smith, P. F.: *Some remarks on multiplication modules.* Arch. Math. **50** (1988), 223–235.

Received May 20, 1994; revised version December 5, 1995