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Flag Transitive $c.Af^*$ Geometries

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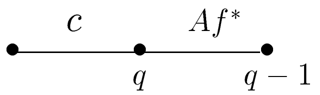
Abstract. In this paper we classify flag-transitive geometries belonging to the following diagram

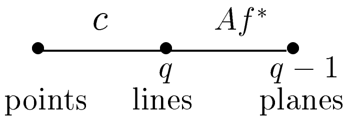


(with $1 < q < \infty$)

It turns out that there are only six simply connected examples, with automorphism groups of the following forms: $2^3.S_4$, $L_3(2)$, $2^6.3.S_6$, $L_3(4).2$, $2^4.S_6$ and $2^9.L_2(17)$. Four more examples arise as quotients of some of these.

1. Introduction

In this paper we study *$c.Af^*$ geometries of order* $q < \infty$, i.e. geometries of rank 3 belonging to the following diagram



where the first stroke denotes circular geometries of order q, i.e. complete graphs with $q+2$ vertices, and the second one denotes dual affine planes of order q. We shall prove that there are precisely ten flag-transitive $c.Af^*$ geometries (three of order 2, five of order 4 and two of order 16). Six of them are simply connected.

Some Terminology and Notation. The reader is referred to [24] for basic notions of diagram geometry. According to [24], all geometries are residually connected, by definition. We denote the incidence relation by $*$. We call the elements of a $c.Af^*$ geometry "points", "lines" and "planes" according to their types (see the above picture) and we freely use phrases as "the point x is on the plane y", "the line z passes through the point x" etc. to mean that $x * y$, $x * z$ etc.

We say that two points (planes) are *collinear* (*dually collinear*) if there is a line incident with both of them. We denote the collinearity relation (the dual collinearity relation) by $\perp$ (by $\top$ respectively). The *collinearity graph* (the *dual collinearity graph*) of a $c.Af^*$ geometry Γ is the graph defined by $\perp$ (by $\top$) on the set of points (planes) of Γ.

The diameter of the collinearity (dual collinearity) graph of Γ will be called the (*dual*) *collinearity diameter* of Γ.

Some of the geometries we consider satisfy one or both of the following properties

(LL) For any two distinct points, there is at most one line incident with both of them.

(T) For any three pairwise collinear points, there is at least one plane incident with all of them.

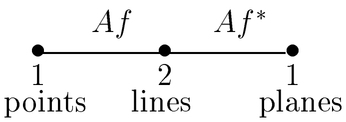
In the context of this paper, (LL) is equivalent to the Intersection Property (IP) ([24], Lemma 7.25). If (LL) holds in Γ, then the point-line system of Γ and the collinearity graph of Γ are the same.

A $c.Af^*$ geometry Γ is said to be *flat* if all its points are incident with all planes. Clearly, (T) holds in every flat geometry, but (LL) fails to hold in them. Indeed, any two points of a flat $c.Af^*$ geometry of order q belong to q^2 planes, each plane contains just one line through them and every line is in q planes. Hence any two points of a flat $c.Af^*$ geometry of order q are in precisely q lines.

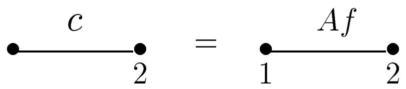
2. Examples

2.1. Bi-affine geometries over GF(2)

Given a plane π and a point p of $PG(3,2)$, let us remove π, p and their residues from $PG(3,2)$. Then we get a geometry belonging to the diagram $Af.Af^*$, with orders $1, 2, 1$:



We call this geometry a *bi-affine geometry* of *order* 2 and *rank* 3, of *flag* or *anti-flag type* according to whether $p \in \pi$ or $p \notin \pi$ (compare [9]). We denote this geometry by $\Gamma^\varepsilon(2)$, with $\varepsilon = +$ or $-$ according to whether $p \in \pi$ or $p \notin \pi$. Since



$\Gamma^{\varepsilon}(2)$ is also a $c.Af^*$ geometry. It satisfies (LL) and it is flag-transitive. $Aut(\Gamma^{\varepsilon}(2))$ is the stabilizer of p and π in the automorphism group $L_4(2)$ of $PG(3,2)$.

$\Gamma^+(2)$ satisfies (T) and $Aut(\Gamma^+(2)) \cong 2^3.S_4$ is flag-transitive. We can factorize $\Gamma^+(2)$ by the (unique) elation of center p and axis π. Thus we obtain a flat geometry with the same diagram, on which $2^2.S_4$ acts flag-transitively. We denote this quotient by $\Gamma^+(2)/2$.

On the other hand, $Aut(\Gamma^-(2)) \cong L_3(2)$, flag-transitive. The collinearity graph of $\Gamma^-(2)$ has diameter 1, (T) does not hold in $\Gamma^-(2)$ and no quotient exists.

2.2. Laguerre structures

Given two sets X, Y with $|X| = q+2$ and $|Y| = q$, a set $\mathcal{F}$ of mappings from X to Y is said to be *sharply* 3*-transitive* (of *degree* $(q+2, q)$) if, given any triple (x_1, x_2, x_3) of distinct elements of X and any triple (y_1, y_2, y_3) of elements of Y, there is precisely one mapping $f \in \mathcal{F}$ such that $f(x_i) = y_i$ for $i = 1, 2, 3$. Such a set $\mathcal{F}$ exists only if q is even [12].

Let Γ be the rank 3 geometry with $X \times Y$ as the point-set, $\mathcal{F}$ as the set of planes and where the lines are the pairs $\{(x_1, y_1), (x_2, y_2)\}$ with $x_i \in X$, $y_i \in Y$ $(i = 1, 2)$ and $x_1 \neq x_2$. The incidence relation is defined as follows. A point $(x, y) \in X \times Y$ (a line $\{(x_1, y_1), (x_2, y_2)\}$) is declared incident with a plane $f \in \mathcal{F}$ when $f(x) = y$ (respectively, $f(x_1) = y_1$ and $f(x_2) = y_2$). A point (x, y) and a line $\{(x_1, y_1), (x_2, y_2)\}$ are declared to be incident when $(x, y) \in \{(x_1, y_1), (x_2, y_2)\}$. This geometry is called a *special Laguerre* 1*-structure* or a *special Laguerre plane* ([27], [28]). We will use the word *Laguerre structure*, for short.

The only examples of Laguerre structures we know are obtained as follows. Let O be a hyperoval in the star of a given point p of $PG(3, q)$, where $q = 2^h$ for some integer h. Take as *points* the points of $PG(3, q)$ different from p and belonging to the $q + 2$ lines of O, as *lines* the lines of $PG(3, q)$ meeting precisely two lines of O and as *planes* those of $PG(3, q)$ not passing through p. The incidence relation is the natural one, inherited from $PG(3, q)$.

Let $\Gamma_O(q)$ be the Laguerre structure obtained in this way. The geometry $PG(3, q)$ can be recovered from $\Gamma_O(q)$ ([14], Theorem 2). Hence $Aut(\Gamma_O(q))$ is the stabilizer of p and O in $P\Gamma L_4(q)$. Namely, $Aut(\Gamma_O(q)) = 2^h : Z_{q-1}A$ where A is the stabilizer of O in $P\Gamma L_2(q)$.

Let π be a plane through p not containing any of the lines of O and let $H_{p,\pi}$ be the group of elations with center p and axis π. If we factorize $\Gamma_O(q)$ by a subgroup $K \neq 1$ of $H_{p,\pi}$ then we obtain a $c.Af^*$ geometry not satisfying (LL). In particular, when $K = H_{p,\pi}$ we get a flat geometry.

For instance, $\Gamma_O(2) = \Gamma^+(2)$ and $\Gamma_O(2)/H_{p,\pi} = \Gamma^+(2)/2$.

Remark. It is worth describing the above construction of $\Gamma_O(q)$ in the dual way. We start from a dual hyperoval O in the plane at infinity of $AG(3, q)$. The planes of $\Gamma_O(q)$ are the points of $AG(3, q)$ whereas the lines (points) of $\Gamma_O(q)$ are the lines (planes) of $AG(3, q)$ whose point (line) at infinity belongs to (is) a line of O. The incidence relation is inherited from $AG(3, q)$.

2.3. The geometry $\Gamma_O(4)$

Clearly, $\Gamma_O(q)$ is flag-transitive if and only if the stabilizer A of O in $P\Gamma L_2(q)$ is 2-transitive on the set of points of O. It is well known that A is 2-transitive on O if and only if $q = 2$ or

4 and that all hyperovals of $PG(2,q)$ ($q = 2$ or 4) are classical. Therefore, if $q = 2$ or 4 then $\Gamma_O(q)$ is flag-transitive and the isomorphism type of $\Gamma_O(q)$ does not depend on the particular choice of O. As the case of $q = 2$ is examined in 2.2, we focus on the case of $q = 4$ here.

We have $Aut(\Gamma_O(4)) = 2^6.3.S_6$. The stabilizer in $Aut(\Gamma_O(4))$ of a plane of $\Gamma_O(4)$ is $3.S_6$ acting as S_6 on the six points of that plane. The stabilizer of a point is $2^4.3.S_5$. The stabilizer of a line can be described as $((2^2.3) \times S_4).2$ and the stabilizer of a chamber is $3 \times S_4$. The following subgroups of $Aut(\Gamma_O(4))$ are also flag-transitive on $\Gamma_O(4)$:

$$2^6.3.A_6, \quad 2^6.3.S_5, \quad 2^6.(3 \times A_5), \quad 2^6.S_5, \quad 2^6.A_5$$

Let $H = H_{p,\pi}$ be as in 2.2. Then H is elementary abelian of order 4, normal in the stabilizer $2^6.3.S_5$ of π in $Aut(\Gamma_O(4))$. Hence $Aut(\Gamma_O(4))/H = 2^4.3.S_5$ is flag-transitive on the (flat) quotient $\Gamma_O(4)/H$ (but the subgroups $2^4.3.A_5$, $2^4.S_5$ and $2^4.A_5$ of $Aut(\Gamma_O(4))/H$ also act flag-transitively on $\Gamma_O(4)/H$). We denote $\Gamma_O(4)/H$ by $\Gamma_O(4)/2^2$. As $\Gamma_O(4)$ is simply connected (Section 4, Corollary 4), every automorphism of $\Gamma_O(4)/2^2$ lifts to an automorphism of $\Gamma_O(4)$. Hence $Aut(\Gamma_O(4)/2^2) = Aut(\Gamma_O(4))/H$.

The group H is central in the subgroup $X = 2^6.A_5$ of the stabilizer $Y = 2^6.3.S_5$ of π in $Aut(\Gamma_O(4))$. The factor group Y/X acts as S_3 on the three 2-subgroups of H. If we factorize $\Gamma_O(4)$ by any of the subgroups of H of order 2 we still get a flag-transitive $c.Af^*$ geometry of order 4. Let us denote this quotient by $\Gamma_O(4)/2$. As $\Gamma_O(4)$ is simply connected, every automorphism of $\Gamma_O(4)/2$ lifts to an automorphism of $\Gamma_O(4)$. Therefore $Aut(\Gamma_O(4)/2) = 2^5.S_5$ (but $2^5.A_5$ is also flag-transitive on $\Gamma_O(4)/2$). Clearly, $\Gamma_O(4)/2$ neither satisfies (LL) nor is flat.

It is not difficult to check that $\Gamma_O(4)/2$ and $\Gamma_O(4)/2^2$ are the only two flag-transitive proper quotients of $\Gamma_O(4)$.

2.4. Gluings

A 1-factorization of a graph $\mathcal{G}$ is a mapping from the set of edges of $\mathcal{G}$ to a given set of *colours* such that every vertex belong to precisely one edge of each colour.

Let $\mathcal{G}$ be a complete graph with $q + 2$ vertices. It is well-known that $\mathcal{G}$ admits a 1-factorization if and only if q is even. Let q be even. Given a 1-factorization φ of $\mathcal{G}$ and an affine plane Π of order q, let α be a bijection from the set of colours of φ to the line at infinity of Π.

Let us take the vertices of $\mathcal{G}$ as *points* and the points of Π as *planes*. As *lines* we take the pairs (e, l) where e is an edge of $\mathcal{G}$ and l is a line of Π with $\alpha(\varphi(e))$ as its points at infinity. Every point is declared to be incident with every plane, whereas the incidence between a point or a plane and a line is defined in the natural way. The geometry obtained in this way is a flat $c.Af^*$ geometry of order q. We call it the *gluing* of $(\mathcal{G}, \varphi)$ with Π *via* α.

The flat quotient $\Gamma_O(q)/H_{p,\pi}$ (notation as in 2.2) is a gluing of the affine plane $AG(2,q)$ with the complete graph with $q+2$ vertices. Indeed, by removing the star of p from π we get a model for the dual of $AG(2,q)$. Thus the lines of π through p are the points at infinity of $AG(2,q)$. On the other hand, they can also be taken as colours for the edges of the complete graph having the lines of O as vertices (every pair of such lines uniquely determines a plane through p, which intersects π in a line ...)

In particular, $\Gamma_O(4)/2^2$ is a gluing of $AG(2,4)$ with the complete graph on six vertices, whereas $\Gamma^+(2)/2$ is a gluing of $AG(2,2)$ with the complete graph on four vertices.

If we want a flag-transitive gluing, we need to assume $q = 2$ or 4 (see [4] and recall that q must be a prime power in a flag-transitive affine plane). Since there is a unique way of gluing $AG(2,4)$ (resp. $AG(2,2)$) with the complete graph on six (four) vertices ([2], 6.2), the quotients $\Gamma_O(4)/2^2$ and $\Gamma^+(2)/2$ already described in 2.3 and 2.1 are the only flag-transitive examples that can be obtained by gluing.

2.5. An example for $L_3(4)$

The following example is due to Sprague [26]. Let $\mathcal{H}$ be one of the three orbits of $L_3(4)$ on the set of hyperovals of $PG(2,4)$. We take $\mathcal{H}$ as the set of planes and the points of $PG(2,4)$ as points. As lines we take the pairs of distinct points of $PG(2,4)$. As incidence relation we take the natural one (symmetrized inclusion). Thus, we obtain a $c.Af^*$ geometry of order 4 satisfying (LL) but not (T). We denote it by $\Gamma_{21}(4)$, where the subscript 21 should remind us of the number of points of this geometry. $\Gamma_{21}(4)$ is flag-transitive and its full automorphism group is $L_3(4).2$ (but $L_3(4)$ also acts flag-transitively on it).

(Clearly, the point-plane system of $\Gamma_{21}(4)$ can also be described as the design obtained from $S(3,6,22)$ by removing a point p and all hexads on it. Compare [26].)

2.6. An example for $2^4.S_6$

The Steiner system $S(3,6,22)$ gives rise to another flag-transitive $c.Af^*$ geometry of order 4. Given a hexad E of $S(3,6,22)$, take the 6 points of E as points, the hexads intersecting E in two points as lines and the 16 points outside E as planes. All points are declared to be incident with all planes. The incidence relation between lines and points or planes is the natural one, inherited from $S(3,6,22)$. We denote this geometry by $\Gamma_6(4)$. It is flag-transitive and its full automorphism group is $2^4.S_6$ (but $2^4.A_6$, $2^4.PGL_2(5)$ and $2^4.L_2(5)$ also act flag-transitively on it).

Clearly, $\Gamma_6(4)$ is flat. However, $\Gamma_6(4) \not\cong \Gamma_O(4)/2^2$. Indeed, the automorphism groups of these two geometries are different. Thus, $\Gamma_6(4)$ is not a gluing.

2.7. The geometry $\Gamma(16)$

The permutation module $U = V(18,2)$ for the linear group $L = L_2(17)$ contains two reducible 9-dimensional L-submodules, say V^+ and V^-, interchanged by an outer automorphism of $L_2(17)$. (See [23]; also [17] or [16] (1.4)). They can be described as follows. Regarding $PG(1,17)$ as the set $\mathrm{GF}(17) \cup \{\infty\}$, let S be the set of non-zero squares in GF(17). The vectors of U can be viewed as characteristic functions of subsets of $PG(1,17)$. The submodule V^+ is generated by the characteristic functions of the images of S under L, whereas V^- is generated by the characteristic functions of the images of $S \cup \{0,\infty\}$ under L. We have $V^+ \cap V^- = C_U(L) = \langle\zeta\rangle$ (where ζ denotes the characteristic function of $PG(1,17)$) and $V^+ + V^-$ is the hyperplane of U consisting of the characteristic functions of subsets of $PG(1,17)$ of even cardinality. For $V = V^+$ or V^-, the L-module $V/\langle\zeta\rangle$ is irreducible.

Note that, if α is a non-square element of GF(17), then αS is the set of non-square elements of GF(17). Hence $PG(1,17)\backslash\alpha S = S \cup \{0,\infty\}$. Let g_α be the element of $PGL_2(17)$ mapping $i \in PG(1,17)$ onto αi (with $\alpha\infty = \infty$, by convention). Then g_α maps the characteristic function of S, which is a vector of V^+, onto a vector of V^-. Therefore, and since g_α normalizes L, it interchanges V^+ and V^-. This makes it clear that the subgroups V^+L and V^-L of $U.L$ are isomorphic. (However, V^+ and V^- are not isomorphic L-modules, as elements of order 8 have different minimal polynomials on V^+ and V^-.) Henceforth, we set $G = V.L$ with $V = V^+$ or V^-. (Which of V^+ or V^- we choose as V makes no difference for the sequel.)

A flag-transitive $c.Af^*$ geometry of order 16 can constructed as follows. We take the vectors of V as *planes*, viewed as functions from $PG(1,17)$ to GF(2). As *points* we take the pairs (i,ε) with $i \in PG(1,17)$ and $\varepsilon \in$ GF(2), declaring a plane u and a point $p = (i,\varepsilon)$ to be incident precisely when $u(i) = \varepsilon$. That is, the set of planes incident with $p = (i,\varepsilon)$ is the affine hyperplane of V of equation $u(i) = \varepsilon$. It is worth noticing that the hyperplane of V of equation $u(i) = 0$ is the commutator $[V, S_i]$ of the (unique) Sylow 17-subgroup S_i of L that fixes i.

We still need to define lines. We shall define them as sets of planes, as follows. Let $p_\infty = (\infty, 0)$ and $p_0 = (0,0)$. There are 2^7 planes incident with both p_∞ and p_0. The stabilizer in L of the unordered pair $\{\infty, 0\}$ is a dihedral group of order 16. Let a be an element of order 8 in that group. Clearly, the centralizer of a^4 in $V_i = [V, S_i]$, which is a-invariant, is at least 4-dimensional. However, as a^4 inverts S_i elementwise, $a^4 s$ is an involution conjugate to a^4, for every $s \in S_i$. Hence, for $s \neq 1$, $s = a^4(a^4 s)$ fixes $C_{V_i}(a^4) \cap C_{V_i}(a^4 s)$, which has to be null as s acts irreducibly on V_i. Therefore $C_{V_i}(a^4)$ is 4-dimensional and hence $[V, a^4] = [V_i, a^4] = C_{V_i}(a^4)$ is also 4-dimensional.

Take as lines of the geometry the images of $[V, a^4]$ under the group VL. Clearly, they are just all cosets of certain 4-spaces in V, namely of the spaces $[V,t]$ for t in the conjugacy class of a^4. We declare them to be incident to exactly the planes which they contain (as vectors of V). We declare a coset $v + [V,t]$ to be incident with the point (i,ε) when t fixes i and $v(i) = \varepsilon$. Clearly, this definition is independent on the choice of the representative v of $v + [V,t]$, as all vectors of $[V,t]$ are contained in $[V, S_i]$ for i fixed by t.

We denote this geometry by $\Gamma(16)$. We have defined it as a shadow geometry on planes, embedded in $AG(9,2)$. Clearly, VL is flag-transitive on $\Gamma(16)$. Let us check that $\Gamma(16)$ is a $c.Af^*$ geometry of order 16.

By the discussion above, all planes (vectors) incident to (contained in) the line $v + [V,t]$ are also incident to the two points $(i, v(i))$ and $(j, v(j))$, where i and j are the two fixed points of t on $PG(1,17)$. Hence the residue of the line $v + [V,t]$ is a generalized digon. Turning to residues of planes (vectors of V), let us consider the residue of the null vector of V. This plane is incident to all points $(i,0)$ ($i \in PG(1,17)$) and to all lines $[V,t]$, $t \in (a^4)^L$. As a 2-point stabilizer in L contains precisely one involution (which is in $(a^4)^L$), for any 2-set $\{i,j\}$ in $PG(1,17)$ there is precisely one line $[V,t]$ incident to the points $(i,0)$ and $(j,0)$. Hence the residues of planes are circular spaces with 18 points.

Consider the residue of the point $p_\infty = (\infty, 0)$. The planes (vectors) in its residue make up the 8-dimensional subspace $[V, S_\infty]$ of V. The lines (cosets of 4-spaces) incident to p_∞ are the subspaces $v + [V,t]$, with $t \in (a^4)^{S_\infty}$ and $v \in [V, S_\infty]$. However, we have proved above that $[V,t] \cap [V,t'] = 0$ for $t,t' \in (a^4)^{S_\infty}$. Hence the seventeen 4-spaces $[V,t]$ ($t \in (a^4)^{S_\infty}$) form a spread of the 8-space $[V,S_i]$. This shows that the residue of p_∞ is a dual affine plane of order 16. (In fact, being flag-transitive, it is a copy of $AG(2,16)$; see [8].)

As we shall see in 7.3, the geometry $\Gamma(16)$ is simply connected. Furthermore, $G = V.L$ is the full automorphism group of $\Gamma(16)$ (compare 7.1, Corollary 44). The center $Z = \langle\zeta\rangle$ of G defines a flag-transitive proper quotient of Γ. We denote it by $\Gamma(16)/2$. This is the unique flag-transitive proper quotient of Γ.

The points of $\Gamma(16)/2$ bijectively correspond to the elements of $\mathrm{GF}(17)\cup\{\infty\}$, whereas the planes of $\Gamma(16)/2$ are the vectors of $W = V/Z$. The lines are the cosets of $[W,g^4] = C_W(g^4)$, where g is any element of L of order 8 with two fixed points on $\mathrm{GF}(17)\cup\{\infty\}$. A point $p \in \mathrm{GF}(17)\cup\{\infty\}$ (a plane $u \in W$) and a line $v + [W,g^4]$ of $\Gamma(16)/2$ are incident if and only if g fixes p (respectively, $(u+v)^{g^4} = u+v$).

Remark. As there are two conjugacy classes of subgroups isomorphic to $L_2(17)$ in $G = VL$ (see [23]), the reader might wonder if the isomorphism type of $\Gamma(16)$ depends on the particular conjugacy class from where the complement L of V has been picked up. But it doesn't. Indeed the elements of $\Gamma(16)$ are vectors of V and cosets of suitable subspaces of V of the form $[V,X]$ with $X \leq L$. These only depend on the projection XV/V of X onto G/V and not on the particular complement of V which X belongs to.

2.8. Another description of $\Gamma^+(2)$, $\Gamma_O(4)$ and $\Gamma_O(4)/2$

A description of the same type as the one given for $\Gamma(16)$ in 2.7 can also be given for $\Gamma^+(2)$, $\Gamma_O(4)$ and $\Gamma_O(4)/2$.

Let $L = PGL_2(3)$ or $PGL_2(5)$ and $p = 3$ or 5, respectively. Then, the 2-point stabilizers of L on $PG(1,p)$ are cyclic and contain a unique involution. Let I denote the class of these involutions.

Let U be the permutation module over GF(2) for L in the action of L on $PG(1,p)$ and let V be the even part of U. The subspace Z spanned by the characteristic function of $PG(1,p)$ is clearly contained in V and V/Z is irreducible. For $p = 3$ it is a natural module for $L_2(2)$ induced by $PGL_2(3)$ on V/Z, for $p = 5$ it is a "natural" module for $\Gamma L_2(4) \cong PGL_2(5)$. In any case, the L-module V is indecomposable.

Let $S \in Syl_p(L)$ with fixed point i on $PG(1,p)$ and $\iota \in I$ fixing the letters i and j. The element ι normalizes precisely two Sylow p-subgroups of L, namely S and the conjugate S^g of S fixing the letter j. The subspace $[V,\iota] = [Z,\iota] \oplus [[V,S],\iota]$ is clearly contained in $[V,S]$. In fact, it is contained precisely in $[V,S]$ and $[V,S]^g = [V,S^g]$. With the same argument as for a^4 in 2.7, we conclude that $dim[V,\iota] = (dim[V,S])/2$ and the spaces $[V,\iota]$, $\iota \in I, \iota \in N_L(S)$, form a spread in $[V,S]$.

Take the vectors of V as planes, the cosets in V of $[V,\iota]$ ($\iota \in I$) as lines and the cosets of $[V,S^g]$ ($g \in L$) as points and define incidence by symmetrized inclusion. Then we get a $c.Af^*$ geometry whose dual affine planes have order $q-1$. Thus, we denote it by $\Gamma(p-1)$. The geometry $\Gamma(p-1)$ admits $G = VL$ as a flag-transitive automorphism group. It turns out that $\Gamma(2) \cong \Gamma^+(2)$ and $\Gamma(4) \cong \Gamma_O(4)/2$. In both cases, G is the full automorphism group.

If we perform the same procedure with the group $L = L_2(17)$ and its permutation module U over GF(2), there are two L-invariant subspaces V^+ and V^- of dimension $(17+1)/2 = 9$

and we construct the geometry $\Gamma(16)$ in $V = V^+$ or V^- (as we have remarked in 2.7, the isomorphism type of $\Gamma(16)$ does not depend on whether $V = V^+$ or $V = V^-$). The same thing happens for the permutation module U over GF(4) for the group $L_2(5)$. There are two L-invariant subspaces V^+ and V^- of dimension $(5+1)/2 = 3$ (see [23]) and the construction in $V = V^+$ or $V = V^-$ gives a geometry isomorphic to $\Gamma_O(4)$ with $VL = 2^6.A_5$ as a flag-transitive group of automorphisms (but $Aut(\Gamma_O(4))$ is larger than VL).

2.9. Summary

Here is the list of the flag-transitive examples constructed in the previous paragraphs.

order	name	automorphism group	description
2	$\Gamma^+(2)$	$2^3.S_4$	2.1. Also 2.8.
2	$\Gamma^+(2)/2$	$2^2.S_4$	2.1
2	$\Gamma^-(2)$	$L_3(2)$	2.1.
4	$\Gamma_O(4)$	$2^6.3.S_6$	2.3. Also 2.8.
4	$\Gamma_O(4)/2$	$2^5.S_5$	2.3. Also 2.8.
4	$\Gamma_O(4)/2^2$	$2^4.3.S_5$	2.3
4	$\Gamma_{21}(4)$	$L_3(4).2$	2.5.
4	$\Gamma_6(4)$	$2^4.S_6$	2.6.
16	$\Gamma(16)$	$2^9.L_2(17)$	2.7. Also 2.8.
16	$\Gamma(16)/2$	$2^8.L_2(17)$	2.7

3. The main theorem

We shall prove the following:

Theorem 1 *There are precisely ten flag-transitive $c.Af^*$ geometries of finite order. They are those listed in* 2.9.

The rest of this paper is devoted to the proof of the above theorem. The case of order 2 is settled by the next proposition, which is a special case of the main theorem of [15]:

Proposition 2 *There are precisely three $c.Af^*$ geometries of order 2, namely* $\Gamma^+(2), \Gamma^+(2)/2$ *and* $\Gamma^-(2)$.

Thus, we will only consider $c.Af^*$ geometries of order $q > 2$. In Section 4 we state some preliminary results, without using flag transitivity. In particular, we obtain that $\Gamma_O(4)$ and $\Gamma_{21}(4)$ are simply connected and we prove that, if Γ is a non-flat $c.Af^*$ geometry of order $q > 2$ with collinearity diameter $\delta_\perp = 1$, then either $\Gamma \cong \Gamma_{21}(4)$ or $q = 18$ (Proposition 18). Thus, the flat case and the case with $\delta_\perp = 1$ only remain to study. The case of $q = 18$ is of no interest in this paper. Indeed, in Section 5 we prove that a flag-transitive $c.Af^*$ geometry has order $q = 2, 4$ or 16.

The case of $q = 4$ is considered in Section 6. We show that $\Gamma_6(4)$ and $\Gamma_O(4)/2^2$ are the unique flat flag-transitive $c.Af^*$ geometries of order 4 and we prove that, if Γ is a flag-transitive $c.Af^*$ geometry of order 4 with collinearity diameter $\delta_\perp > 1$, then Γ is a cover of $\Gamma_O(4)/2^2$. Hence $\Gamma = \Gamma_O(4)$ or $\Gamma_O(4)/2$, since $\Gamma_O(4)$ is simply connected.

The case of $q = 16$ is discussed in Section 7. In that section, given a flag-transitive $c.Af^*$ geometry Γ of order 16, we prove that either $\Gamma = \Gamma(16)/2$ or Γ is a double cover of $\Gamma(16)/2$. Therefore $\Gamma(16)$ is simply connected and $\Gamma = \Gamma(16)$ or $\Gamma(16)/2$.

4. Some preliminary results

In this section Γ is a $c.Af^*$ geometry of order $q > 2$. We do not assume flag transitivity. We assume $q > 2$ only because the case of $q = 2$ is covered by Proposition 2. However, this assumption is not really necessary for the sequel (but it will make some statements a bit shorter).

The following is easy to prove ([24], Chapter 7, Exercise 7.42):

Lemma 3 *The geometry* Γ *is a Laguerre structure if and only if it satisfies both* (LL) *and* (T).

The pair of properties (LL) and (T) is preserved under taking covers. On the other hand, the collinearity diameter of a Laguerre structure is 2. Hence (LL) fails to hold in any proper quotient of a Laguerre structure. Therefore:

Corollary 4 *All Laguerre structures are simply connected.*

(In particular, $\Gamma_O(4)$ is simply connected.) The next result is due to Sprague [26], [27]:

Proposition 5 *If* (LL) *holds in* Γ*, then either* Γ *is a Laguerre structure or* $\Gamma \cong \Gamma_{21}(4)$.

(One more case was allowed in [26] and [27], with $q = 10$. However, that case is impossible since no projective planes of order 10 exist [20]). Laguerre structures of order q and optimal $(q+2, 3)$-codes of order q are equivalent objects ([28], Theorem 4). It is well known that there is just one optimal $(6, 3)$-code of order 4. Therefore:

Proposition 6 *If* $q = 4$ *and* (LL) *holds in* Γ*, then* $\Gamma \cong \Gamma_O(4)$ *or* $\Gamma_{21}(4)$.

Corollary 7 *The geometry* $\Gamma_{21}(4)$ *is simply connected.*

(Obvious, by Proposition 6 and because (LL) is preserved when taking covers.) We need some definitions before to state the next lemmas. Given an element x of Γ, the set of points incident with x will be denoted by $\sigma_0(x)$. We call it the 0-*shadow* of x. Similarly, the 1-*shadow* $\sigma_1(x)$ and the 2-*shadow* $\sigma_2(x)$ of x are respectively the set of lines and the set of planes incident with x. We write $\sigma_i(x, y)$ for $\sigma_i(x) \cap \sigma_i(y)$, $\sigma_i(x, y, z)$ for $\sigma_i(x) \cap \sigma_i(y) \cap \sigma_i(z)$ etc.

We denote the residue of an element x by Γ_x, as in [24]. When x is a point, Γ_x can be viewed both as an affine plane or as dual affine plane, according to whether we take the lines and the planes of Γ_x as lines and points or, dually, as points and lines, respectively. We choose the latter point of view. Thus, Γ_x is an affine plane. We denote its dual by Γ_x^*.

We denote the collinearity graph (the dual collinearity graph) of Γ by $\mathcal{G}_\perp(\Gamma)$ (by $\mathcal{G}_\top(\Gamma)$) and the collinearity diameter (dual collinearity diameter) by $\delta_\perp$ (by $\delta_\top$). Given two points a and b, we denote their distance in $\mathcal{G}_\perp(\Gamma)$ by $\delta(a, b)$. Given a point a, the neighbourhood of a in $\mathcal{G}_\perp(\Gamma)$ is denoted by $a^\perp$.

Two lines (or a line and a point) are said to be *coplanar* if there is a plane incident with both of them.

Lemma 8 *Given a point a of Γ, a line x on a and a plane u incident with a but not with x, the line x is coplanar with all but one lines of u on a.*

(Obvious, since Γ^*_a is an affine plane.)

Lemma 9 *We have $\delta_\perp \leq 2$.*

Proof. Let a and d points at distance 3, if possible, and let (a, b, c, d) be a path of $\mathcal{G}_\perp(\Gamma)$ from a to d. Let x, y, z be lines joining a with b, b with c and c with d, respectively. Let us take a plane u on y and a point e of u other than b or c and let y_b, y_c be the lines of u joining e with b and c, respectively. The lines x and y are not coplanar, as a and c are at distance 2. Hence x is coplanar with y_b by Lemma 7. Therefore $a \perp e$. Similarly, $e \perp d$. Hence $\delta(a, d) \leq 2$, a contradiction. q.e.d.

Lemma 10 *We have $|\sigma_0(u) \setminus a^\perp| \leq 1$ for every plane u and every point a.*

Proof. The statement is trivial if $a * u$. Let $a \notin \sigma_0(u)$ and assume $a^\perp \not\supseteq \sigma_0(u)$. Let $b \in \sigma_0(u) \setminus a^\perp$. We have $\delta(a, b) = 2$ by Lemma 9. Let $c \in a^\perp \cap b^\perp$ and let x be a line on c and b. The line x is coplanar with all but one lines of u on b, by Lemma 8. Hence there are a line y of u on b and a plane v incident with both x and y. Let z be a line joining c with a. By Lemma 8, z is coplanar with all lines of v on c but one. However, z and x are not coplanar, since $a \not\perp b$. Therefore z is coplanar with the line of v joining c with the point d of y other than b. Hence $a \perp d$. Let x' be a line joining a with d. By Lemma 8, x' is coplanar with all lines of u on d but one. Clearly, y is the only line of u on d non-coplanar with x'. Therefore a is collinear with all points of u but b. q.e.d.

Given two points a and b of Γ, let us write $a\Phi b$ to mean that $a \not\perp b$ or $a = b$.

Lemma 11 *Φ is an equivalence relation.*

Proof. Let $a\Phi b\Phi c$ and $a\perp c$, if possible. Let u be a plane on a and c. By Lemma 10, b is non-collinear with at most one point of u, thus it is collinear with at least one of a and c, contrary the hypothesis that $a\Phi b\Phi c$. q.e.d.

Given two distinct points a, b we denote by $\nu_{a,b}$ the number of lines joining a with b.

Lemma 12 *The number $\nu_{a,b}$ does not depend on the choice of the collinear pair (a, b).*

Proof. Let a, b, c be a triple of coplanar points, let u be a plane on a, b, c and let x_a, x_b, x_c be the lines of u joining b with c, c with a and a with b, respectively. Let z be any other line on a and b. By Lemma 8, z is coplanar with x_a, since it cannot be coplanar with x_c. Let v be the plane on z and x_a and let $f(z)$ be the line of v joining a with c. We have $f(z) \neq x_b$, otherwise u and v are two different planes on c with two distinct lines x_a, x_b in common, contrary to the fact that Γ^*_c is an affine plane.

It is now clear that, if we also set $f(x_c) = x_b$, then we get an injective mapping from the set of lines on a and b to the set of lines on a and c. Therefore $\nu_{a,b} \leq \nu_{a,c}$. Similarly, $\nu_{a,c} \leq \nu_{a,b}$. Hence $\nu_{a,b} = \nu_{a,c}$. The conclusion follows from the connectedness of Γ. q.e.d.

We write ν for $\nu_{a,b}$ and we call ν the *collinearity index* of Γ. Clearly $\nu \leq q$, since point-residues are dual affine planes of order q and two lines of Γ with the same points cannot be coplanar. Obviously we also have $\nu \geq 1$, and $\nu = 1$ if and only (LL) holds.

Lemma 13 *The geometry* Γ *is flat if and only if* $\nu = q$.

Proof. Let $\nu = q$. Then two lines on a point a have the same points if and only if they are parallel in Γ_a^*. Therefore $|a^\perp| = q+1$ and any two points of $a^\perp$ are collinear since they belong to non-parallel lines of Γ_a^*, i. e. to coplanar lines of Γ. Hence $\mathcal{G}_\perp(\Gamma)$ is a complete graph and Γ has $q+2$ points, namely it is flat.
Conversely, if Γ is flat, then $|\sigma_2(a,b)| = q^2$ for any two distinct points a, b. Hence $\nu = q$. q.e.d.

Given a point a in Γ, an easy computation shows that $|a^\perp| = (q+q^2)/\nu$. Denoted by φ the size of the equivalence class of a with respect to the relation Φ (see Lemma 11), the point-set of Γ has size $\varphi + (q+q^2)/\nu$. Hence φ does not depend on the choice of the point a. Therefore:

Lemma 14 *The graph* $\mathcal{G}_\perp(\Gamma)$ *is a complete* t*-partite graph with classes of size* φ *and*

$$t \;=\; 1 + \frac{q+q^2}{\varphi\nu}$$

(Note that φ might be 1. In that case $\mathcal{G}_\perp(\Gamma)$ is a complete graph, that is $\delta_\perp = 1$). It follows from the above that $\varphi\nu$ divides $q+q^2$. Moreover:

Lemma 15 *The number* $\nu\varphi$ *divides* q.

Proof. Let a be a point of Γ. As $\mathcal{G}_\perp(\Gamma)$ is a complete t-partite graph, $a^\perp$ is a join of classes of Φ. The relation "to join a with points of the same class of Φ" is an equivalence relation on the set of lines through a. It is in fact a refinement of the parallelism of Γ_a^* and its equivalence classes have size $\nu\varphi$. Parallelism classes in Γ_a^* have size q. Hence $\nu\varphi$ divides q. q.e.d.

Lemma 16 *One of the following holds:*

(i) $\varphi\nu = q$;

(ii) $\nu = \varphi = 1$ *and* $q = 4$;

(iii) $\varphi = 1$, $\nu = 3$ *and* $q = 18$.

Proof. Since Γ has $\varphi + (q+q^2)/\nu$ points, the number of planes of Γ is

$$\frac{(\varphi + \frac{q+q^2}{\nu})q^2}{q+2}$$

Therefore $q+2$ divides $q^2(\varphi\nu + q + q^2)$. The greatest common divisor of $q+2$ and q^2 is a divisor of 4. Consequently, $q+2$ divides $4(\varphi\nu + q + q^2)$. That is, $q+2$ divides $4\varphi\nu + 8$. Moreover, $\nu\varphi$ divides q by Lemma 15. Consequently, $m\varphi\nu + 2$ divides $4\varphi\nu + 8$, with $m = q/\varphi\nu$. It is straightforward to prove that the latter condition holds only if m and $\varphi\nu$ are as in one of the following cases:

(1) $m = 1$;
(2) $m = 3$ and $\nu\varphi = 2$ (hence $q = 6$);
(3) $m = 4$ and $\nu\varphi = 1$ (hence $q = 4$);
(4) $m = 6$ and $\nu\varphi = 3$ (hence $q = 18$);

(5) $m = 7$ and $\nu\varphi = 2$ (hence $q = 14$).

Cases (2) and (5) are impossible by the well known Bruck-Ryser condition for the existence of a projective plane of order q. In case (4) we might have either $\nu = 3$ and $\varphi = 1$ or $\nu = 1$ and $\varphi = 3$. However, if $\nu = 1$, then Γ is a Laguerre structure by Proposition 5. The collinearity graph of a Laguerre structure of order q is a complete $(q+2)$-partite graph with all classes of size q. Therefore the case of $\nu = 1$, $\varphi = 3$ and $q = 18$ is impossible. The statement is proved. q.e.d.

Corollary 17 *If $\delta_\perp = 2$, then $\mathcal{G}_\perp(\Gamma)$ is a complete $(q+2)$-partite graph and the 0-shadows of the planes are maximal cliques of $\mathcal{G}_\perp(\Gamma)$.*

(Easy, by Lemmas 14 and 16.)

Proposition 18 *If $\delta_\perp = 1$, then one of the following holds:*

(i) Γ *is flat;*

(ii) $\Gamma \cong \Gamma_{21}(4)$*;*

(iii) $q = 18$ *and* $\nu = 3$.

Proof. We have $\delta_\perp = 1$ if and only if $\varphi = 1$. Furthermore, (LL) holds if and only if $\nu = 1$. The statement follows from Lemma 16 and Proposition 5. q.e.d.

Lemma 19 *If Γ does not satisfy* (LL) *and $q \neq 18$, then $|\sigma_2(a, x)| = \nu$ for any non-incident point-line pair (a, x) with $\sigma_0(x) \subseteq a^\perp$.*

Proof. Let (LL) fail to hold (i.e. $\nu > 1$) and $q \neq 18$. Then $q = \nu\varphi$ by Lemma 16. If $\delta_\perp = 1$, then Γ is flat by Proposition 18. Hence $\nu = q$ by Lemma 13 and the statement follows.

Let $\delta_\perp = 2$ (compare Lemma 9) and let (a, x) be a non-incident point-line pair with $\sigma_0(x) \subseteq a^\perp$. Let $\{b, c\} = \sigma_0(x)$. Every plane incident with x and a uniquely determines a line on a and b. Conversely, given any line y on a and b, the lines x and y are non-parallel in Γ_b^* since $a \perp c$ and $q = \nu\varphi$. Hence they are coplanar in Γ. There are ν lines on a and b. The statement is now clear. q.e.d.

Lemma 20 *We have $|\sigma_0(u, v)| = 2|\sigma_1(u, v)|$ for any two distinct planes u, v.*

Proof. Let x, y be distinct lines incident with both u, v. Then x and y have no points in common. Indeed, if a were a common point of x and y, then x, y would be distinct lines of Γ_a^* meeting in two distinct points u, v, but this is impossible.
On the other hand, any point a incident with both u and v is incident with precisely one line $x \in \sigma_1(u, v)$ (namey the line of Γ_a^* through the points u and v). Whence the statement.q.e.d.

Proposition 21 *We have $q \equiv 0$ $(mod\ 2)$.*

Proof. When $q = 18$ there is nothing to prove. Thus, let $q \neq 18$.

If (LL) holds (that is, $\nu = 1$), then the statement follows from Proposition 5 (recall that Laguerre structures have even order [12]). If $q = \nu$ (that is, if Γ is flat), then $q \equiv 0 \pmod 2$ by Lemma 20.

Let $1 < \nu < q$. Given a plane u and two points a, b of it, let a', b' be points distinct from a and b and such that $a \not\perp a'$ and $b \not\perp b'$ (they exist by Lemma 14). By Lemma 14, we have $a' \perp b'$ and every point of u different from a and b is collinear with both a' and b'. Let x be a line on a' and b'. For every point $c \in \sigma_0(u) \setminus \{a, b\}$, there are ν planes of $\sigma_2(x, c)$ by Lemma 19. Hence, νq is the number of point-plane flags (c, v) with $c \in \sigma_0(u) \setminus \{a, b\}$ and $v * x$. However, we can count these flags in another way too, by choosing a plane v on x and counting the points incident with both u and v. Therefore $q\nu = \sum_{v \in \sigma_2(x)} |\sigma_0(u, v)|$. However, the number $|\sigma_0(u, v)|$ is even (possibly 0) for every $v \in \sigma_2(x)$, by Lemma 20. Hence $\varphi\nu$ is even. On the other hand, $q = \varphi\nu$ by Lemma 16. Therefore q is even. q.e.d.

5. Feasible orders

From now G is a flag-transitive subgroup of $Aut(\Gamma)$, with Γ a $c.Af^*$ geometry of order q. Given an element x of Γ, we denote by G_x the stabilizer of x in G and by K_x the elementwise stabilizer of Γ_x. We set $\overline{G}_x = G_x/K_x$. That is, $\overline{G}_x$ is the action of G_x on Γ_x. Given a flag (x, y), we set $G_{x,y} = G_x \cap G_y$, we denote by $K_{x,y}$ the elementwise stabilizer of the residue of (x, y) and we set $\overline{G}_{x,y} = G_{x,y}/K_{x,y}$.

As in the previous section, given a point p of Γ we denote by Γ_p^* the dual of Γ_p. We shall prove the following,

Proposition 22 *We have $q = 2$, 4 or 16 and the affine plane Γ_p^* is desarguesian. Furthermore, when $q = 16$, given a point p and a plane u of Γ we have $L_2(17) \leq \overline{G}_u \leq PGL_2(17)$ and $\overline{G}_p \leq A\Gamma L_1(2^8)$.*

Proof. Since the order of a flag-transitive affine plane is a prime power [7] and q is even by Proposition 21, we have $q = 2^h$ for some $h \geq 1$. It is well known that for $q = 2$ or 4 there is a unique affine plane of order q, which is desarguesian. Thus, we may assume $q > 4$, that is $h > 2$.

Given a plane u of Γ, the flag transitivity of G on Γ implies the 2-transitivity of $\overline{G}_u$ on the $q + 2$ points of Γ_u. By comparing the list of 2-transitive groups ([3]; also [21]) and recalling that q is a power of 2, it is not difficult to prove that the following are the only possibilities for $\overline{G}_u$:

$$A_{q+2} \leq \overline{G}_u \leq S_{q+2} \tag{1.a}$$

$$L_n(s) \leq \overline{G}_u \leq P\Gamma L_n(s) \tag{1.b}$$

with

$$q + 2 \;=\; \frac{s^n - 1}{s - 1}$$

in $(1.b)$, for some prime power s and some integer $n > 1$. On the other hand, given a point p of u, $\Gamma_{p,u}$ is the bundle of lines of Γ_p^* on u, which is a point of the affine plane Γ_p^*. Therefore,

$$|\overline{G}_{u,p}| \;\leq\; |P\Gamma L_2(q)| \tag{2}$$

(see [1], for instance). Our proof will go on as follows. By comparing (2) and (1.a), (1.b) we will get some control over $\overline{G}_{u,p}$. Then we shall turn to $\overline{G}_p$ and we will obtain the conclusion.

In case (1.a) we have $\overline{G}_{u,p} \geq A_{q+1}$. Therefore $(2^h+1)! \leq 2^{h+1}(2^{2h}-1)h$, by (2) and by the fact that $q = 2^h$. The latter inequality forces $h \leq 2$, contrary to the assumption $h > 2$. Thus, (1.a) is ruled out and (1.b) holds.

Let s, n be as in (1.b). As $2^h + 2 = (s^n-1)/(s-1)$, s is odd, n is even and

$$(3) \qquad 2^h = \frac{s^n - 2s + 1}{s-1}$$

The stabilizer of p in the subgroup $L_n(s)$ of $\overline{G}_u$ is $E : GL_{n-1}(s)/Z$, with Z the group of n^{th}-roots of unit in GF(s) and E elementary abelian of order s^{n-1}. Clearly, $|Z| \leq n$. By (2) we get the following:

$$(4) \qquad s^{n(n-1)/2}\prod_{i=1}^{n-1}(s^i-1) \leq n2^h(2^{2h}-1)h$$

Let $n \geq 6$. Then

$$\prod_{i=1}^{n-1}(s^i-1) = (s-1)\prod_{i=2}^{n-1}(s^i-1) > (s-1)\prod_{i=2}^{n-1}s^{i-1} =$$

$$= (s-1)\prod_{i=1}^{n-2}s^i = (s-1)s^{(n-2)(n-1)/2}$$

Comparing this and (4) we get

$$(s-1)s^{(n-1)^2} < n2^h(2^{2h}-1)h$$

By substituting (3) in the above inequality we obtain the following

$$(s-1)s^{(n-1)^2} < nh\frac{(s^n-2s+1)(s^n-3s+2)(s^n-s)}{(s-1)^3}$$

Hence

$$(5) \qquad (5)(s-1)^4 s^{(n-1)^2-3n} < nh$$

On the other hand, $h < n\log_2 s$, by (3). Since s is odd and > 1, we have $\log_2 s < s-1$. Therefore $h < n(s-1)$. Comparing (5) we obtain

$$(s-1)^3 s^{(n-1)^2-3n} < n^2$$

However, this inequality does not hold for $n \geq 6$ and $s \geq 3$. Therefore $n < 6$. As n is even, $n = 2$ or 4. Assume $n = 4$. Then

$$2^h = s^3 + s^2 + s - 1$$

by (3). As s is odd, the right side of the above equality is congruent to 2 or 6 modulo 8. Hence $h = 1$ and $s = 1$, contradiction.

Therefore $n = 2$. Also, $s = 2^h + 1$ by (3) and $q = 2^h$. On the other hand, s is a prime power. Thus, by a classical result of number theory [11] either s is prime or $s = 9$. In the first case we have

$$(6.a) \qquad Z_s : Z_{(s-1)/2} \leq \overline{G}_{p,u} \leq Z_s : Z_{s-1}$$

whereas

$$(6.b) \qquad 3^2 : Z_4 \leq \overline{G}_{p,u} \leq (3^2 : Z_8).2$$

in the latter case.

Now we turn to Γ_p^*. By [1] one of the following holds:

$$(7.a) \qquad \overline{G}_{p,u} \cong Suz(2^{h/2}).Z_{h/2} \text{ with } \Gamma_p^* \text{ a Lüneburg plane}$$

$$(7.b) \qquad \overline{G}_{p,u} \leq \Gamma L_1(2^{2h})$$

$$(7.c) \qquad \overline{G}_{p,u} \leq P\Gamma L_2(2^h) \text{ with } \Gamma_p^* \text{ a desarguesian affine plane}$$

(note that (7.b) and (7.c) are not incompatible).

Assume (7.a). Then $h \equiv 2 \pmod 4$ and $h \geq 6$. Hence (6.a) holds for $\overline{G}_{u,p}$. According to that, $Suz(2^{h/2})$ should contain a cyclic group of order 2^h. This is false. Indeed the largest cyclic subgroups of a Sylow 2-subgroup of a Suzuki group have order 4. Thus, (7.a) is ruled out.

Let (7.b) hold. By combining (7.b) with any of (6.a) or (6.b) we get that $2^{h-1}(2^h + 1)$ divides $(2^{2h} - 1)2h$. This forces $h = 4$ (recall that $h > 2$ by assumption). Therefore $\Gamma_p^* = AG(2, 16)$, since this is the unique flag-transitive affine plane of order 16 (see [8]). Furthermore, $\overline{G}_u$ and $\overline{G}_p$ are as in the second claim of the proposition, by (1.b) and (7.b).

In order to finish the proof we only need to show that (7.c) forces $h = 4$ (whence $G_p \leq A\Gamma L_1(2^8)$ by (6.a)). Assume (7.c) and let $s = 9$, as in (6.b). Then $\overline{G}_{p,u} \geq Z_4$. Hence $P\Gamma L_2(8)$ contains Z_4. However this is false (the Sylow 2-subgroup of $P\Gamma L_2(8)$ is an elementary abelian subgroup of order 8). Thus, (6.b) is ruled out. Therefore (6.a) holds and $h \neq 3$.

By (6.a), the Sylow 2-subgroup of $\overline{G}_{p,u}$ contains a cyclic group of order 2^{h-1}, whereas cyclic subgroups of the Sylow 2-subgroup of $P\Gamma L_2(2^h)$ have order at most 2^{r+1}, where r is the exponent of the maximal power of 2 that divides h (possibly, $r = 0$). Consequently $h - 1 \leq r + 1$, that is $r \geq h - 2$. On the other hand, $2^r \leq h$. Therefore $h \leq 4$. Hence $h = 4$, as $h \neq 3$ by the above and $h > 2$ by assumption. q.e.d.

6. The case of $q = 4$

Let $q = 4$. The geometry Γ is assumed to be flag-transitive in this section, but we will exploit this hypothesis only in a very few places.

The case with (LL) is classified in Proposition 6. Thus, we assume that (LL) fails to hold in Γ, namely $\nu > 1$. Therefore $\nu = 2$ or 4, since ν divides q (Lemma 16). Note that Γ is flat when $\nu = 4$ (Lemma 13). In 6.2 we prove that, if Γ is flat then Γ is isomorphic to either $\Gamma_O(4)/2^2$ or $\Gamma_6(4)$. We consider the case of $\nu = 2$ in 6.3, proving that if $\nu = 2$ then Γ is a cover of $\Gamma_O(4)/2^2$. On the other hand, $\Gamma_O(4)$ is simply connected by Corollary 4. Hence Γ is a quotient of $\Gamma_O(4)$. In fact, $\Gamma \cong \Gamma_O(4)/2$ as $\nu = 2$.

6.1. More definitions and notation

We say that two distinct lines *overlap* if they meet in two points (i.e., they have the same points). Clearly, "being overlapping or equal" is an equivalence relation on the set of lines of Γ. We denote it by $\|_0$. All equivalence classes of $\|_0$ have size ν. Furthermore, overlapping lines have no planes in common. Thus, if $x \|_0 y$ and $a \in \sigma_0(x) = \sigma_0(y)$, then x and y are parallel lines of the affine plane Γ_a^*. Given two lines x, y, we write $x \|_2 y$ to mean that $\sigma_2(x) = \sigma_2(y)$. This is also an equivalence relation. We shall denote the set of points of Γ by P.

6.2. The flat case

Let $\nu = 4$. Then Γ is flat (Lemma 13). We can assume that P is the set $\{1, 2, 3, 4, 5, 6\}$ of the first six positive integers. The number of planes of Γ is 16 and the number of lines is 60.

If u and v are distinct planes of Γ, then $\sigma_1(u, v)$ consists of three lines (Lemma 20), which partition P $(= \sigma_0(u) = \sigma_0(v))$ in three pairs.

Lemma 23 *Given a line x of Γ, let u_1, u_2, u_3, u_4 be the four planes on x. If $|\sigma_1(u_1, u_2, u_3)| = 3$ then $\sigma_1(u_1, u_2, u_3) \subseteq \sigma_1(u_4)$.*

Proof. Let $\sigma_1(u_1, u_2, u_3) = \{x, y, z\}$. By flag transitivity, some $g \in G_x$ maps u_3 onto u_4. We have $|g(\{u_1, u_2, u_3\}) \cap \{u_1, u_2, u_3\}| = 2$. Let

$$\{u, u'\} = g(\{u_1, u_2, u_3\}) \cap \{u_1, u_2, u_3\}$$

Then $\sigma_1(u_4, u, u') = \sigma_1(u, u')$. However, $\sigma_1(u, u') = \{x, y, z\}$ because u and u' are two of the planes u_1, u_2, u_3 and $\sigma_1(u_i, u_j) = \{x, y, z\}$ for $i, j = 1, 2, 3$, $i \neq j$. Thus u_4 is incident with all of x, y, z. q.e.d.

Lemma 24 *Given a line x of Γ, let u_1, u_2, u_3 be three of the four planes on x. If $|\sigma_1(u_1, u_2, u_3)| \geq 2$ then $|\sigma_1(u_1, u_2, u_3)| = 3$.*

Proof. Let $\sigma_1(u_1, u_2, u_3) \supseteq \{x, y\}$ and let z_i be the third line of $\sigma_1(u_i, u_3)$, for $i = 1, 2$. The condition $\sigma_0(z_i) = P \setminus (\sigma_0(x) \cup \sigma_0(y))$ uniquely determines z_i in $\sigma_1(u_3)$. Hence $z_1 = z_2$. q.e.d.

Lemma 25 *Given a line x of Γ, let u_1, u_2, u_3, u_4 be the four planes on x. One of the following occurs*

(i) $|\sigma_1(u_1, u_2, u_3, u_4)| = 3$;

(ii) $\sigma_1(u_i, u_j, u_k) = \{x\}$ *for any 3-subset $\{i, j, k\}$ of $\{1, 2, 3, 4\}$.*

Moreover, if one of the above cases occurs for some line x, then the same holds for every line x.

(Obvious, by Lemmas 24 and 23 and flag transitivity). By this lemma, either all classes of $\|_2$ have size 3 (case (i) of the lemma) or $\|_2$ is the identity (case (ii)).

6.2.1. Case (i)

Assume that the classes of $\|_2$ have size 3.

We denote by $\|_0 \cdot \|_2$ the product of the relations $\|_0$ and $\|_2$, in this order (that is, $x \|_0 \cdot \|_2 y$ for two lines x, y when $x \|_0 z \|_2 y$ for some line z). We say that $\|_0$ and $\|_2$ *commute* if $\|_0 \cdot \|_2 = \|_2 \cdot \|_0$.

Lemma 26 *The relations $\|_0$ and $\|_2$ commute.*

Proof. Given two overlapping lines x and x', let u_1, u_2, u_3, u_4 be the four planes incident with x and let u'_1, u'_2, u'_3, u'_4 be those incident with x'. Let $\sigma_1(u_1, u_2, u_3, u_4) = \{x, y, z\}$ and $\sigma_1(u'_1, u'_2, u'_3, u'_4) = \{x', y', z'\}$. We need to prove that y' overlaps y or z.

Suppose that y' does not overlap any of y or z. Then the same holds for z'. We can assume without loss that $\sigma_0(x) = \sigma_0(x') = \{1, 2\}$, $\sigma_0(y) = \{3, 4\}$ and $\sigma_0(z) = \{5, 6\}$. Therefore x', y', z' determine one of the following two partitions:

$$\{\{1,2\},\{3,5\},\{4,6\}\}, \quad \{\{1,2\},\{3,6\},\{4,5\}\}$$

We can assume without loss that they determine the first partition. Given $i = 1, 2, 3$ or 4, we denote by P_i the partition of P formed by the three lines of $\sigma_1(u_1, u'_i)$. None of the pairs $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$ is a member of P_i because u_1, u_2, u_3, u_4 are the four planes incident with each of x, y and z and none of them is equal to any of u'_1, u'_2, u'_3, u'_4 (indeed these are the planes on x', and x' overlaps x). Similarly, neither $\{3, 5\}$ nor $\{4, 6\}$ is a class of P_i. Thus, the following partitions only remain to consider:

$$\begin{array}{ll} \{\{1,3\},\{2,6\},\{4,5\}\}, & \{\{1,6\},\{2,3\},\{4,5\}\}, \\ \{\{1,4\},\{2,5\},\{3,6\}\}, & \{\{1,5\},\{2,4\},\{3,6\}\}. \end{array}$$

Furthermore, $P_i \neq P_j$ if $i \neq j$, otherwise $\sigma_1(u_1, u'_i) = \sigma_1(u_1, u'_j) = \sigma_1(u'_i, u'_j) = \{x', y', z'\}$, whereas $x' \notin \sigma_1(u_1)$. Thus, we can assume that

$$\begin{array}{ll} P_1 = \{\{1,3\},\{2,6\},\{4,5\}\}, & P_2 = \{\{1,6\},\{2,3\},\{4,5\}\}, \\ P_3 = \{\{1,4\},\{2,5\},\{3,6\}\}, & P_4 = \{\{1,5\},\{2,4\},\{3,6\}\}. \end{array}$$

Consequently, the line of u_1 incident with the points 4 and 5 is incident with both u'_1 and u'_2. However, this is impossible. Indeed $\sigma_1(u'_1, u'_2) = \{x', y', z'\}$, but none of these lines has the pair $\{4, 5\}$ as its 0-shadow. q.e.d.

By Lemma 26, the product $\|_0 \cdot \|_2$ of the relations $\|_0$ and $\|_2$ is the equivalence relation generated by them. Let us denote it by $\|$. The classes of $\|$ have size 12. Hence $\|$ has precisely five classes. Each class of $\|$ uniquely determines a partition of P in three pairs. The five partitions of P obtained in this way form a 1-factorization of the complete graph that has P as the set of vertices. Furthermore, given a point a, two distinct lines x, y on a overlap if and only if they are parallel in the affine plane Γ_a^* (indeed the classes of $\|_0$ have size 4).

It is now clear that Γ is a gluing of $AG(2,4)$ with the complete graph on six vertices (compare 2.4). As we have remarked at the end of 2.4, there is a unique way of gluing $AG(2,4)$ with the complete graph on six vertices and that gluing is precisely $\Gamma_O(4)/2^2$. Thus,

Proposition 27 *If $\|_2$ has classes of size 3, then $\Gamma \cong \Gamma_O(4)/2^2$.*

6.2.2. Case (ii)

We now assume that $\|_2$ is the identity. Chosen a point a of Γ, we define a rank 2 geometry Π as follows. The points of Π are the five points of Γ other than a and the planes of Γ. The lines of Π are the 20 lines of Γ on a and a symbol ∞. The incidence relation of Π is the one inherited from Γ, with the additional clause that ∞ is declared to be incident with precisely the points of Γ other than a.

Clearly, if we drop ∞ and its points from Π then we get Γ_a^*, which is $AG(2,4)$. Therefore $\Pi = PG(2,4)$.

For every line x of Γ not on a we set $O_x = \sigma_0(x) \cup \sigma_2(x)$. We denote by $\mathcal{O}$ the family of these sets.

Lemma 28 *For every line x of Γ not on a, the set O_x is a hyperoval of the projective plane Π.*

Proof. Clearly, $|O_x| = 6$. We have $\sigma_1(u,v,w) = \{x\}$ for any three distinct elements $u, v, w \in O_x$ because $\|_2$ is the identity relation. Therefore u, v, w cannot be collinear in Π. q.e.d.

Lemma 29 *We have $O_x \neq O_y$ for any two distinct lines x, y of Γ not on a.*

Proof. If $O_x = O_y$, then $\sigma_2(x) = \sigma_2(y)$. Hence $x = y$ because $\|_2$ is the identity relation. q.e.d.

It is well known that there are 168 hyperovals in $\Pi = PG(2,4)$ and that the relation "to intersect in an even number of points" (actually, 0, 2 or 6) is an equivalence relation on that set of hyperovals. The relation splits the set into three classes, say $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, each of size 56.

Lemma 30 *The family $\mathcal{O}$ is contained in one of the classes $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$.*

Proof. Let x, y be two distinct lines of Γ not on a and let $z \in O_x \cap O_y$. This element might be a point or a plane of Γ.

Let z be a point of Γ, that is $z \in \sigma_0(x) \cap \sigma_0(y)$. If x and y have the same points, then they cannot be incident with any common plane. Therefore $O_x \cap O_y = \sigma_0(x) = \sigma_0(y)$, and this is a pair of points. On the other hand, let z be the only point incident with both x and y. Then x and y are not parallel in the affine plane Γ_z^*, since $\nu = 4$. This forces $\sigma_2(x) \cap \sigma_2(y)$ to be nonempty. However, $|\sigma_2(x) \cap \sigma_2(y)| = 1$ as x and y have a point in common. Therefore $|O_x \cap O_y| = 2$.

Assume now that x and y have no points in common. Hence z is a plane of Γ. As $\|_2$ is the identity relation, the three planes on x determine on $\sigma_0(z)$ three distinct partitions in pairs and the pair $\sigma_0(x)$ belongs to each of these partitions. It is clear that only one of these partitions contains $\sigma_0(y)$. Therefore only one of the three planes on x other than z is incident with y. This shows that $|O_x \cap O_y| = 2$. q.e.d.

Assume $\mathcal{O} \subseteq \mathcal{O}_1$, to fix ideas. Since Γ contains precisely 40 lines not on a, we have $|\mathcal{O}| = 40$. It is well known that every line of $PG(2,4)$ meets 40 hyperovals in each of the classes. On the other hand, every hyperoval of $\mathcal{O}$ meets ∞. Therefore, $\mathcal{O}$ is the family of the hyperovals of $\mathcal{O}_1$ intersecting ∞.

Let $\mathcal{S}$ be the rank 2 geometry that has a and the points of Π as points and the lines of Π and the hyperovals of Π as blocks, with the incidence relation inherited from Π and the additional clause $a * \infty$. Then $\mathcal{S}$ is a model of $S(3,6,22)$. It is now clear that Γ consists of the six points of the hexad ∞, the 16 points of Π not on ∞ (these are taken as planes) and the 40 hexads of $\mathcal{S}$ intersecting the hexad ∞ in two points. That is,

Proposition 31 *We have* $\Gamma \cong \Gamma_6(4)$.

By Propositions 27 and 31, $\Gamma_O(4)/2^2$ and $\Gamma_6(4)$ are the only flat flag-transitive $c.Af^*$ geometries of order 4.

6.3. The case of $\nu = 2$

Let $\nu = 2$. Hence $\varphi = 2$, by Lemma 16. That is, every point is non-collinear with precisely one point. Given a point a, we denote by $\pi(a)$ the unique point non-collinear with it. We call $\pi(a)$ the *antipodal point* of a.

Since $\varphi = 2$, we have $\delta_\perp = 2$ and $|P| = 12$. Therefore Γ has 32 planes and 120 lines.

Two lines incident with two distinct planes have no points in common. Hence two distinct planes are incident with at most three common lines. We say that two planes u, v are *i-secant* if $|\sigma_1(u,v)| = i$. Given a line-plane flag (x,u), let n_i be the number of planes incident with x and i-secant with u. Note that the numbers n_1, n_2, n_3 do not depend on the choice of the flag (x,u), by flag transitivity.

We denote by m_i the number of planes i-secant with u. By flag transitivity, the numbers m_0, m_1, m_2, m_3 do not depend on the choice of the plane u. Clearly, $m_0 + m_1 + m_2 + m_3 = 31$, since 32 is the number of planes of Γ.

Lemma 32 *We have* $m_3 = n_3 = 0$.

Proof. Since any two planes on the same point a meet in a line on a, two planes are 3-secant if and only if they have the same set of points. Thus, "being 3-secant or equal" is an equivalence relation. Let us denote it by $\|^*$. Given a point a, its stabilizer G_a in G preserves the equivalence relation induced by $\|^*$ on the set of planes on a. However, G_a acts primitively on the set of planes of Γ_a ([7], 2.3.7). Therefore either all planes on a belong to the same class of $\|^*$, or no two of them correspond in $\|^*$. In the latter case $m_3 = 0$ (hence $n_3 = 0$).

Let the first case occur, if possible. By connectedness, any two planes of Γ correspond in $\|^*$. That is, all planes of Γ are incident with the same set of points. Namely, Γ is flat, contrary to the hypothesis that $\nu = 2$. q.e.d.

Lemma 33 *We have* $m_1 = m_2 = 15$, $n_1 = 1$, $n_2 = 2$.

Proof. By counting the flags (v, y) with y a line of u and v a plane i-secant with u, we get $m_1 = 15n_1$ and $2m_2 = 15n_2$. Clearly, $n_1 + n_2 + n_3 = 3$. Hence $n_1 + n_2 = 3$ because $n_3 = 0$ (Lemma 32). Therefore $m_1 + 2m_2 = 45$. Hence either $m_1 = m_2 = 15$ or $m_1 = 45$ and $m_2 = 0$. In the latter case, any two planes have at most one line in common. This forces (LL) to hold in Γ, by Lemma 7.25 of [24] in the dual of Γ. However, (LL) fails to hold in Γ, as we have assumed $\nu = 2$. Therefore $m_1 = m_2 = 15$. q.e.d.

Lemma 34 *We have $\delta_\top = 2$ and $m_0 = 1$.*

Proof. By Lemma 14, given any two planes u, v, some points of u are collinear with some points of v. Let x be a line joining a point a of u with a point b of v and let w be a plane on x. A line incident with u and w can be found in Γ_a. In Γ_b we find a line incident with w and v. Hence u and v have distance ≤ 2 in the dual collinearity graph of Γ. Therefore $\delta_\top \leq 2$.

On the other hand, $m_0 + m_1 + m_2 = 31$. Hence $m_0 = 1$, by Lemma 33. Consequently, $\delta_\top = 2$. q.e.d.

As $m_0 = 1$, for every plane u there is precisely one plane that is not dually collinear with u. We call it the *antipodal plane* of u and we denote it by $\pi(u)$.

Lemma 35 *Given a plane u, the planes that are 1-secant with u are 2-secant with $\pi(u)$, whereas those that are 2-secant with u are 1-secant with $\pi(u)$.*

Proof. Let v be a plane 2-secant with both u and $\pi(u)$. Then

$$|\sigma_0(u, v)| = |\sigma_0(\pi(u), v)| = 4$$

Hence $\sigma_0(u, \pi(u), v) \neq \emptyset$, which is impossible because $\sigma_0(u, \pi(u)) = \emptyset$. Therefore no planes are 2-secant with both u and $\pi(u)$. The conclusion follows from Lemma 33. q.e.d.

Lemma 36 *The function π mapping every point and every plane onto its antipodal is an automorphism of the point-plane system of Γ and it commutes with every automorphism of Γ.*

Proof. Clearly, π^2 is the identity. Whence π is bijective. It is also clear that it commutes with every automorphism of Γ.

By contradiction, assume that π is not an automorphism of the point-plane system of Γ. Then $\pi(a) \notin \sigma_0(\pi(u))$ for some point-plane flag (a, u). Then $\pi(b) \notin \sigma_0(\pi(u))$ for every point $b * u$, by flag transitivity. That is, every point of u is collinear with all points of $\pi(u)$.

Let N be the number of 5-tuples (b, c, x, y, v) with b a point of u, c a point of $\pi(u)$, v a plane incident with both b and c, x the line on b incident with both u and v, and y the line on c incident with both v and $\pi(u)$. Since all points of u are collinear with all points of $\pi(u)$, there are precisely 36 choices for the pair (b, c). Given one of these pairs, there are eight planes incident with both b and c. If v is one of them, the lines x and y of Γ_b and Γ_c joining v with u and $\pi(u)$ respectively are uniquely determined. Therefore $N = 36 \cdot 8$.

However, we can count N in the following way, too. There are 30 planes distinct from u and $\pi(u)$. By Lemma 35, fifteen of these planes are 1-secant with u and 2-secant with $\pi(u)$. The remaining ones are 2-secant with u and 1-secant with $\pi(u)$. Hence there are $30 \cdot 2$ triples

(x, y, v) with v a plane different from u and $\pi(u)$ and x, y lines of v incident with u and $\pi(u)$, respectively. Each of these lines has two points. Therefore $N = 30 \cdot 4$. However, we have previously obtained $N = 36 \cdot 8$. The contradiction is reached. q.e.d.

Lemma 37 *Given a line-plane flag (x, u), let a, b be the two points of x and let v be another plane on x. Then $\pi(v)$ is incident with the line of $\pi(u)$ that is incident with $\pi(a)$ and $\pi(b)$.*

Proof. Both $\pi(a)$ and $\pi(b)$ are incident with both $\pi(u)$ and $\pi(v)$, by Lemma 36. Let y be the line of $\pi(u)$ incident with $\pi(a)$ and $\pi(b)$. By contradiction, assume that $y \notin \sigma_1(\pi(v))$.

Then $\pi(v)$ and $\pi(u)$ are 2-secant and $\sigma_1(\pi(u), \pi(v))$ consists of two lines y_a, y_b containing $\pi(a)$ and $\pi(b)$, respectively. By Lemma 35, $\pi(u)$ is 1-secant with v, as it is 2-secant with $\pi(v)$. Therefore v is 2-secant with u, by Lemma 35. Let a', b', a'', b'' be the points of $\sigma_0(u) \setminus \sigma_0(x)$, with $a', b' \in \sigma_0(u, v)$. We have $\sigma_0(\pi(u), \pi(v)) = \{\pi(a), \pi(b), \pi(a'), \pi(b')\}$ by Lemma 36, and we can assume that $\{\pi(a), \pi(a')\} = \sigma_0(y_a)$ and $\{\pi(b), \pi(b')\} = \sigma_0(y_b)$.

As $\overline{G}_u$ is 2-transitive on the six points of u, it contains $L_2(5)$, acting on those points as on $PG(1, 5)$. Thus, there is an element $\bar{g} \in \overline{G}_u$ fixing both a and b and acting with two orbits of size 2 on the remaining four points of u.

Let u', v' be the two planes on x different from both u and v. By Lemma 33, one of them is 1-secant with u and 2-secant with v, the other one is 2-secant with u and 1-secant with v. We can assume that u' and v' are 1-secant with u and with v, respectively. Then $\sigma_0(u, v') = \{a'', b''\}$.

Let g be an element of G_u inducing $\bar{g}$ on $\sigma_0(u)$. Clearly, g fixes u' and either it fixes both v and v' or it permutes them. In the latter case $\bar{g}$ permutes the pairs $\{a', b'\}$ and $\{a'', b''\}$. Assume that this is the case. In the affine plane $\Gamma_a^* = AG(2, 4)$, u, u', v, v' are the four points of the line x. As g fixes two of them, it acts as a semilinear transformation on x. Therefore the action of $\overline{G}_a$ on the line at infinity of the affine plane Γ_a^* contains $P\Gamma L_2(4) \cong PGL_2(5)$. Hence $\overline{G}_u \geq PGL_2(5)$. Consequently, the stabilizer of a and b in $\overline{G}_u$ contains some element $\bar{f}$ of order 4 such that $\bar{f}^2 = \bar{g}$. Let f be an element of G_u inducing $\bar{f}$ on $\sigma_0(u)$. Clearly, f fixes u' and permutes v and v'. Hence $\bar{f}$ interchanges the pairs $\{a', b'\}$ and $\{a'', b''\}$. However, this forces $\bar{g}$ to fix both those pairs, contrary to the hypothesis that g permutes v and v'.

Therefore g fixes both v and v' and permutes a' with b' and a'' with b''. It also fixes $\pi(a)$, $\pi(b)$, $\pi(u)$ and $\pi(v)$ and permutes $\pi(a')$ with $\pi(b')$, since g commutes with π. However, this forces g to map the lines y_a, y_b onto lines different from both y_a and y_b. Thus, g cannot fix $\pi(v)$. We have reached the contradiction. q.e.d.

Given a line x, let a, b be its two points and let u be any of the four planes on x. We denote by $\pi(x)$ the line of $\pi(u)$ joining $\pi(a)$ with $\pi(b)$. By Lemma 37, this definition is consistent. Thus we have extended π to an automorphism of Γ. Clearly, $\pi^2 = 1$ and π belongs to the center of $Aut(\Gamma)$. It is also clear that π defines a quotient of Γ. We denote this quotient by Γ/π.

Proposition 38 *We have $\Gamma/\pi \cong \Gamma_O(4)/2^2$.*

Proof. As Γ is flag-transitive and π belongs to the center of $Aut(\Gamma)$, the geometry Γ/π is also flag-transitive. Clearly, it is flat. We need to prove that Γ/π is as in 6.2.1. Then Proposition 27 will give us the conclusion.

Given a plane u of Γ and a line x of it, let v, v' be the two planes on x that are 2-secant with u. Let a, b, a', b', a'', b'' be the points of u. We can assume that $\{a, b\} = \sigma_0(x)$, $\{a, b, a', b'\} = \sigma_0(u, v)$ and $\{a, b, a'', b''\} = \sigma_0(u, v')$. Let x', x'' be the lines of u joining a' with b' and a'' with b'', respectively.

The plane $\pi(v)$ is 2-secant with $\pi(u)$, whence it is 1-secant with u, by Lemma 35. Obviously, $\sigma_0(u, \pi(v)) = \sigma_0(u) \setminus \sigma_0(u, v) = \{a'', b''\}$. Therefore $\{x, x'\} = \sigma_1(u, v)$ and $\{x''\} = \sigma_1(u, \pi(v))$. Similarly, $\{x, x''\} = \sigma_1(u, v')$ and $\{x'\} = \sigma_1(u, \pi(v'))$. Hence the images of u, v, v' in Γ/π are incident with the images of x, x' and x''. Thus, Γ/π is as in 6.2.1. q.e.d.

Corollary 39 *We have* $\Gamma \cong \Gamma_O(4)/2$.

Proof. The Laguerre structure $\Gamma_O(4)$ is the universal cover of $\Gamma_O(4)/2^2$, by Corollary 4. Hence Γ is a quotient of $\Gamma_O(4)$, by Proposition 38 and by the definition of universal covers. On the other hand, $\Gamma_O(4)/2$ is the only flag-transitive non-flat proper quotient of $\Gamma_O(4)$. Therefore $\Gamma \cong \Gamma_O(4)/2$. q.e.d.

7. The case of $q = 16$

Let $q = 16$. As in Section 5, G stands for a flag-transitive subgroup of $Aut(\Gamma)$.

7.1. Preliminaries

Choose a chamber (p, l, u) of Γ, with p, l and u a point, a line and a plane, respectively. By Proposition 22,

$$L_2(17) \leq \overline{G}_u \leq L_2(17).2 \ (= PGL_2(17)) \tag{1}$$

and $\overline{G}_p \leq A\Gamma L_1(2^8)$. On the other hand, $2^8 : Z_{17} \leq \overline{G}_p$ by flag-transitivity. Therefore,

$$2^8 : Z_{17} \leq \overline{G}_p \leq A\Gamma L_1(2^8) \ \ (= (2^8 : (Z_{17} \times Z_5 \times Z_3)).Z_8) \tag{2}$$

Lemma 40 *We have* $K_{p,u} = K_u$.

Proof. Clearly, $K_u \leq K_{p,u}$. On the other hand, the subgroup $K_{p,u}$ fixes p, u and all lines of u on p. Hence it fixes all points of u. Therefore $K_{p,u} \leq K_u$. q.e.d.

Let $(\overline{G}_u)_p$ be the stabilizer of p in $\overline{G}_u$. We have $Z_{17}.Z_8 \leq (\overline{G}_u)_p \leq Z_{17}.Z_{16}$ by (1). Furthermore $(\overline{G}_u)_p = \overline{G}_{p,u}$ by Lemma 40. Recalling that $\overline{G}_{p,u}$ is a factor of $(\overline{G}_p)_u$ and comparing (2), we obtain the following,

$$(\overline{G}_u)_p = Z_{17}.Z_8 \ \ (= \overline{G}_{u,p}) \tag{3}$$

Hence

$$\overline{G}_u = L_2(17) \tag{4.1}$$

$$Z_{17}.Z_8 \leq (\overline{G}_p)_u \leq \Gamma L_1(2^8) \ \ (= (Z_{17} \times Z_5 \times Z_3).Z_8) \tag{4.2}$$

Lemma 41 *We have $K_p = 1$.*

Proof. Let $g \in K_p$. Then g fixes all points of $p^\perp$. Let q be one of them. Given a line m joining p with q, let v be a plane on m. Since $g \in K_p$, g fixes v and all lines of v on p. Hence it fixes all points of v. Consequently, it fixes all lines of v. However, v is any of the planes on m. Therefore g fixes m, all planes on m and all lines incident with any of those planes. It is now clear that g acts trivially on Γ_q. That is, $g \in K_q$. Hence $K_p \leq K_q$. Conversely, $K_q \leq K_p$. Therefore $K_p = K_q$. By connectedness, $K_x = K_p$ for every point x of Γ. This forces $K_p = 1$. q.e.d.

Lemma 42 *We have $K_u = 1$.*

Proof. Since $K_u = K_{u,p}$, we have $K_u \leq Z_5 \times Z_3$ by (3), (4.2) and Lemma 41. Thus, $K_u = 1, Z_3, Z_5$ or $Z_5 \times Z_3$. On the other hand, $G_u = K_u.L_2(17)$ by (4.1). As $L_2(17)$ is simple, this extension must be central. The Schur multiplier of $L_2(17)$ is 2 (see [5]). Hence the extension $K_u.L_2(17)$ is a direct product, $G_u = K_u \times L_2(17)$.

This forces K_u to be central in $G_{p,u}$. However $G_{p,u} = K_u.Z_{17}.Z_8 < \Gamma L_1(2^8)$ by (3) and Lemma 41 and this group does not contain any central subgroup of order 3, 5 or 15. Therefore $K_u = 1$. q.e.d.

Corollary 43 *We have $K_{p,u} = 1$.*

(Easy, by Lemmas 40 and 42.) By Lemmas 41 and 42, by Corollary 43 and by (3), (4.1) and (2) we get the following:

$$G_u = L_2(17) \tag{5.1}$$

$$G_p = 2^8 : Z_{17}.Z_8 \leq A\Gamma L_1(2^8) \tag{5.2}$$

Furthermore $G_l = G_{p,l}G_{l,u} = G_{l,u}G_{p,l}$, with

$$G_{p,l} = 2^4 : Z_8, \quad G_{l,u} = Z_8.2, \text{ and} \tag{5.3}$$

$$Z_8 = G_{p,l,u} \ (= G_{p,l} \cap G_{l,u}). \tag{5.4}$$

Therefore,

$$G_l = 2^4 : Z_8.Z_2. \tag{5.5}$$

Corollary 44 *We have $G = Aut(\Gamma)$.*

(Indeed (5.1) leaves just one possibility for G_u.)

7.2. The structure of G

We have $16 = \nu\varphi$ by Lemma 16 and the collinearity graph $\mathcal{G}_\perp(\Gamma)$ of Γ is a complete 18-partite graph with all classes of size φ (Lemma 14). Thus, $\varphi = 2^t$ for some $t \in \{0,1,2,3,4\}$ and Γ has 18φ points and $(18\varphi \cdot 2^8)/18 = 2^8\varphi = 2^{8+t}$ planes.

Let X_1, X_2,..., X_{18} be the eighteen classes of $\mathcal{G}_\perp(\Gamma)$ and let V be the kernel of the action of G on $\mathcal{X} = \{X_i\}_{i=1}^{18}$.

Lemma 45 *The group V acts regularly on the set of planes of Γ.*

Proof. By Corollary 17, every plane of Γ is incident with one point in every class of $\mathcal{X}$. This fact and Lemma 42 imply that V acts semi-regularly on the set of planes of Γ. Let O_1, O_2,... O_k be the orbits of V on the set of planes of Γ.

Given a point p, assume that the set $\sigma_2(p)$ of the planes of Γ on p intersects at least two of O_1, O_2,..., O_k in at least two elements. Since V is normal in G, it is normalized by G_p. Hence the orbits of V intersecting $\sigma_2(p)$ non-trivially are imprimitivity classes for G_p in its action on $\sigma_2(p)$, which is the set of the points of the affine plane Γ_p^*. However, this is impossible ([7], 2.3.7). Therefore, either $\sigma_2(p)$ intersects every orbit of V in at most one point or it is contained in one orbit of V.

Assume the first case, that is $|\sigma_2(p) \cap O_i| \leq 1$ for $i = 1, 2, ...k$. Then $G_p \cap V = 1$. Therefore, G_p acts faithfully on the eighteen classes of $\mathcal{X}$. On the other hand, G_p contains a normal subgroup W of order 2^8 and an element b of order 17. Clearly, W stabilizes the class of $\mathcal{X}$ containing p. On the other hand, b permutes the classes of $\mathcal{X}$ fixed by W, since it normalizes W. However, b has order 17, stabilizes the class of $\mathcal{X}$ containing p and cyclically permutes the remaining classes of $\mathcal{X}$. Therefore the class containing p is the unique member of $\mathcal{X}$ that is stabilized by W. However, this is impossible, because W is a 2-group and $\mathcal{X}$ has even size.

Therefore $\sigma_2(p)$ is contained in one orbit of V. For instance, $\sigma_2(p) \subseteq O_1$. Then, given a point p' collinear with p, the planes incident with p and p' belong to O_1. Hence $\sigma_2(p') \subseteq O_1$. By connectedness, all planes of Γ belong to O_1. That is, V is transitive (whence regular) on the set of planes of Γ. q.e.d.

Corollary 46 *We have $G = V : G_u$ (split extension) and $|V| \leq 2^{8+t}$.*

(Easy, by Lemma 45 and because 2^{8+t} is the number of planes of Γ.) Henceforth we set $V_p = G_p \cap V$, for short.

Corollary 47 *We have $V_p = O_2(G_p) = 2^8$ and V acts transitively on each of the classes of $\mathcal{X}$.*

Proof. Let X be the class of $\mathcal{X}$ containing the point p. We have $|X| = \varphi$ and $|V| = 2^{8+t}$ by Lemma 45. Therefore $|V_p| \geq 2^8$, with $|V_p| = 2^8$ if and only if V is transitive on X. On the other hand, V_p is normal in G_p and $O_2(G_p) = 2^8$ by (5.2). Therefore $V_p = O_2(G_p)$. q.e.d.

We have $G_{p,l,u} = Z_8$ by (5.4). Let a be a generator of $G_{p,l,u}$. Then

Corollary 48 *We have $G_l = [V_p, a^4]\langle a\rangle\langle c\rangle$ for some involution $c \in G_{l,u} \backslash G_{p,l,u}$.*

Proof. We have $G_{p,l} = [O_2(G_p), a^4]\langle a\rangle$ by (5.3). On the other hand, $|G_l : G_{p,l}| = 2$ (see (5.5)) and $G_{l,u}$ contains some involutions c permuting the two points of l. Corollary 47 gives us the conclusion. q.e.d.

Lemma 49 *We have $V = V_pV_{p'}$ for any two collinear points p and p'.*

Proof. Clearly, $V_p \cap V_{p'}$ is in the center of V, as V_p and $V_{p'}$ generate V. Let p, p' be distinct collinear points. By Corollary 47, the groups V_p and $V_{p'}$ act regularly on $\sigma_2(p)$ and on $\sigma_2(p')$, respectively. Therefore, given two planes $u, u' \in \sigma_2(p) \cap \sigma_2(p')$, there are elements $g \in V_p$ and $g' \in V_{p'}$ mapping u onto u'. However, $g = g'$ because V is regular on the set of planes of Γ. Therefore $V_p \cap V_{p'}$ is regular on $\sigma_2(p) \cap \sigma_2(p')$. On the other hand, $|\sigma_2(p) \cap \sigma_2(p')| = 2^4\nu$, since there are ν lines incident with both p and p' and every line is incident with 2^4 planes. Hence $|V_p \cap V_{p'}| = 2^4\nu$. Therefore,

$$|V_pV_{p'}| \;=\; \frac{2^{16}}{2^4\nu} \;=\; 2^{8+t} = |V|$$

Hence $V = V_pV_{p'}$. q.e.d.

Lemma 50 *The group V is elementary abelian.*

Proof. As V is a 2-group, we have $Z(V) \neq 1$. Assume $Z(V) \neq V$ and let O_1, O_2,..., O_k be the orbits of $Z(V)$ on the set of planes of Γ. As in the proof of Lemma 45, one can prove that $\sigma_2(p)$ meets each of O_1, O_2,..., O_k in at most one element. Therefore $Z(V) \cap G_p = 1$.

Let p' be a another point, collinear with p. We have $V_p \cap V_{p'} \leq Z(V)$. Indeed both V_p and $V_{p'}$ are abelian by Corollary 47 and they generate V by Lemma 49. Hence $Z(V) \cap G_p \neq 1$, contrary to the conclusion that we have previously obtained from the hypothesis that $Z(V) \neq V$. Therefore V is abelian. It is also elementary abelian. Indeed V_p and $V_{p'}$ generate V and all their elements other than 1 are involutions. q.e.d.

By Lemmas 45 and 50 we obtain the following:

Corollary 51 *Let $m = 8 + t$. Then V is an m-dimensional* GF(2)*-module for $G_u = L_2(17)$.*

Lemma 52 *One of the following holds:*

(i) *We have $t = 0$ (that is, V is 8-dimensional), $Z(G) = 1$, the G_u-module V is irreducible and Γ is flat.*

(ii) *We have $t = 1$ (hence V is 9-dimensional), $Z(G)$ is a 1-dimensional G_u-submodule of V and $V/Z(G)$ is irreducible; furthermore, $V = Z(G) \times V_p$ for any point p of Γ (but V is an indecomposable module for G_u).*

Proof. The module V is at most 12-dimensional, by Corollary 51. Therefore, and by [17], a G_u-invariant section of V is either trivial or isomorphic to one of the two irreducible 8-dimensional modules called V^+ and V^- in 2.7. By Lemma 49, $V = V_pV_{p'}$ with p and p' distinct collinear points. Hence V is indecomposable with central submodule $Z = Z(G)$, we have $V = V_p \times Z$ and V/Z is an 8-dimensional irreducible module for G_u. (We keep the multiplicative notation for V here, thus writing $V_p \times Z$ instead of $V_p \oplus Z$.)

By (5.4), given a chamber (p, l, u) of Γ, we have $G_{p,l,u} = \langle a \rangle$ with $a^8 = 1$. Let p' be the point of l other than p. As $V = V_p \times Z = V_{p'} \times Z$, we have $[V, a] = [V_p, a] = [V_{p'}, a]$. Hence $[V, a] \leq V_p \cap V_{p'}$. However, $|C_{V_p}(a)| = 2$ by (5.2). Hence $[V_p, a] \cong V_p/C_{V_p}(a)$ is 7-dimensional. Therefore $V_p \cap V_{p'}$ is at least 7-dimensional. Consequently, $V = V_pV_{p'}$ is at most 9-dimensional. That is, either $dim(V) = 8$ and $Z = 1$ or $dim(V) = 9$ and $Z = Z_2$.

If $dim(V) = 8$ (that is, $t = 0$), then Γ has 2^8 planes, whence it is flat. We are in case (i). When $dim(V) = 9$ we have case (ii). q.e.d.

Corollary 53 *There are at most* 2^9 *planes in* Γ.

(Easy, by the previous lemma and Lemma 45.)

7.3. End of the proof

By Lemma 52, either $Z(G) = 1$ and Γ is flat or $V = V_p \times Z(G)$ with $Z(G) = Z_2$.

Proposition 54 *If* $Z(G) = 1$, *then* $\Gamma = \Gamma(16)/2$.

Proof. Let $Z(G) = 1$. By Lemma 52(i), V is an irreducible 8-dimensional module for $G_u = L_2(17)$. This description uniquely determines G (see 2.7).
The planes of Γ bijectively correspond to the vectors of V and G_u acts on the eighteen points of Γ as $L = L_2(17)$ does on $PG(1, 17)$. The module V is in fact a section of the permutation module of L. We have $G_p = V : G_{p,u}$, $G_{p,u} = Z_{17}Z_8$ and $G_{p,l,u} = Z_8$. Let a be a generator of $G_{p,l,u}$ and let c be an involution of $G_{l,u}$ permuting the two points of l. Then $G_l = [W, a^4]\langle a \rangle \langle c \rangle$ by Corollary 48. By comparing 2.7, one can easily see that $\Gamma \cong \Gamma(16)/2$. q.e.d.

Proposition 55 *If* $Z(G) = Z_2$, *then* $\Gamma = \Gamma(16)$.

Proof. Let $Z = Z(G) = Z_2$. We have $Z \cap G_u = 1$ as V is regular on the set of planes. Furthermore, $Z \cap G_p = 1$ as $V = V_p \times Z$. Also, Z stabilizes all classes of $\mathcal{X}$, since V is just the elementwise stabilizer of $\mathcal{X}$. This and the fact that $Z \cap G_p = 1$ imply that $Z \cap G_l = 1$ for any line l. Thus, Z defines a quotient of Γ. Clearly, Γ/Z is flat and G/Z is flag-transitive on Γ/Z. Therefore $\Gamma/Z \cong \Gamma(16)/2$ by Proposition 57.

On the other hand, $\Gamma(16)$ is the universal cover of $\Gamma(16)/2$. (Indeed, a proper cover of $\Gamma(16)$ would have more than 2^9 planes, contrary to Corollary 53.) Therefore Γ is a quotient of $\Gamma(16)$. However, Γ has as many planes as $\Gamma(16)$. Hence $\Gamma = \Gamma(16)$. q.e.d.

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