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On the Pole Placement Problem for Linear Systems of Arbitrary Genus

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Abstract. Let X be a smooth compact Riemann surface of genus $g > 0$. Here we show that for every rank $m \geq 2$ there is in general no solution to the pole placement problem.

V. G. Lomadze introduced in [3] linear systems associated to divisors and vector bundles on an arbitrary complete, smooth, and irreducible algebraic curve X of genus $g > 0$. We consider the pole placement problem (or PPP for short) in the setting and with the notations of [3]. Let F be a rank m effective vector bundle on X; the PPP asks if there is always a completely reachable linear system s with m imputs, F as associated Martin-Hermann vector bundle and the determinant $\mathrm{ch}(F)$ of F as characteristic divisor $c(s)$. In this paper for every integer $m \geq 2$ and for every such curve X we will give a negative solution to PPP for the case of m imputs. Recall that for genus 0 PPP has always a solution. For the case of one input (i.e. the rank $m = 1$ case) in [3, Theorem 3 of §7], it was shown that PPP has always a solution. The question of the solubility of PPP was raised in [3, §7]. Indeed we will show (see Theorem 1 and Theorem 2 for precise statements) that for $g > 0$ and $m \geq 2$ the non solubility of PPP is rather the norm, not an exception. We stress that in [3], the PPP was raised only for the output number $p = 0$ and state feedback (i.e. controllable (A, B) systems if $g = 0$).

We will use heavily the notions introduced in [3]. We take as base field an infinite perfect field k. For a summary of properties of sheaves on X, see [5, Ch. 5, §2].

First, we will give a construction of data for which PPP has no solution.

Theorem 1. *Fix a smooth complete irreducible algebraic curve X of genus $g > 0$, an integer $m \geq 2$ and an effective divisor D on X with $H^0(X, \mathbf{O}(D)) \geq 2$. Then there is a rank $m \geq 2$*

effective vector bundle F on X with Chern class $\mathrm{Ch}(F)$ *of F in the class of D but for which PPP fails for the pair (F, D) (i.e. there is no surjection $f : F \longrightarrow D$ with* $\mathrm{Ker}(f) \cong \mathbf{O}^m$*).*

Proof. Taking as F a vector bundle of the form $G \oplus \mathbf{O}^{m-2}$ we may always reduce to the case $m = 2$. Fix a point $P \in X$ not in the support of D. We have $\dim(H^0(X, \mathbf{O}(D))) - 1 \leq \dim(H^0(X, \mathbf{O}(D-P))) \leq \dim(H^0(X, \mathbf{O}(D)))$. Since by assumption P is not a base point of the complete linear series associated to $H^0(X, \mathbf{O}(D))$, we have $\dim(H^0(X, \mathbf{O}(D-P))) < \dim(H^0(X, \mathbf{O}(D)))$. Since $\dim(H^0(X, \mathbf{O}(D))) \geq 2$, we have $\dim(H^0(X, \mathbf{O}(D-P))) = \dim(H^0(X, \mathbf{O}(D))) - 1 > 0$. Hence there is an effective divisor D' such that $D' + P$ is linearly equivalent to D. Take as F the vector bundle $\mathbf{O}^{m-2} \oplus \mathbf{O}(P) \oplus \mathbf{O}(D')$. Since P is a positive divisor, we have $\dim(H^0(X, \mathbf{O}(P))) > 0$. Any two linearly independent sections of $O(P)$ induces (after deleting the base points, if any) a surjective morphism $X \longrightarrow \mathbf{P}^1$ of degree ≤ 1. Since $g > 0$, this is impossible. Hence we have $\dim(H^0(X, \mathbf{O}^{m-2} \oplus \mathbf{O}(P))) = m - 1$. This implies that if we take any generically injective map $j : \mathbf{O}^m \longrightarrow F$, then P is in the support of $\mathrm{Coker}(j)$. Hence $\mathrm{Coker}(j) \neq D$. □

Just by the definition (see [3]) PPP always holds for pairs (F, D) with $\dim(H^0(X, \mathbf{O}(D))) = 1$.

In the proof of Theorem 1 we constructed a bundle F such that PPP fails for (F, D) (i.e. F is not the Martin-Hermann vector bundle of a completely reachable linear system with D as pole divisor). Here we will show that this failure is the rule, while the pairs (F, D) obtained by a completely reachable linear system form a "small set"("it depends on less parameters", or, more precisely, these linear systems are parametrized one to one by the disjoint union of finitely many varieties of dimension smaller than the dimension of a variety parametrizing one to one "most" pairs without PPP).

Theorem 2. *Fix a smooth complete irreducible algebraic curve X of genus $g>0$, an integer $m \geq 2$ and an effective divisor D on X; set $d := \deg(D) > 0$ and $h := dim(H^0(X, \mathbf{O}(D)))$. Set $W := \mathrm{Ext}^1(D, \mathbf{O}^m)$ i.e. W is the vector space of dimension md parametrizing all the equivalence classes of extensions*

$$0 \longrightarrow \mathbf{O}^m \longrightarrow F \longrightarrow D \longrightarrow 0. \tag{1}$$

(a) The subset of W such that in the corresponding extension (1) the middle term F is locally free is a non empty Zariski open subset U of W. There is a non empty Zariski open subset V of U such that for every $t \in V$ the corresponding bundle F_t has $\dim(H^0(X, F_t)) = \max\{m, d + m(1-g)\}$.

(b) If $d \leq mg$ for every s, t in V the corresponding bundles are isomorphic (as abstract vector bundles) if and only if s and t are in the same orbit for the action of the group $\mathrm{GL}(m)$ *induced by the natural action of* $\mathrm{GL}(m)$ *on* $\mathbf{O}^m$.

(c) If $d \leq mg$ for every effective divisor D' linearly equivalent to D but with $D' \neq D$ and for every $t \in V$ the pole placement problem fails for the pair (F_t, D').

Proof.

Proof of part (a): Note that for every sheaf which fits as middle term in the extension (1) we have $\dim(H^0(X, F)) \geq m$. Note that for every sheaf which fits as middle term in the extension (1) we have $\chi(F) = d + m(1-g)$ by Riemann-Roch. Hence $\max\{m, d + m(1-g)\}$

is a lower bound of $\dim(H^0(X, F))$ for every such F. The set U is Zariski open because the condition "locally free" is an open condition for coherent sheaves on X; this is very well known and can be checked for instance using [1, Ex. II.5.8] or that a sheaf on X is locally free if and only if it has no torsion ([5, middle of page 132]). By the theorem of semicontinuity for cohomology ([1, Th. III.12.8]), the subset V is Zariski open. If $d \geq mg$ the non-emptyness of V was proved in [4, Lemma 5 in §2 and Proposition in §3], which shows the existence of F with $H^1(X, F) = 0$ (or see [6, Prop. 1.8.1(i)], or [2, Th. 4.6], for stronger results). Hence we will assume $d < mg$. Fix $P \in X$, P not in the support of D; set $A := D + (mg - d)P$ (which is an effective divisor of degree mg). Apply the quoted result of [4] (or [6] or [2]) to the extension of the divisor A with $\mathbf{O}^m$ to obtain a rank m vector bundle G with $\dim(H^0(X, G)) = m$ and which fits in the exact sequence

$$0 \longrightarrow \mathbf{O}^m \longrightarrow G \longrightarrow A \longrightarrow 0. \tag{2}$$

Let $u : G \to (mg - d)P$ be the surjection of sheaves obtained composing the surjection $G \to A$ in the exact sequence (2) with the surjection $A \longrightarrow (mg - d)P$ of skyscraper sheaves. Set $F := \mathrm{Ker}(u)$. By construction F fits in the exact sequence (1). Since $H^0(X, F) \subseteq H^0(X, G)$, we have $\dim((H^0(X, F)) = m$.

Proof of part (b): Note that the "if" part is trivial. For the "only if" part we fix s and t in V with $F_t \cong F_s$. Since $\dim(H^0(X, F_t)) = dim(H^0(X, F_s)) = m$, all the global sections of F_t and F_s come from global sections of $\mathbf{O}^m$. Hence every isomorphism of F_t with F_s induces an automorphism of $H^0(X, \mathbf{O}^m)$ i.e. an element of $\mathrm{GL}(m)$, as wanted.

Proof of part (c): Every surjection of sheaves $F_t \longrightarrow D'$ with kernel isomorphic to $\mathbf{O}^m$ gives a dimension m linear subspace $H^0(X, \mathbf{O}^m)$ of the vector space $H^0(X, F_t)$. Since they have the same dimension by part (b), this surjection induces an isomorphism as torsion sheaves of D' and D, i.e. $D' = D$ as divisor (not only as equivalence class). □

The trivial example given by the vector bundle $\mathbf{O}^{m-1} \oplus \mathbf{O}(D)$ (reduction to the rank 1 case proved in [3, Th. 3 in §7]) shows that for every $m \geq 2$, every X and every effective divisor D on X there is a rank m vector bundle F on X with $\mathrm{Ch}(F)$ in the class of D and such that (F, D') has PPP for every D' in the class of D.

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