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# Representations of the Generalized Symmetric Groups

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## Introduction

The main aim of this paper is to construct a full set of irreducible, inequivalent representations of the generalised symmetric groups $G(m,1,n)$ in terms of $m$-sets of partitions of $n$ and combinatorial concepts connected with generalised Young tableaux, etc. As a matter of fact, the irreducible representations of the generalised symmetric groups $G(m,1,n)$ were studied by Osima [5], Puttaswamaiah [7], Read [9] and Hughes [2], respectively. However, there are well-known constructions of irreducible representations and of irreducible modules, called Specht modules, for the symmetric groups $\mathcal{S}_n$ which are based on elegant combinatorial concepts connected with Young tableaux, etc. (see, e.g. [3]). Therefore, in this paper we show how the Specht approach to the irreducible representations of the symmetric groups can be extended to deal with the generalised symmetric groups $G(m,1,n)$. As a convention, throughout this paper, we assume that $\xi$ is a primitive $m$-th root of unity.

## 1. Diagrams, Tableaux and Tabloids

The *generalised symmetric group* $G(m,1,n)$ is the group of all permutations $\sigma$ of $\{\xi^k i \mid i = 1, \ldots, n;\ k = 1, \ldots, m\}$ such that $\sigma(\xi^k i) = \xi^k\sigma(i)$ for $i = 1, \ldots, n;\ k = 1, \ldots, m$. The order of $G(m,1,n)$ is $m^n n!$

Let $\mathcal{S}_n$ be the symmetric group on $n$ symbols and $C_m$ be the cyclic group of order $m$. Then $G(m,1,n)$ is the semi-direct product of $C_m^n$ with $\mathcal{S}_n$, written as $C_m^n \times \mathcal{S}_n$, where $C_m^n$ is the direct product of $n$ copies of $C_m$. The group $G(m,1,n)$ will sometimes be denoted by $W_n^m$. The group $G(m,1,n)$ has the following presentation (see [1]):

$$\begin{aligned} G(m,1,n) \;&=\; \langle r_1, \ \ldots, \ r_{n-1}, \ w_1, \ \ldots, \ w_n \mid r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = e, \ |i-j| \geq 2, \\ &\qquad w_i^m = e, \ w_i w_j = w_j w_i, \ r_i w_i = w_{i+1} r_i, \ r_i w_j = w_j r_i, \ j \neq i, i+1 \rangle. \end{aligned}$$

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We may identify $r_i$ $(i = 1, \ldots, n-1)$ with the transposition $(i, i+1)$ and therefore the group generated by $\{r_1, \ldots, r_{n-1}\}$ is the symmetric group $\mathcal{S}_n$. The $w_i$ $(i = 1, \ldots, n)$ may be identified with the mapping $\begin{pmatrix} i \\ \xi i \end{pmatrix}$ : $\{1, \ldots, n\} \to \mathbf{C}\backslash\{0\}$ defined by $i \to \xi i$ and $j \to j$, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$.

The above relations imply that if $\tau \in \mathcal{S}_n$, then $\tau w_i \tau^{-1} = w_{\tau(i)}$, $i = 1, \ldots, n$. If $C_m^n$ is the subgroup of $G(m,1,n)$ generated by $\{w_1, \ldots, w_n\}$, then $C_m^n \lhd G(m,1,n)$ and $G(m,1,n)$ permutes the letters $\{1, \ldots, n\}$ as well as multiplying any number of them by some power of $\xi$. Furthermore, for the group $G(m,1,n)$ if we take $m = 1$ and 2 then we have the symmetric group $\mathcal{S}_n$ and the hyperoctahedral group $O_n$ respectively (see [9]).

If $\tau = \begin{pmatrix} 1 & 2 & \ldots & n \\ \tau(1) & \tau(2) & \ldots & \tau(n) \end{pmatrix} \in \mathcal{S}_n$, let

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \xi^{s_1}\tau(1) & \xi^{s_2}\tau(2) & \ldots & \xi^{s_n}\tau(n) \end{pmatrix} = \tau \prod_{i=1}^{n} w_i^{s_i} \in G(m,1,n),$$

where $1 \le s_i \le m$. Define $\varphi :\ G(m,1,n) \to \mathcal{S}_n$ by $\varphi(\sigma) = \tau$. Then $\varphi$ is an epimorphism and $\ker\varphi = C_m^n$. Define the *sign* of $\sigma \in G(m,1,n)$ to be $sgn(\sigma) = sgn(\varphi(\sigma))$.

**Definition 1.1** *An element $\sigma \in G(m,1,n)$ is even if $\varphi(\sigma)$ is an even element of $\mathcal{S}_n$ and odd otherwise. In terms of the generators $r_i$, $w_i$ given above, $\sigma$ is even if and only if the number of $r_i$ appearing in any expression for $\sigma$ is even (see [8]).*

Any element $\sigma \in G(m,1,n)$ may be written uniquely (up to reordering) as the product of disjoint cycles $\sigma = \theta_1 \ldots \theta_t$, where

$$\theta_i = \begin{pmatrix} b_{i1} & b_{i2} & \ldots & b_{ik_i} \\ \xi^{s_{i1}}b_{i2} & \xi^{s_{i2}}b_{i3} & \ldots & \xi^{s_{ik_i}}b_{i1} \end{pmatrix},$$

$b_{ij} \in \{1, \ldots, n\}$, $s_{ij} \in \{1, \ldots, m\}$, $k_i$ is the length of the cycle $\theta_i$, $i = 1, \ldots, t$.

Let $f(\theta_i) = \sum_{j=1}^{k_i} s_{ij}$, and put $f(\sigma) = \sum_{i=1}^{t} f(\theta_i)$. Define $a_{pq}(\sigma)$ to be the number of cycles $\theta_i$ of $\sigma$ of length $q$ such that $f(\theta_i) \equiv p\ (mod\ m)$ for $1 \le p \le m$, $1 \le q \le n$. The $m \times n$ matrix $(a_{pq}(\sigma))$ is called the *type* of $\sigma$, denoted by $Ty(\sigma)$ (see [9]). Then it is well known that $\sigma$, $\pi \in G(m,1,n)$ are conjugate in $G(m,1,n)$ if and only if $Ty(\sigma) = Ty(\pi)$ (see [4]). Let

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \xi^{r_1}\gamma(1) & \xi^{r_2}\gamma(2) & \ldots & \xi^{r_n}\gamma(n) \end{pmatrix} = \gamma \prod_{i=1}^{n} w_i^{r_i} \in G(m,1,n)$$

where $\gamma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \gamma(1) & \gamma(2) & \ldots & \gamma(n) \end{pmatrix} \in \mathcal{S}_n$, $1 \le r_i \le m$. Then we have

$$\begin{aligned} \sigma\pi &= \begin{pmatrix} 1 & 2 & \ldots & n \\ \xi^{s_1}\tau(1) & \xi^{s_2}\tau(2) & \ldots & \xi^{s_n}\tau(n) \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & n \\ \xi^{r_1}\gamma(1) & \xi^{r_2}\gamma(2) & \ldots & \xi^{r_n}\gamma(n) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \ldots & n \\ \xi^{(r_1+s_{\gamma(1)})}\tau\gamma(1) & \xi^{(r_2+s_{\gamma(2)})}\tau\gamma(2) & \ldots & \xi^{(r_n+s_{\gamma(n)})}\tau\gamma(n) \end{pmatrix} \\ &= \tau\gamma \prod_{i=1}^{n} w_i^{(r_i+s_{\gamma(i)})}. \end{aligned}$$

Hence, $f(\sigma\pi) \equiv f(\sigma) + f(\pi)$ $(mod\ m)$ for any $\sigma,\ \pi \in G(m,1,n)$. Furthermore, we assume that $f(\tau) = 0$ for all $\tau \in \mathcal{S}_n$. Thus, $f(\sigma^{-1}) = mk - f(\sigma)$ for any $\sigma \in G(m,1,n)$, where $k \in \mathbf{Z}$.

**Definition 1.2** *An m-set of partitions of n, $(\lambda^{(1)},\ \ldots,\ \lambda^{(m)})$, consists of m partitions $\lambda^{(1)},\ \ldots,\ \lambda^{(m)}$ such that $\lambda^{(i)} = (\lambda_1^{(i)},\ \ldots,\ \lambda_{s_i}^{(i)})$ is a partition of $l_i$, where* $\sum_{j=1}^{s_i} \lambda_j^{(i)} = l_i$ and $\sum_{i=1}^{m} l_i = n$.

For any $i$, if $l_i = 0$ then it is still necessary to write $\lambda^{(i)} = 0$. Let $(\lambda^{(1)},\ \ldots,\ \lambda^{(m)})$ be an $m$-set of partitions of $n$, then $(\lambda^{(1)},\ \ldots,\ \lambda^{(m)})$ will be abbreviated to $\lambda^{[m]}$. In the $\lambda^{[m]}$, the part corresponding to $\lambda^{(i)}$ $(1 \leq i \leq m)$ will be called the *i-th constituent* of the $\lambda^{[m]}$. Let $\mathcal{P}^{(m)}(n)$ denote the set of all $m$-sets of partitions of $n$. Saeed-ul-Islam ([11]) has proved that the number of conjugacy classes in $G(m,1,n)$ is given by

$$|\mathcal{P}^{(m)}(n)| = \sum p(l_1)\ldots p(l_m),$$

where the summation is taken over all $m$-tuples $\{l_1,\ \ldots,\ l_m\}$ such that $\sum_{i=1}^{m} l_i = n$ and $p(l_i)$ denotes the number of partitions of $l_i,\ i = 1,\ \ldots,\ m$.

Furthermore, Osima ([5]) gives that the number of ordinary irreducible representations of $G(m,1,n)$ is $|\mathcal{P}^{(m)}(n)|$.

The group $G(m,1,n)$ has $2m$ one dimensional representations $1,\ \sigma_1,\ \sigma_2,\ \ldots,\ \sigma_{m-1},\ \eta,\ \epsilon_1,\ \ldots,\ \epsilon_{m-1}$ defined by

(a) $1(g) = 1$ for all $g \in G(m,1,n)$,
(b) $\sigma_k(r_i) = 1$ $(i = 1,\ \ldots,\ n-1)$, $\sigma_k(w_j) = \xi^k$ $(j = 1,\ \ldots,\ n)$,
(c) $\eta(r_i) = -1$ $(i = 1,\ \ldots,\ n-1)$, $\eta(w_j) = 1$ $(j = 1,\ \ldots,\ n)$,
(d) $\epsilon_k(r_i) = -1$ $(i = 1,\ \ldots,\ n-1)$, $\epsilon_k(w_j) = \xi^k$ $(j = 1,\ \ldots,\ n)$,

where $k = 1,\ \ldots,\ m-1$ and $\xi$ is a primitive $m$-th root of unity. Note that $1, \eta$ are real representations (called the *trivial* and *sign* representations respectively) and if $m$ is even then $\sigma_{\frac{m}{2}},\ \epsilon_{\frac{m}{2}}$ are also real representations.

**Definition 1.3** *Let $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$. The generalized Young diagram of $\lambda^{[m]}$ is $[\lambda^{[m]}] = ([\lambda^{(1)}],\ \ldots,\ [\lambda^{(m)}])$, where $[\lambda^{(i)}]$ is the usual Young diagram of $\lambda^{(i)},\ i = 1,\ \ldots,\ m$. For any i, if $\lambda^{(i)} = 0$ then we write $[\lambda^{(i)}] = \emptyset$.*

For example if $\lambda^{[4]} = (3^2 1, 21, 0, 2^2 1) \in \mathcal{P}^{(4)}(15)$, then we shall draw the generalized Young diagram of $\lambda^{[4]}$ as in the following:

$$[\lambda^{[4]}] = \left( \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \end{array}, \begin{array}{cc} \bullet & \bullet \\ \bullet & \\ & \end{array}, \emptyset, \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array} \right)$$

**Definition 1.4** *Let $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$. A $\lambda^{[m]}$-tableau t is an array of the symbols $\{\xi^k i \mid i = 1,\ \ldots,\ n;\ k = 1,\ \ldots,\ m\}$ obtained by replacing each node in $[\lambda^{[m]}]$ by one of the symbols $\{\xi^k i \mid i = 1,\ \ldots,\ n;\ k = 1,\ \ldots,\ m\}$ with the moduli of all the numbers appearing being distinct.*

A $\lambda^{[m]}$-tableau $t$ will sometimes be written $(t_{\lambda^{(1)}}, \ldots, t_{\lambda^{(m)}})$. Clearly there are $m^n n!$ $\lambda^{[m]}$-tableaux for any generalized Young diagram $[\lambda^{[m]}]$.

Let $t(i,j,k) = t_{\lambda^{(k)}}(i,j)$, where $t_{\lambda^{(k)}}(i,j)$ stands for the entry of $t_{\lambda^{(k)}}$ in position $(i,j)$, as in a matrix, where $k = 1, \ldots, m$. Thus, $t(i,j,k)$ stands for the entry of $t$ in position $(i,j,k)$ where $k = 1, \ldots, m$. In the tableau $t$, the part corresponding to $t_{\lambda^{(k)}}$ $(1 \le k \le m)$ will be called the *k-th constituent* of the tableau $t$.

The group $G(m,1,n)$ acts on a $\lambda^{[m]}$-tableau $t = (t(i,j,k))$ as follows: $\sigma t = (\sigma t(i,j,k)) = (\sigma t_{\lambda^{(1)}}(i,j), \ldots, \sigma t_{\lambda^{(m)}}(i,j))$ for all $\sigma \in G(m,1,n)$.

For example, let $\lambda^{[3]} = (21, 0, 2^2) \in \mathcal{P}^{(3)}(7)$. Then

$$t = \left( \begin{array}{cc} 2 & \xi 7 \\ \xi^2 1 & \end{array}, \emptyset, \begin{array}{cc} \xi^2 5 & 6 \\ 3 & \xi^2 4 \end{array} \right)$$

is a $\lambda^{[3]}$-tableau. If

$$\sigma = \left( \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \xi 1 & 3 & \xi^2 4 & 7 & 2 & 5 & \xi^2 6 \end{array} \right) \in G(3,1,7)$$

then

$$\sigma t = \sigma \left( \begin{array}{cc} 2 & \xi 7 \\ \xi^2 1 & \end{array}, \emptyset, \begin{array}{cc} \xi^2 5 & 6 \\ 3 & \xi^2 4 \end{array} \right) = \left( \begin{array}{cc} 3 & 6 \\ 1 & \end{array}, \emptyset, \begin{array}{cc} \xi^2 2 & 5 \\ \xi^2 4 & \xi^2 7 \end{array} \right)$$

**Definition 1.5** *Let $t$ be a $\lambda^{[m]}$-tableau. A row permutation of $t$ is an element $\sigma$ of $W_n^m$ such that $\sigma$ permutes the symbols in each row of $t$ and multiplies any number of symbols in $t_{\lambda^{(k)}}$ $(k = 2, \ldots, m)$ by a power of $\xi$. A column permutation of $t$ is an element $\sigma$ of $W_n^m$ such that $\sigma$ permutes the symbols in each column of $t$ and multiplies any number of symbols in $t_{\lambda^{(1)}}$ by a power of $\xi$.*

We can now define the *row stabilizer* $R_t$ and *column stabilizer* $C_t$ of $t$ as follows. Let $R_t$ be the group of row permutations of $t$. Thus, $R_t$ is isomorphic to

$$\prod_{i=1}^{m} \left( W^{m_i}_{\lambda_1^{(i)}} \times \ldots \times W^{m_i}_{\lambda_{s_i}^{(i)}} \right),$$

where

$$m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2, \ \ldots, \ m \end{cases}$$

and is a subgroup of $W_n^m$.

Let $C_t$ be the group of column permutations of $t$. Thus, $C_t$ is isomorphic to

$$\prod_{i=1}^{m} \left( W^{m_i}_{\mu_1^{(i)}} \times \ldots \times W^{m_i}_{\mu_{r_i}^{(i)}} \right),$$

where

$$m_i = \begin{cases} m & \text{if } i = 1 \\ 1 & \text{if } i = 2, \ \ldots, \ m \end{cases},$$

$(\mu_1^{(i)}, \ldots, \mu_{r_i}^{(i)})$ is the partition conjugate to $(\lambda_1^{(i)}, \ldots, \lambda_{s_i}^{(i)})$, and $C_t$ is a subgroup of $W_n^m$.

Now, we define an equivalence relation $\sim$ on the set of $\lambda^{[m]}$-tableaux as follows.

**Definition 1.6** *Two $\lambda^{[m]}$-tableaux $t_1$ and $t_2$ are row equivalent, written $t_1 \sim t_2$, if there exists $\sigma \in R_{t_1}$ such that $t_2 = \sigma t_1$. The equivalence class which contains the $\lambda^{[m]}$-tableau $t$ is $\{t\}$ and is called a $\lambda^{[m]}$- tabloid.*

If $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$ with notation as in Definition 1.2, then the number of $\lambda^{[m]}$-tabloids is

$$m^{l_1} \frac{n!}{\prod_{i=1}^{m} \left( \prod_{j=1}^{s_i} \lambda_j^{(i)}! \right)}.$$

Since $\{t\} = \{t' \mid t' = \sigma t \text{ for some } \sigma \in R_t\}$, the number of $\lambda^{[m]}$-tableaux in any given equivalence class is

$$|R_t| = \left( \prod_{i=1}^{s_1} \lambda_i^{(1)}! \right) \left( \prod_{i=2}^{m} \prod_{j=1}^{s_i} m^{\lambda_j^{(i)}} \lambda_j^{(i)}! \right).$$

Furthermore, it can be verified that if $\sigma \in W_n^m$ then $R_{\sigma t} = \sigma R_t \sigma^{-1}$ and $C_{\sigma t} = \sigma C_t \sigma^{-1}$.

**Definition 1.7** *If $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$, let $M^{\lambda^{[m]}}$ be the vector space over the complex field* **C** *whose basis elements are the $\lambda^{[m]}$-tabloids.*

Now, we define the action of $W_n^m$ on the vector space $M^{\lambda^{[m]}}$ as follows. Let $\tau \in \mathcal{S}_n$ and $1 \leq s_i \leq m$. If $\sigma = \tau \prod_{i=1}^{n} w_i^{s_i} \in W_n^m$, then

$$\sigma\{t\} = \xi^{\phi(\sigma,t)}\{\sigma t\}$$

where $\phi(\sigma, t) = \sum_{i=1}^{n} k_i s_i$ if $i$ appears in the $k_i$-th constituent of the tableau $t$. Note that if $\sigma \in \mathcal{S}_n$ then since $\phi(\sigma, t) \equiv 0 \ (mod\ m)$ we have $\sigma\{t\} = \{\sigma t\}$. Now we show that this is a group action on the vector space $M^{\lambda^{[m]}}$. Let

$$\pi = \gamma \prod_{i=1}^{n} w_i^{r_i} \in W_n^m,$$

where $\gamma \in \mathcal{S}_n$ and $1 \leq r_i \leq m$. Then we have

$$\begin{aligned} \sigma\pi &= \tau\gamma \prod_{i=1}^{n} w_i^{(r_i + s_{\gamma(i)})} \text{ and so} \\ (\sigma\pi)\{t\} &= \xi^{\phi(\sigma\pi,t)}\{\sigma\pi t\}, \end{aligned}$$

where $\phi(\sigma\pi, t) = \sum_{i=1}^{n} k_i(r_i + s_{\gamma(i)})$ if $i$ appears in the $k_i$-th constituent of the tableau $t$. (Note that at this point, we have not used the relation $\xi^m = 1$). For $\pi = \gamma \prod_{i=1}^{n} w_i^{r_i} \in W_n^m$ if $i$ appears in the $k_i$-th constituent of the tableau $t$ then $\gamma(i)$ appears in the same $k_i$-th constituent of the tableau $\pi t$.

Let $\gamma(i) = j$ $(1 \leq i, j \leq n)$. Then for $\sigma = \tau \prod_{j=1}^{n} w_j^{s_j} \in W_n^m$, $w_j^{s_j} = w_{\gamma(i)}^{s_{\gamma(i)}}$ and so

$$\sigma(\pi\{t\}) = \sigma\left(\xi^{\phi(\pi,t)}\{\pi t\}\right) = \xi^{\phi(\pi,t)}\xi^{\phi(\sigma,\pi t)}\{\sigma\pi t\},$$

where $\phi(\sigma,\pi t)=\sum_{i=1}^{n} k_i s_{\gamma(i)}$ and $\gamma(i)$ appears in the $k_i$-th constituent of the tableau $\pi t$. Thus,

$$\begin{aligned}\phi(\sigma\pi,t) &= \phi(\pi,t)+\phi(\sigma,\pi t) \text{ and so}\\ (\sigma\pi)\{t\} &= \sigma(\pi\{t\}).\end{aligned}$$

This action is well-defined, for if $\{t_1\}=\{t_2\}$ then there exists $\sigma\in R_{t_1}$ such that $t_2=\sigma t_1$. Then for any $\pi\in W_n^m$, $\pi\sigma\pi^{-1}\in\pi R_{t_1}\pi^{-1}=R_{\pi t_1}$, so

$$\xi^{\phi(\pi,t_1)}\{\pi t_1\}=\xi^{\phi(\pi,\sigma t_1)}\{\pi\sigma t_1\}=\xi^{\phi(\pi,t_2)}\{\pi t_2\}.$$

Extending this action to be linear on $M^{\lambda^{[m]}}$ turns $M^{\lambda^{[m]}}$ into a $\mathbf{C}W_n^m$-*module*, and so we have the following theorem.

**Theorem 1.8** *The $\mathbf{C}W_n^m$-module $M^{\lambda^{[m]}}$ is isomorphic to the permutation module of $W_n^m$ on the subgroup $R_t$. $M^{\lambda^{[m]}}$ is a cyclic $\mathbf{C}W_n^m$-module generated by any one $\lambda^{[m]}$-tabloid and*

$$dim_{\mathbf{C}}M^{\lambda^{[m]}}=m^{l_1}\frac{n!}{\prod_{i=1}^{m}\left(\prod_{j=1}^{s_i}\lambda_j^{(i)}!\right)}$$

## 2. Dominance Partial Ordering

Now we impose the following partial $\trianglerighteq$ order on $\mathcal{P}^{(m)}(n)$, which we call the *generalized dominance partial order.*

**Definition 2.1** *Let $\lambda^{[m]}$, $\mu^{[m]}\in\mathcal{P}^{(m)}(n)$. Then $\lambda^{[m]}$ dominates $\mu^{[m]}$, written $\lambda^{[m]}\trianglerighteq\mu^{[m]}$, if one of the following conditions is satisfied:*

*(i)* $|\lambda^{(1)}|>|\mu^{(1)}|$,

*(ii)* $|\lambda^{(j)}|=|\mu^{(j)}|$ $(1\le j\le k)$ *and* $|\lambda^{(k+1)}|>|\mu^{(k+1)}|$ *for some* $k\in\{1,\ 2,\ \ldots,\ m-2\}$,

*(iii)* $|\lambda^{(i)}|=|\mu^{(i)}|$ *and* $\lambda^{(i)}\trianglerighteq\mu^{(i)}$ *for all* $i=1,\ \ldots,\ m$.

**Lemma 2.2** *Let $t$ be a $\lambda^{[m]}$-tableau and $t'$ be a $\mu^{[m]}$-tableau such that for all $a$, if $a$ occurs in $t_{\lambda^{(i)}}$ then $\xi^{s_i}a$ occurs in $t'_{\mu^{(i)}}$, where $i\in\{1,\ \ldots,\ m-1\}$ and $s_i\in\{1,\ \ldots,\ m\}$. Suppose that $a,\ b$ belong to the same row of $t'_{\mu^{(i)}}$ implies that $c,\ d$ belong to different columns of $t_{\lambda^{(i)}}$ where $c=\xi^{\rho_1}a,\ d=\xi^{\rho_2}b,\ (1\le\rho_1,\ \rho_2\le m)$, and $i\in\{1,\ \ldots,\ m\}$. Then $\lambda^{[m]}\trianglerighteq\mu^{[m]}$.*

*Proof.* By hypothesis, we have that $|\lambda^{(i)}|=|\mu^{(i)}|$ for all $i=1,\ \ldots,\ m$. Thus, by using a similar argument to that in the proof of the corresponding lemma (Lemma 2.2.4 in [10] for the symmetric groups) the result can be deduced. q.e.d.

**Corollary 2.3** *Suppose $|\lambda^{(i)}|=|\mu^{(i)}|$ for all $i=1,\ \ldots,\ m-1$ and $\lambda^{[m]}\not\trianglerighteq\mu^{[m]}$. Let $t$ be a $\lambda^{[m]}$-tableau and $t'$ be a $\mu^{[m]}$-tableau such that for all $a$, if $a$ occurs in $t_{\lambda^{(i)}}$ then $\xi^{s_i}a$ occurs in $t'_{\mu^{(i)}}$, where $i\in\{1,\ \ldots,\ m-1\}$ and $s_i\in\{1,\ \ldots,\ m\}$. Then there exist $a,\ b$ in the same row of $t'_{\mu^{(i)}}$ and $c,\ d$ in the same column of $t_{\lambda^{(i)}}$, where $c=\xi^{\rho_1}a,\ d=\xi^{\rho_2}b,\ (1\le\rho_1,\ \rho_2\le m)$, and $i\in\{1,\ \ldots,\ m\}$.*

## 3. Specht Modules

We now construct a complete set of irreducible modules of $G(m,1,n)$ over the complex field $\mathbf{C}$; these are the so-called Specht modules, $S^{\lambda^{[m]}}$.

**Definition 3.1** *Let $t$ be a $\lambda^{[m]}$-tableau. Define $\kappa_t \in \mathbf{C}W_n^m$ by*

$$\kappa_t = \sum_{\sigma \in C_t} \xi^{-f(\sigma)}(sgn\ \sigma)\sigma,$$

*where $f(\sigma)$ is defined as in section 1. The $\lambda^{[m]}$-polytabloid $e_t$ associated with the tableau $t$ is given by $e_t = \kappa_t\{t\}$.*

**Lemma 3.2** *Let $t$ be a $\lambda^{[m]}$-tableau and $\pi \in W_n^m$. Then*
*(i) $\kappa_{\pi t} = \pi\kappa_t\pi^{-1}$*
*(ii) $\pi e_t = \xi^{\phi(\pi,t)}e_{\pi t}$,*
*(iii) if $\pi \in C_t$ then $\pi e_t = \xi^{f(\pi)}(sgn\ \pi)e_t$.*

*Proof.* (i) If $\pi \in W_n^m$ then

$$\begin{aligned} \kappa_{\pi t} &= \sum_{\sigma \in C_{\pi t}} \xi^{-f(\sigma)}(sgn\ \sigma)\sigma = \sum_{\sigma \in \pi C_t \pi^{-1}} \xi^{-f(\sigma)}(sgn\ \sigma)\sigma \\ &= \sum_{\tau \in C_t} \xi^{-f(\pi\tau\pi^{-1})} sgn(\pi\tau\pi^{-1})\pi\tau\pi^{-1} \\ &= \pi\kappa_t\pi^{-1}. \end{aligned}$$

(ii) $\pi e_t = \pi\kappa\{t\} = \kappa_{\pi t}\pi\{t\} = \xi^{\phi(\pi,t)}\kappa_{\pi t}\{\pi t\} = \xi^{\phi(\pi,t)}e_{\pi t}$.
(iii) If $\pi \in C_t$ then

$$\begin{aligned} \pi e_t &= \sum_{\sigma \in C_t} \xi^{-f(\sigma)}(sgn\ \sigma)\pi\sigma\{t\} = \sum_{\tau \in C_t} \xi^{-f(\pi^{-1}\tau)} sgn(\pi^{-1}\tau)\tau\{t\} \\ &= \xi^{f(\pi)}(sgn\ \pi)e_t. \end{aligned}$$

q.e.d.

We are now in a position to define the Specht modules.

**Definition 3.3** *Let $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$. The Specht module $S^{\lambda^{[m]}}$ for the $m$-set of partitions $\lambda^{[m]}$ is the submodule of $M^{\lambda^{[m]}}$ spanned by $\lambda^{[m]}$-polytabloids.*

Because of part (ii) of Lemma 3.2, we have the following.

**Theorem 3.4** *$S^{\lambda^{[m]}}$ is a cyclic $\mathbf{C}W_n^m$-module generated by any one polytabloid.*

If $\nu = \sum_i \alpha_i\{t_i\} \in M^{\lambda^{[m]}}$, where $\alpha_i \in \mathbf{C}$, then we say that $\{t_i\}$ appears in $\nu$ if $\alpha_i \neq 0$.

**Remark 3.5** The action of $G(m,1,n)$ on $M^{\lambda^{[m]}}$ has already been defined in section 1. Now, we explain that this is a natural action because of the following reason. For the group $G(m,1,n)$ if we take $n = 1$ then

$$G(m,1,1) = C_m^1 = \langle w_1 \mid w_1^m = e\rangle$$

and $\mathcal{P}^{(m)}(1)$ consists of precisely $m$ distinct $m$-sets of partitions of 1, i.e.

$$\mathcal{P}^{(m)}(1) = \{\lambda_i^{[m]}\ (i = 1,\ \ldots,\ m) \mid 1 \text{ appears in the } i-\text{th constituent of } \lambda_i^{[m]}\}.$$

For $\lambda_1^{[m]} = (1,\ 0,\ \ldots,\ 0)$ we have

$$\begin{aligned} M^{\lambda_1^{[m]}} &= \langle \{\xi^k 1,\ \emptyset,\ \ldots,\ \emptyset\} \mid k = 1,\ \ldots,\ m\rangle \text{ and} \\ S^{\lambda_1^{[m]}} &= \langle e_t \mid e_t = \sum_{k=1}^{m} \{\xi^k 1,\ \emptyset,\ \ldots,\ \emptyset\}\rangle. \end{aligned}$$

For $i = 2,\ 3,\ \ldots,\ m$,

$$M^{\lambda_i^{[m]}} = S^{\lambda_i^{[m]}} = \langle \{t\} \mid 1 \text{ appears in the } i-\text{th constituent of } t\rangle.$$

Now define $X_i : G(m,1,1) \to GL(1, \mathbf{C})$ by $X_i(w_1^j) = \xi^{ij}$ for $i,\ j = 1,\ \ldots,\ m$. Then $X_1,\ X_2,\ \ldots,\ X_m = 1$ are $m$ distinct irreducible matrix representations of $G(m,1,1)$ having degree 1. Let $Y_i$ be the matrix representation of $G(m,1,1)$ afforded by $S^{\lambda_i^{[m]}}$ for $i = 1,\ \ldots,\ m$. Then $X_i$ and $Y_i$ are equivalent irreducible representations of $G(m,1,1)$ for $i = 1,\ \ldots,\ m$.

The following lemma is a simple generalization of the corresponding result (Lemma 1.4 in [6]) for the symmetric groups.

**Lemma 3.6** *Let $t$ and $t'$ be $\lambda^{[m]}$-tableaux. The following conditions are equivalent:*

*(i) $\{t'\}$ is contained in $e_t$,*

*(ii) there exist $\rho \in R_{t'}$, $\pi \in C_t$ such that $\rho t' = \pi t$,*

*(iii) $a,\ b$ belong to the same row of $t'_{\lambda^{(i)}}$ implies that $c,\ d$ belong to different columns of $t_{\lambda^{(i)}}$, where $c = \xi^{\rho_1} a,\ d = \xi^{\rho_2} b,\ (1 \le \rho_1,\ \rho_2 \le m)$, and $i \in \{1,\ \ldots,\ m\}$.*

**Lemma 3.7** *Let $t$ be a $\lambda^{[m]}$-tableau and $t'$ be a $\mu^{[m]}$-tableau. Suppose there exist $a,\ b$ in the same row of $t'_{\mu^{(i)}}$ with $c,\ d$ in the same column of $t_{\lambda^{(i)}}$, where $c = \xi^{\rho_1} a,\ d = \xi^{\rho_2} b,\ (1 \le \rho_1,\ \rho_2 \le m)$, and $i \in \{1,\ \ldots,\ m\}$. Then $C_t \cap R_{t'} \ne \emptyset$.*

*Proof.* Let $a = \xi^{s_1} k$ and $b = \xi^{s_2} l$, where $1 \le s_1,\ s_2 \le m,\ 1 \le k,\ l \le n$ and $k \ne l$. Then we may assume without loss of generality that $c = \xi^{\rho_1 + s_1} k$ and $d = \xi^{\rho_2 + s_2} l$, where $1 \le \rho_1,\ \rho_2 \le m$.

First of all, let $i = 1$. Then

$$(a,b) = (k,l) w_k^{m+s_2-s_1} w_l^{m+s_1-s_2} \in C_t \cap R_{t'}.$$

Now let $i \in \{2,\ 3,\ \ldots,\ m\}$. Then

$$(c,d) = (k,l) w_k^{m+(\rho_2+s_2)-(\rho_1+s_1)} w_l^{m+(\rho_1+s_1)-(\rho_2+s_2)} \in C_t \cap R_{t'}. \qquad \text{q.e.d.}$$

**Lemma 3.8** *Let $t$ be a $\lambda^{[m]}$-tableau and $t'$ be a $\mu^{[m]}$-tableau. If there exist $a,\ b$ in the same row of $t'_{\mu^{(i)}}$ such that $c,\ d$ are in the same column of $t_{\lambda^{(i)}}$, where $c = \xi^{\rho_1} a,\ d = \xi^{\rho_2} b,\ (1 \le \rho_1,\ \rho_2 \le m)$, and $i \in \{1,\ \ldots,\ m\}$. Then $\kappa_t \{t'\} = 0$.*

*Proof.* In the first place, let $i = 1$. Then by Lemma 3.7, $(a, b) \in C_t \cap R_{t'}$ and $(e-(a, b))\{t'\} = 0$, and furthermore $(a, b)$ generates a subgroup of order 2 of $C_t$. Select signed coset representatives $\sigma_1, \ \ldots, \ \sigma_r$ for this subgroup of $C_t$; then

$$\kappa_t\{t'\} = \left(\sum_{i=1}^{r} \xi^{-f(\sigma_i)}(sgn \ \sigma_i)\sigma_i\right)(e-(a, b))\{t'\} = 0.$$

Secondly, let $i \in \{2, \ 3, \ \ldots, \ m\}$. Then, by Lemma 3.7, $(c, d) \in C_t \cap R_{t'}$ and $(e-(c, d))\{t'\} = 0$, and so $(c, d)$ generates a subgroup of order 2 of $C_t$. Select once again signed coset representatives $\sigma_1, \ \ldots, \ \sigma_s$ for this subgroup of $C_t$; then

$$\kappa_t\{t'\} = \left(\sum_{i=1}^{s} \xi^{-f(\sigma_i)}(sgn \ \sigma_i)\sigma_i\right)(e-(c, d))\{t'\} = 0. \qquad \text{q.e.d.}$$

**Lemma 3.9** *Suppose* $|\lambda^{(i)}| = |\mu^{(i)}|$ *for all* $i = 1, \ \ldots, \ m-1$ *and* $\lambda^{[m]} \ntrianglerighteq \mu^{[m]}$. *Let* $t$ *be a* $\lambda^{[m]}$*-tableau and* $t'$ *be a* $\mu^{[m]}$*-tableau such that for all* $a$, *if* $a$ *occurs in* $t_{\lambda^{(i)}}$ *then* $\xi^{s_i}a$ *occurs in* $t'_{\mu^{(i)}}$, *where* $i \in \{1, \ \ldots, \ m-1\}$ *and* $s_i \in \{1, \ \ldots, \ m\}$. *Then* $\kappa_t\{t'\} = 0$.

*Proof.* The result follows from Corollary 2.3 and Lemma 3.8. q.e.d.

**Lemma 3.10** *Let* $t$ *be a* $\lambda^{[m]}$*-tableau and* $t'$ *be a* $\mu^{[m]}$*-tableau such that for all* $a$, *if* $a$ *occurs in* $t_{\lambda^{(i)}}$ *then* $\xi^{s_i}a$ *occurs in* $t'_{\mu^{(i)}}$, *where* $i \in \{1, \ \ldots, \ m-1\}$ *and* $s_i \in \{1, \ \ldots, \ m\}$. *Suppose that* $\kappa_t\{t'\} \neq 0$. *Then* $\lambda^{[m]} \trianglerighteq \mu^{[m]}$.

*Proof.* Suppose that $a$ and $b$ are two elements in the same row of $t'_{\mu^{(i)}}$, where $i \in \{1, \ \ldots, \ m\}$. Then $c$ and $d$ cannot be in the same column of $t_{\lambda^{(i)}}$, where $c = \xi^{\rho_1}a, \ d = \xi^{\rho_2}b, \ (1 \leq \rho_1, \ \rho_2 \leq m)$ and $i \in \{1, \ \ldots, \ m\}$, for if so, then by Lemma 3.8, $\kappa_t\{t'\} = 0$, contradicting our hypothesis. Thus, Lemma 2.2 yields $\lambda^{[m]} \trianglerighteq \mu^{[m]}$. q.e.d.

**Definition 3.11** *Let* $t$ *and* $t'$ *be* $\lambda^{[m]}$*-tableaux. We say that* $t$ *agrees with* $t'$ *if whenever* $a$ *occurs in* $t_{\lambda^{(i)}}$ *then* $\xi^{s_i}a$ *occurs in* $t'_{\lambda^{(i)}}$ *for some* $s_i \in \{1, \ \ldots, \ m\}$, *for all* $a$ *and* $i \in \{1, \ \ldots, \ m\}$.

With this definition, we can prove the following lemma and its corollary by using Lemma 3.6 and Lemma 3.8.

**Lemma 3.12** *Let* $t$ *and* $t'$ *be* $\lambda^{[m]}$*-tableaux such that* $t$ *agrees with* $t'$.
*(i) If* $\{t'\}$ *is not contained in* $e_t$ *then* $\kappa_t\{t'\} = 0$.
*(ii) If* $\{t'\}$ *is contained in* $e_t$ *then* $\kappa_t\{t'\} = \pm e_t$.

**Corollary 3.13** *Let* $t$ *be a* $\lambda^{[m]}$*-tableau and* $u \in M^{\lambda^{[m]}}$. *Suppose that* $u$ *is a linear combination of all tabloids* $\{t'\}$ *such that* $t$ *agrees with each* $t'$. *Then* $\kappa_t u$ *is a multiple of* $e_t$.

**Remark 3.14** Let $t$ and $t'$ be $\lambda^{[m]}$-tableaux such that $t$ agrees with $t'$. Since $e_{t'}$ is a linear combination of all tabloids such that $t$ agrees with all of them then by Corollary 3.13 we have $\kappa_t e_{t'} = \beta e_t$, where $\beta \in \mathbf{C}$.

**Lemma 3.15** *Let* $t$ *be a* $\lambda^{[m]}$*-tableau. Then* *(i)* $\kappa_t\kappa_t = |C_t|\kappa_t$, *(ii)* $\kappa_t e_t = |C_t|e_t$.

*Proof.* We know that $\kappa_t = \sum_{\sigma \in C_t} \xi^{-f(\sigma)}(sgn\ \sigma)\sigma$. Then

$$\begin{aligned}
\kappa_t \kappa_t &= \left(\sum_{\sigma \in C_t} \xi^{-f(\sigma)}(sgn\ \sigma)\sigma\right)\left(\sum_{\pi \in C_t} \xi^{-f(\pi)}(sgn\ \pi)\pi\right) \\
&= \sum_{\sigma,\pi \in C_t} \xi^{-f(\sigma)}(sgn\ \sigma)\xi^{-f(\pi)}(sgn\ \pi)\sigma\pi \\
&= \sum_{\sigma,\gamma \in C_t} \xi^{-f(\sigma)}(sgn\ \sigma)\xi^{-f(\sigma^{-1}\gamma)} sgn(\sigma^{-1}\gamma)\gamma, \quad \text{(where } \sigma\pi = \gamma) \\
&= \sum_{\sigma,\gamma \in C_t} \xi^{-f(\gamma)}(sgn\ \gamma)\gamma \\
&= \sum_{\sigma \in C_t} \sum_{\gamma \in C_t} \xi^{-f(\gamma)}(sgn\ \gamma)\gamma \\
&= |C_t|\kappa_t,
\end{aligned}$$

thus proving (i), and (ii) follows from (i). q.e.d.

**Lemma 3.16** *Let $t$ be a $\lambda^{[m]}$-tableau and $w_i \in W_n^m$, $1 \le i \le n$. The following conditions are equivalent:*

*(i) $i$ occurs in the $k$-th constituent of $t$,*

*(ii) $i$ occurs in the $k$-th constituent of every tabloid appearing in $e_t$,*

*(iii) $w_i e_t = \xi^k e_t$.*

*Proof.* By definition of the column stabilizer $C_t$ of $t$, the equivalence of (i) and (ii) follows directly from the formula

$$e_t = \sum_{\sigma \in C_t} \xi^{-f(\sigma)}(sgn\ \sigma)\sigma\{t\}.$$

(ii)⇒(iii): Assume (ii) so that $i$ occurs in the $k$-th constituent of every tabloid appearing in $e_t$. Then by definition of the action of $W_n^m$ on $M^{\lambda^{[m]}}$ we have $w_i e_t = \xi^k e_t$.
(iii)⇒(ii): Let $w_i e_t = \xi^k e_t$. Suppose that $i$ occurs in the $l$-th constituent of every tabloid appearing in $e_t$ such that $k \neq l$. Then we have $w_i e_t = \xi^l e_t$, which is a contradiction. q.e.d.

We now give the main theorem of this section.

**Theorem 3.17** *Let $\mathbf{C}$ be the field of complex numbers and $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$. Then*

$$Hom_{\mathbf{C}W_n^m}(S^{\lambda^{[m]}}, S^{\lambda^{[m]}}) \cong \mathbf{C}.$$

*Proof.* Let $I$ be the identity function of $S^{\lambda^{[m]}}$. We show that every $\mathbf{C}W_n^m$-endomorphism of $S^{\lambda^{[m]}}$ is a scalar multiple of $I$.

Let $\theta$ be an endomorphism of $S^{\lambda^{[m]}}$ and let $t$ be a $\lambda^{[m]}$-tableau. Suppose that for all $k = 1, \ldots, m$ we have that $\xi^{\rho_1^{(k)}} i_1^{(k)}, \ldots, \xi^{\rho_{s_k}^{(k)}} i_{s_k}^{(k)}$ occur in the $k$-th constituent of the tableau $t$, where $\rho_j^{(k)} \in \{1, \ldots, m\}$ and $i_j^{(k)} \in \{1, \ldots, n\}$ for $j = 1, 2, \ldots, s_k$. Suppose that $e_{t^*}$ is involved in $\theta(e_t)$ for some $e_{t^*} \neq \alpha e_t$, where $\alpha \in \mathbf{C}$ and $t^*$ is a $\lambda^{[m]}$-tableau. By Lemma 3.16, we have

$$w_{i_j^{(k)}}\theta(e_t) = \theta(w_{i_j^{(k)}} e_t) = \xi^k \theta(e_t)$$

for $j = 1, 2, \ldots, s_k$ and for all $k = 1, \ldots, m$.

If there exists at least one entry, say $\xi^{\rho_j^{(k)}} i_j^{(k)}$, in the $k$-th constituent of $t$ such that $\xi^{\rho_j^{(k)}} i_j^{(k)}$ occurs in the $l$-th constituent of $t^*$, where $j = 1, 2, \ldots, s_k$, $k = 1, \ldots, m$ and $k \neq l$, then by using Lemma 3.16, we have $w_{i_j^{(k)}} e_{t^*} = \xi^l e_{t^*}$. This implies that for $w_{i_j^{(k)}} \in W_n^m$,

$$\theta(w_{i_j^{(k)}} e_t) \neq w_{i_j^{(k)}} \theta(e_t).$$

This means $\theta$ does not preserve the action of $W_n^m$. Thus $\theta(e_t)$ is at most a linear combination of all $\lambda^{[m]}$-polytabloids $e_{t'}$ such that $t$ agrees with each $t'$, i.e.

$$\theta(e_t) = \sum \alpha_{t'} e_{t'}$$

where the summation is taken over all $\lambda^{[m]}$-tableaux $t'$ such that $t$ agrees with each one of them. Then by Lemma 3.15 (ii) and Remark 3.14 we have

$$|C_t| \theta(e_t) = \theta(\kappa_t e_t) = \kappa_t \theta(e_t) = \sum \alpha_{t'} \kappa_t e_{t'} = \sum \alpha_{t'} \beta e_t$$

where $\beta \in \mathbf{C}$. Thus, $\theta(e_t) = \alpha e_t$ where $\alpha = \frac{1}{|C_t|} \sum \alpha_{t'} \beta$, and $\theta = \alpha I$ for some scalar $\alpha$ which proves the theorem. q.e.d.

**Corollary 3.18** *Let* $\mathbf{C}$ *be the field of complex numbers and* $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$. *Then* $S^{\lambda^{[m]}}$ *is irreducible.*

We have constructed an irreducible Specht module $S^{\lambda^{[m]}}$ for each $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$. We shall next show that they are mutually non-isomorphic.

As in the case of symmetric groups, we define an inner product $\langle \cdot, \cdot \rangle$ on $M^{\lambda^{[m]}}$ by setting

$$\langle \{t\}, \{t'\} \rangle = \begin{cases} 1 & \text{if } \{t\} = \{t'\} \\ 0 & \text{if } \{t\} \neq \{t'\} \end{cases}$$

and extend to be bilinear on $M^{\lambda^{[m]}}$.

**Proposition 3.19** *Let* $\lambda^{[m]}, \mu^{[m]} \in \mathcal{P}^{(m)}(n)$ *such that* $|\lambda^{(1)}| > |\mu^{(1)}|$ *or* $|\lambda^{(j)}| = |\mu^{(j)}|$ $(1 \leq j \leq k)$ *and* $|\lambda^{(k+1)}| > |\mu^{(k+1)}|$ *for some* $k \in \{1, \ldots, m-2\}$. *Suppose the field of scalars is* $\mathbf{C}$ *and* $\theta \in Hom(S^{\lambda^{[m]}}, S^{\mu^{[m]}})$. *Then* $\theta = 0$.

*Proof.* Let $\theta \in Hom(S^{\lambda^{[m]}}, S^{\mu^{[m]}})$. Let $t$ be a $\lambda^{[m]}$-tableau and $t'$ be a $\mu^{[m]}$-tableau. Then by hypothesis there exists at least one entry, say $\xi^{\rho_1} i$, in the $p$-th constituent of $t$ such that $\xi^{\rho_2} i$ occurs in the $q$-th constituent of $t'$, where $i \in \{1, \ldots, n\}$, $\rho_1, \rho_2 \in \{1, \ldots, m\}$ and $p \neq q$. Then by Lemma 3.16 we have

$$w_i e_t = \xi^p e_t \text{ and } w_i e_{t'} = \xi^q e_{t'}.$$

This implies that $\theta(w_i e_t) \neq w_i \theta(e_t)$, which means that $\theta$ does not preserve the action of $W_n^m$. Thus there is no alternative but $\theta(e_t) = 0$ for all $\lambda^{[m]}$-tableaux $t$. Therefore $\theta = 0$ which implies that $S^{\lambda^{[m]}}$ and $S^{\mu^{[m]}}$ are not isomorphic when $|\lambda^{(1)}| > |\mu^{(1)}|$ or $|\lambda^{(j)}| = |\mu^{(j)}|$ $(1 \leq j \leq k)$ and $|\lambda^{(k+1)}| > |\mu^{(k+1)}|$ for some $k \in \{1, \ldots, m-2\}$. q.e.d.

**Proposition 3.20** *Let $\lambda^{[m]}$, $\mu^{[m]} \in \mathcal{P}^{(m)}(n)$ such that $|\lambda^{(i)}| = |\mu^{(i)}|$ for all $i = 1, \ldots, m-1$. Suppose the field of scalars is $\mathbf{C}$ and $\theta \in Hom(S^{\lambda^{[m]}}, M^{\mu^{[m]}})$ is non-zero. Thus $\lambda^{[m]} \trianglerighteq \mu^{[m]}$ and if $\lambda^{[m]} = \mu^{[m]}$ then $\theta$ is multiplication by a scalar.*

*Proof.* Since $\theta \neq 0$, there is some basis vector $e_t$ such that $\theta(e_t) \neq 0$. Because $\langle \cdot, \cdot \rangle$ is an inner product with complex scalars, $M^{\lambda^{[m]}} = S^{\lambda^{[m]}} \oplus \left(S^{\lambda^{[m]}}\right)^{\perp}$. Thus we can extend $\theta$ to an element of $Hom(M^{\lambda^{[m]}}, M^{\mu^{[m]}})$ by setting $\theta\left(\left(S^{\lambda^{[m]}}\right)^{\perp}\right) = 0$. So

$$0 \neq \theta(e_t) = \theta(\kappa_t\{t\}) = \kappa_t\theta(\{t\}) = \kappa_t\left(\sum \alpha_{t'}\{t'\}\right),$$

where the summation is taken over all $\mu^{[m]}$-tableaux $t'$ such that for all $a$, if $a$ occurs in $t_{\lambda^{(i)}}$ then $\xi^{s_i}a$ occurs in $t'_{\mu^{(i)}}$, where $i \in \{1, \ldots, m-1\}$ and $s_i \in \{1, \ldots, m\}$. (Here, we need this restriction otherwise $\theta$ does not preserve the action of $W_n^m$). By Lemma 3.10 we have $\lambda^{[m]} \trianglerighteq \mu^{[m]}$.

In the case $\lambda^{[m]} = \mu^{[m]}$ the above summation is taken over all $\lambda^{[m]}$-tableaux $t'$ such that $t$ agrees with each $t'$. Hence, Corollary 3.13 yields $\theta(e_t) = \beta e_t$ for some $\beta \in \mathbf{C}$. So for any $\sigma \in W_n^m$,

$$\begin{aligned} \theta(e_{\sigma t}) &= \theta(\xi^{m-\phi(\sigma,t)}\sigma e_t) = \xi^{m-\phi(\sigma,t)}\sigma\theta(e_t) \\ &= \xi^{m-\phi(\sigma,t)}\sigma(\beta e_t) = \beta e_{\sigma t}. \end{aligned}$$

Thus $\theta$ is multiplication by a scalar. q.e.d.

**Theorem 3.21** *The $S^{\lambda^{[m]}}$ ($\lambda^{[m]} \in P^{(m)}(n)$) give a complete set of irreducible $\mathbf{C}W_n^m$-modules.*

*Proof.* The $S^{\lambda^{[m]}}$ are irreducible by Corollary 3.18. Since we have the right number of modules for a full set, it suffices to show that they are pairwise inequivalent. Now, let $\lambda^{[m]}$, $\mu^{[m]} \in \mathcal{P}^{(m)}(n)$. If $|\lambda^{(1)}| > |\mu^{(1)}|$ or $|\lambda^{(j)}| = |\mu^{(j)}|$ ($1 \leq j \leq k$) and $|\lambda^{(k+1)}| > |\mu^{(k+1)}|$ for some $k \in \{1, \ldots, m-2\}$ then by Proposition 3.19, $S^{\lambda^{[m]}}$ and $S^{\mu^{[m]}}$ are not isomorphic. If $|\lambda^{(i)}| = |\mu^{(i)}|$ for all $i = 1, \ldots, m-1$ and $S^{\lambda^{[m]}} \cong S^{\mu^{[m]}}$, then there exists a non-zero homomorphism $\theta \in Hom(S^{\lambda^{[m]}}, M^{\mu^{[m]}})$, since $S^{\mu^{[m]}} \subseteq M^{\mu^{[m]}}$. Thus, by Proposition 3.20, $\lambda^{[m]} \trianglerighteq \mu^{[m]}$. Similarly, $\mu^{[m]} \trianglerighteq \lambda^{[m]}$, so $\lambda^{[m]} = \mu^{[m]}$. q.e.d.

## 4. A Basis for Specht Modules

As in the case of the symmetric group, generally the polytabloids that generate $S^{\lambda^{[m]}}$ are not linearly independent. We now determine a subset which forms a basis for $S^{\lambda^{[m]}}$.

**Definition 4.1** *A $\lambda^{[m]}$-tableau $t$ is said to be standard if the entries of $t$ are all positive integers, and $t_{\lambda^{(i)}}$ is a standard tableau for all $i = 1, \ldots, m$. A tabloid $\{t\}$ is a standard tabloid if there is a standard tableau in the equivalence class $\{t\}$. A polytabloid $e_t$ is a standard polytabloid if $t$ is a standard tableau.*

**Theorem 4.2** *The set $\{e_t \mid t \text{ is a standard } \lambda^{[m]} - \text{tableau}\}$ is a $\mathbf{C}$-basis for $S^{\lambda^{[m]}}$. The dimension of $S^{\lambda^{[m]}}$ is the number of standard $\lambda^{[m]}$-tableaux.*

We will spend the next two sections proving this theorem. First we shall establish that the $e_t$ are linearly independent. For this case, we shall need a partial order on tabloids associated with the same $m$-set of partitions of $n$.

Let $t = (t_{\lambda^{(1)}}, \ldots, t_{\lambda^{(m)}})$ be a $\lambda^{[m]}$- tableau. Let $|t_{\lambda^{(k)}}|$ denote the modulus of $t_{\lambda^{(k)}}$ for all $k = 1, \ldots, m$, that is, if $t_{\lambda^{(k)}} = (t_{\lambda^{(k)}}(i,j))$, where $t_{\lambda^{(k)}}(i,j))$ stands for the entry of $t_{\lambda^{(k)}}$ in position $(i,j)$ then $|t_{\lambda^{(k)}}| = (|t_{\lambda^{(k)}}(i,j)|)$ for all $k = 1, \ldots, m$.

Let $(s_i(t_{\lambda^{(1)}}))_{1\le i\le n}$ be the sequence associated with $t_{\lambda^{(1)}}$ such that for all $i$, $s_i(t_{\lambda^{(1)}})$ =the power of the complex coefficient of the $i$ if $i$ occurs in $t_{\lambda^{(1)}}$, 0 if $i$ is not contained in $t_{\lambda^{(1)}}$, or if $i$ occurs in $t_{\lambda^{(1)}}$ without complex coefficient. (Note that at this point, we do not use the relation $\xi^{\frac{m}{2}} = -1$ if $m$ is even.)

For example, if $\lambda^{[3]} = (31, 2^2, 21) \in \mathcal{P}^{(3)}(11)$ and

$$t = \left( \begin{matrix} \xi^2 1 & \xi 7 & 3 \\ \xi 9 & & \end{matrix} , \begin{matrix} \xi^2 6 & 4 \\ 11 & \xi 5 \end{matrix} , \begin{matrix} 8 & \xi 2 \\ 10 & \end{matrix} \right)$$

then

$$|t_{\lambda^{(1)}}| = \begin{matrix} 1 & 7 & 3 \\ 9 & & \end{matrix} , \quad |t_{\lambda^{(2)}}| = \begin{matrix} 6 & 4 \\ 11 & 5 \end{matrix} , \quad |t_{\lambda^{(3)}}| = \begin{matrix} 8 & 2 \\ 10 & \end{matrix}$$

and $(s_i(t_{\lambda^{(1)}}))_{1\le i\le 11} = (2,0,0,0,0,0,1,0,1,0,0)$.

Given any tableau $t = (t_{\lambda^{(1)}}, \ldots, t_{\lambda^{(m)}})$, let $m_{ir}(t_{\lambda^{(k)}})$ denote the number of entries less than or equal to $i$ ($i = 1, \ldots, n$) in the first $r$ rows of $|t_{\lambda^{(k)}}|$, where $k = 1, \ldots, m$.

**Definition 4.3** *Let $\lambda^{[m]} \in \mathcal{P}^{(m)}(n)$. Let $\{t\}$ and $\{s\}$ be $\lambda^{[m]}$-tabloids. Then we write*
*(a) $\{t_{\lambda^{(1)}}\} \trianglelefteq \{s_{\lambda^{(1)}}\}$ if one of the following conditions is satisfied:*

1. *some entries of $t_{\lambda^{(1)}}$ are complex (or negative) and the entries in $s_{\lambda^{(1)}}$ are all positive integers (in this case $\{t_{\lambda^{(1)}}\} \triangleleft \{s_{\lambda^{(1)}}\}$),*
2. *$|t_{\lambda^{(1)}}| = |s_{\lambda^{(1)}}|$ and $s_i(t_{\lambda^{(1)}}) \le s_i(s_{\lambda^{(1)}})$ for all $i = 1, \ldots, n$,*
3. *$|t_{\lambda^{(1)}}| \ne |s_{\lambda^{(1)}}|$ and for all $i$ $(1 \le i \le n)$ and $r$ $m_{ir}(t_{\lambda^{(1)}}) \le m_{ir}(s_{\lambda^{(1)}})$, and*

*(b) $\{t_{\lambda^{(k)}}\} \trianglelefteq \{s_{\lambda^{(k)}}\}$ if for all $i$ $(1 \le i \le n)$ and $r$ $m_{ir}(t_{\lambda^{(k)}}) \le m_{ir}(s_{\lambda^{(k)}})$, where $k = 2, 3, \ldots, m$.*

*Hence, $\{s\}$ dominates $\{t\}$, written $\{t\} \trianglelefteq \{s\}$, if $\{t_{\lambda^{(k)}}\} \trianglelefteq \{s_{\lambda^{(k)}}\}$ for all $k = 1, \ldots, m$.*

Let $\{t\}$ be a $\lambda^{[m]}$-tabloid. Suppose that $a$, $b$ occur in $t_{\lambda^{(j)}}$ such that $a = \xi^{s_1}k$, $b = \xi^{s_2}l$, $k < l$ and $a$ is in the $p$-th row and $b$ is in the $q$-th row of $t_{\lambda^{(j)}}$, where $1 \le s_1,\ s_2 \le m$, $k,\ l \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then the definition of $m_{ir}(t_{\lambda^{(j)}})$ gives

$$(*) \qquad m_{ir}((a,b)t_{\lambda^{(j)}}) - m_{ir}(t_{\lambda^{(j)}}) = \begin{cases} 1 & \text{if } q \le r < p \text{ and } k \le i < l \\ -1 & \text{if } p \le r < q \text{ and } k \le i < l \\ 0 & \text{otherwise} \end{cases}$$

where $j \in \{1, \ldots, m\}$.

Just as for partitions, there is a dominance lemma for tabloids.

**Lemma 4.4** *(Dominance Lemma for Tabloids) Let $\{t\}$ be a $\lambda^{[m]}$- tabloid. If there exist $a$, $b$ in $t_{\lambda^{(j)}}$ such that $a = \xi^{s_1}k$, $b = \xi^{s_2}l$, $k < l$ and $a$ appears in a lower row than $b$, where $1 \le s_1,\ s_2 \le m$, $k,\ l \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ then $\{t\} \triangleleft (a,b)\{t\}$.*

*Proof.* Let $a$ and $b$ be two elements in $t_{\lambda^{(j)}}$ such that $a = \xi^{s_1}k,\ b = \xi^{s_2}l$, where $1 \leq k < l \leq n,\ 1 \leq s_1,\ s_2 \leq m$ and $j \in \{1,\ \ldots,\ m\}$. Then

$$(a, b) = (k, l)w_k^{m+s_2-s_1}w_l^{m+s_1-s_2} \text{ and } (a, b)\{t\} = \{(a, b)t\}.$$

Therefore the result follows from $(*)$. q.e.d.

The dominance lemma puts a restriction on which tabloids can appear in a standard polytabloid.

**Corollary 4.5** *If $t$ is standard and $\{s\}$ appears in $e_t$, then $\{s\} \trianglelefteq \{t\}$.*

*Proof.* Let $s = \pi t$, where $\pi \in C_t$, so that $\{s\}$ appears in $e_t$. We induct on the number of *column inversions* in $s$. Let $\pi = \gamma\tau \in C_t$ such that $\tau \in \mathcal{S}_n$ and $\gamma = \prod_i w_i^{s_i} \in C_m^n$, where each $i$ appears in $t_{\lambda^{(1)}}$ and $1 \leq s_i \leq m$. We now consider all the possible cases in terms of $\pi \in C_t$.

(i) Firstly, if $\tau = e$ then $\pi = \gamma$. Thus, since $s = \gamma t$, $\gamma$ multiplies any number of symbols in $t_{\lambda^{(1)}}$ by a power of $\xi$. Hence, some entries of $s_{\lambda^{(1)}}$ are complex (or negative) and the entries in $s_{\lambda^{(j)}}$ do not change for all $j = 2,\ 3,\ \ldots,\ m$. Then we have $\{s_{\lambda^{(1)}}\} \triangleleft \{t_{\lambda^{(1)}}\}$ by part (a)(1) of Definition 4.3 and $\{s_{\lambda^{(j)}}\} = \{t_{\lambda^{(j)}}\}$ for all $j = 2,\ 3,\ \ldots,\ m$. Therefore, $\{s\} \triangleleft \{t\}$.

(ii) Secondly, if $\gamma = e$ then $\pi = \tau$. Since $s = \tau t$ then in the same column of $s_{\lambda^{(j)}}$ there are positive integers $k < l$ such that $k$ is in a lower row than $l$, where $j \in \{1,\ \ldots,\ m\}$. Thus, by Lemma 4.4 $\{s\} \triangleleft (k, l)\{s\}$. Since $(k, l)\{s\}$ is involved in $e_t$ and $(k, l)\{s\}$ has fewer inversions than $\{s\}$, induction shows that $(k, l)\{s\} \trianglelefteq \{t\}$. Therefore, $\{s\} \triangleleft \{t\}$.

(iii) Finally, if $\tau,\ \gamma \neq e$ then $\pi = \gamma\tau$ and $s = \gamma\tau t$. By (i), some entries of $s_{\lambda^{(1)}}$ are complex (or negative) and so we have at once $\{s_{\lambda^{(1)}}\} \triangleleft \{t_{\lambda^{(1)}}\}$. Furthermore, in the same column of $s_{\lambda^{(j)}}$ there are entries $a = \xi^{s_1}k$ and $b = \xi^{s_2}l$ such that $k < l$ and $a$ appears in a lower row than $b$, where $1 \leq s_1,\ s_2 \leq m,\ k,\ l \in \{1,\ \ldots,\ n\}$ and $j \in \{1,\ \ldots,\ m\}$. Thus, by Lemma 4.4 $\{s\} \triangleleft (a, b)\{s\}$. Since $(a, b)\{s\}$ is involved in $e_t$ and $(a, b)\{s\}$ has fewer inversions than $\{s\}$, induction shows that $(a, b)\{s\} \trianglelefteq \{t\}$. Therefore, $\{s\} \triangleleft \{t\}$. Hence, we have the required result. q.e.d.

The previous corollary says that $\{t\}$ is the maximum tabloid in $e_t$.

**Lemma 4.6** *Let $v_1,\ \ldots,\ v_k$ be elements of $M^{\lambda^{[m]}}$. Suppose, for each $v_i$, we can choose a tabloid $\{t_i\}$ appearing in $v_i$ such that*

1. *$\{t_i\}$ is maximum in $v_i$, and*
2. *the $\{t_i\}$ are all distinct.*

*Then $v_1,\ \ldots,\ v_k$ are linearly independent over* **C**.

*Proof.* Choose the labels such that $\{t_1\}$ is maximal among the $\{t_i\}$. Thus conditions 1 and 2 ensure that $\{t_1\}$ appears only in $v_1$. (If $\{t_1\}$ occurs in $v_i,\ i > 1$, then $\{t_1\} \triangleleft \{t_i\}$, contradicting the choice of $\{t_1\}$.) It follows that in any linear combination

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

we must have $c_1 = 0$ since there is no other way to cancel $\{t_1\}$. By induction on $k$, the rest of the coefficients must also be zero. q.e.d.

We now have all the ingredients to prove independence of the standard basis.

**Proposition 4.7** *The set* $\{e_t \,|\, t$ *is a standard* $\lambda^{[m]}$*-tableau*$\}$ *is linearly independent over* $\mathbf{C}$.

*Proof.* By Corollary 4.5, $\{t\}$ is maximum in $e_t$, and by hypothesis they are all distinct. Thus Lemma 4.6 applies. q.e.d.

## 5. The Garnir Relations

In this section we shall show that standard $\lambda^{[m]}$-polytabloids span $S^{\lambda^{[m]}}$; that is, if $t$ is an arbitrary $\lambda^{[m]}$- tableau, then $e_t$ is a linear combination of standard $\lambda^{[m]}$-polytabloids.

To determine the Garnir element of $t$ associated with $e_t$, we use the following relations.

**Lemma 5.1** *Let* $t$ *be a* $\lambda^{[m]}$*-tableau.*

*(i) Let* $a$ *and* $b$ *be two elements in the same column of* $t_{\lambda^{(j)}}$ *such that* $a = \xi^{s_1}k,\ b = \xi^{s_2}l$, *where* $1 \leq s_1,\ s_2 \leq m,\ k,\ l \in \{1,\ \ldots,\ n\}$ *and* $j \in \{1,\ \ldots,\ m\}$. *Then* $(a,b) = (k,l)w_k^{m+s_2-s_1}w_l^{m+s_1-s_2}$ *generates a subgroup of order 2 of* $C_t$ *and*

$$(e + (a,b))e_t = 0 \quad (\textit{alternacy relation}).$$

*(ii) For any entry* $a$ *in the tableau* $t$ *such that* $a = \xi^s i$, *where* $1 \leq s < m,\ 1 \leq i \leq n$,

$$(e + \alpha w_i^{m-s})e_t = 0 \quad (\textit{primitive root change relation}),$$

*where* $\alpha = -\xi^{-j(m-s)}$ *if* $a$ *appears in* $t_{\lambda^{(j)}}$ *for* $j \in \{1,\ \ldots,\ m\}$.

*Proof.* (i) follows immediately from $(a,b)e_t = -e_t$. To prove (ii), we use the following fact

$$w_i^{m-s}e_t = \xi^{\phi(w_i^{m-s},t)}e_t,$$

where $\phi(w_i^{m-s},t) = j(m-s)$ if $a$ appears in $t_{\lambda^{(j)}}$ for $j \in \{1,\ \ldots,\ m\}$. q.e.d.

**Remark 5.2** The previous lemma says that we can find the elements of $W_n^m$ which remove any complex (or negative) entries of $t$ associated with $e_t$ (primitive root change relation) and which make $t$ column standard (alternacy relation), i.e. the tableau $t$ associated with $e_t$ may be reorganized so that no complex (or negative) entries remain and all columns are standard.

Now we shall find elements of the group algebra of $G(m,1,n)$ which annihilate the given polytabloid $e_t$. Let $t$ be a $\lambda^{[m]}$-tableau associated with $e_t$ such that the entries of $t$ were reorganized by the primitive root change and alternacy relations. Suppose that $t$ has entries $t(i,j,k)$ and $t(i,j+1,k)$, where $k \in \{1,\ \ldots,\ m\}$. Let $A$ be the set of entries in column $j$ of $t$ below and including $t(i,j,k)$, and let $B$ be the set of entries in column $j+1$ of $t$ above and including $t(i,j+1,k)$. Suppose that $A = \{a_i\ :\ i \in I\}$ and $B = \{b_j\ :\ j \in J\}$, where $I$ and $J$ are index sets. Let $\mathcal{S}_A,\ \mathcal{S}_B,\ \mathcal{S}_{A\cup B}$ be the subgroups of $G(m,1,n)$. Let $\sigma_1,\ \ldots,\ \sigma_r$ be coset representatives for $\mathcal{S}_A \times \mathcal{S}_B$ in $\mathcal{S}_{A\cup B}$, and let

$$\mathcal{S}_{A\cup B} = \biguplus_{j=1}^{r} \sigma_j(\mathcal{S}_A \times \mathcal{S}_B) \ \text{ and } \ \mathcal{G}_{A,B} = \sum_{j=1}^{r}(sgn\ \sigma_j)\sigma_j.$$

$\mathcal{G}_{A,B}$ is called a *Garnir element associated with* $t$.

**Remark 5.3** (i) If the elements of $A$ and $B$ belong to $t_{\lambda^{(1)}}$, then we assume that $\mathcal{S}_A,\ \mathcal{S}_B,\ \mathcal{S}_{A\cup B}$ are the subgroups of $G(m,1,n)$ generated by the elements in $A,\ B,\ A\cup B$ respectively. Let $C_{A\cup B}$ $(\subset C_m^n)$ be the subgroup of $\mathcal{S}_{A\cup B}$ generated by $\{w_{a_i},\ w_{b_j} \mid i\in I,\ j\in J\}$. Since $C_{A\cup B}$ is also a subgroup of $\mathcal{S}_A\times\mathcal{S}_B$, all the coset representatives $\sigma_1,\ \ldots,\ \sigma_r$ are in $\mathcal{S}_n$.
(ii) If the elements of $A$ and $B$ belong to the $t_{\lambda^{(j)}}$, where $j\in\{2,\ 3,\ \ldots,\ m\}$ then we assume that $\mathcal{S}_A,\ \mathcal{S}_B,\ \mathcal{S}_{A\cup B}$ are the subgroups of $G(m,1,n)$ generated by permutations on $A,\ B,\ A\cup B$ respectively.
(iii) The coset representatives $\sigma_1,\ \ldots,\ \sigma_r$ are, of course, not unique, but for practical purposes note that we may take $\sigma_1,\ \ldots,\ \sigma_r$ so that $\sigma_1 t,\ \ldots,\ \sigma_r t$ are all the tableaux which agree with $t$ except in the positions occupied by $A\cup B$, and whose entries increase vertically downwards in the positions occupied by $A\cup B$.

Let $H$ be any subset of $W_n^m$. Define

$$\bar{H}=\sum_{\sigma\in H}\xi^{-f(\sigma)}(sgn\ \sigma)\sigma$$

and if $H=\{\sigma\}$ then we write $\bar{\sigma}=\xi^{-f(\sigma)}(sgn\ \sigma)\sigma$ for $\bar{H}$.

**Lemma 5.4** *Let $H\le W_n^m$ be a subgroup. Let $a=\xi^{s_1}k$ and $b=\xi^{s_2}l$, where $1\le s_1,\ s_2\le m$, and $k,\ l\in\{1,\ 2,\ \ldots,\ n\},\ k\neq l$.*

*(i) If the element $(a,b)=(k,l)w_k^{m+s_2-s_1}w_l^{m+s_1-s_2}\in H$ then we can factor $\bar{H}=c(e-(a,b))$, where $c\in \mathbf{C}W_n^m$.*

*(ii) If $t$ is a $\lambda^{[m]}$-tableau with $a,\ b$ in the same row of $t_{\lambda^{(j)}}$, where $j\in\{1,\ 2,\ \ldots,\ m\}$, and $(a,b)\in H$, then $\bar{H}\{t\}=0$.*

*Proof.* (i) Consider the subgroup $K=\{e,\ (a,b)\}$ of $H$. Select coset representatives $\sigma_1,\ \ldots,\ \sigma_s$ for $K$ in $H$ and write $H=\biguplus_{i=1}^{s}\sigma_i K$. But then

$$\bar{H}=\left(\sum_{i=1}^{s}\bar{\sigma}_i\right)(e-(a,b)),$$

as desired.
(ii) By hypothesis, $(a,b)\{t\}=\{(a,b)t\}=\{t\}$. Thus,

$$\bar{H}\{t\}=c(e-(a,b))\{t\}=c(\{t\}-\{t\})=0. \qquad \text{q.e.d.}$$

**Proposition 5.5** *Let $t,\ A,\ B$ be as in the definition of a Garnir element. If $|A\cup B|$ is greater than the number of elements in column $j$ of $t$ then $\mathcal{G}_{A,B}e_t=0$.*

*Proof.* Let

$$\overline{\mathcal{S}_A\times\mathcal{S}_B}=\sum_{\sigma\in\mathcal{S}_A\times\mathcal{S}_B}\xi^{-f(\sigma)}(sgn\ \sigma)\sigma \quad\text{and}\quad \overline{\mathcal{S}_{A\cup B}}=\sum_{\sigma\in\mathcal{S}_{A\cup B}}\xi^{-f(\sigma)}(sgn\ \sigma)\sigma.$$

Consider any $\sigma\in C_t$. If the elements of $A$ and $B$ are in the $t_{\lambda^{(1)}}$ then by the hypothesis there must be positive integers $a,\ b\in A\cup B$ such that $c,\ d$ belong to the same row of $\sigma t_{\lambda^{(1)}}$ where $c=\xi^{s_1}a,\ d=\xi^{s_2}b,\ 1\le s_1,\ s_2\le m$. Thus,

$$(c,d)=(a,b)w_a^{m+s_2-s_1}w_b^{m+s_1-s_2}\in\mathcal{S}_{A\cup B},$$

by Remark 5.3 (i). If the elements of $A$ and $B$ are in the $t_{\lambda^{(j)}}$, where $j \in \{2, 3, \ldots, m\}$ then by the hypothesis there must be positive integers $a, b \in A \cup B$ such that $a, b$ belong to the same row of $\sigma t_{\lambda^{(j)}}$, where $j \in \{2, 3, \ldots, m\}$. Therefore, $(a, b) \in \mathcal{S}_{A\cup B}$ by Remark 5.3 (ii).

In terms of the elements of $A$ and $B$ appearing in the $t_{\lambda^{(j)}}$, where $j \in \{1, 2, \ldots, m\}$, $(c, d)$ or $(a, b)$ belong to the same group $\mathcal{S}_{A\cup B}$. But then $\overline{\mathcal{S}_{A\cup B}}\{\sigma t\} = 0$ by Lemma 5.4. Since this is true for every $\sigma$ appearing in $\kappa_t$, we have $\overline{\mathcal{S}_{A\cup B}} e_t = 0$.

Now $\mathcal{S}_{A\cup B} = \biguplus_{j=1}^{r} \sigma_j(\mathcal{S}_A \times \mathcal{S}_B)$ so $\overline{\mathcal{S}_{A\cup B}} = \mathcal{G}_{A,B}\overline{\mathcal{S}_A \times \mathcal{S}_B}$ by Remark 5.3. Since $\mathcal{S}_A \times \mathcal{S}_B \subset C_t$ then $\overline{\mathcal{S}_A \times \mathcal{S}_B}$ is a factor of $\kappa_t$ and $\overline{\mathcal{S}_A \times \mathcal{S}_B} e_t = |\mathcal{S}_A \times \mathcal{S}_B| e_t$. Therefore,

$$0 = \overline{\mathcal{S}_{A\cup B}} e_t = |\mathcal{S}_A \times \mathcal{S}_B| \mathcal{G}_{A,B} e_t.$$

Thus, $\mathcal{G}_{A,B} e_t = 0$. q.e.d.

We shall now use the Garnir relations, primitive root change relation and alternacy relation to prove that any polytabloid can be written as a linear combination of standard polytabloids.

**Theorem 5.6** *The set* $\{e_t \mid t$ *is a standard* $\lambda^{[m]}$*−tableau*$\}$ *spans* $S^{\lambda^{[m]}}$.

*Proof.* First we write $[t]$ for the column equivalence class of $t$, that is

$$[t] = \{t' \mid t' = \sigma t \text{ for some } \sigma \in C_t\}.$$

The column equivalence classes are partially ordered in a way similar to the partial order in Definition 4.3 on the row equivalence classes.

Let $t$ be any $\lambda^{[m]}$-tableau associated with $e_t$. Then we may assume that $e_t$ may be written as a linear combination of column standard polytabloids by Lemma 5.1. Therefore, because of Remark 5.2, we may suppose that the entries in $t$ are all positive integers and are in increasing order down the columns. Suppose that $t$ is not standard. This means that some constituents of $t$ are not standard, say $r$ constituents are not standard where $r \leq m$. Suppose that the $k$-th constituent of $t$ is not standard. By induction, we may assume that $e_s$ can be written as a linear combination of the polytabloids $e_{t'}$ such that the $k$-th constituent of each $t'$ is standard when $[s] \rhd [t]$ and prove the same result for $e_t$ by considering the $k$-th constituent of $t$.

Since we have supposed that the $k$-th constituent of $t$ is not standard then there must be some adjacent pair of columns in the $k$-th constituent of $t$, say the $j$-th and $(j+1)$-th columns, with entries $a_1 < a_2 < \ldots < a_p$ and $b_1 < b_2 < \ldots < b_q$ with $a_i > b_i$ for some $i$. Thus we have the following situation in the $k$-th constituent of $t$:

$$\begin{array}{ccc}
a_1 & & b_1 \\
 & & \wedge \\
a_2 & & b_2 \\
 & & \wedge \\
\vdots & & \vdots \\
 & & \wedge \\
a_i & > & b_i \\
\wedge & & \\
a_{i+1} & & \vdots \\
\wedge & & b_q \\
\vdots & & \\
\wedge & & \\
a_p & &
\end{array}$$

Take $A = \{a_i,\ a_{i+1},\ \ldots,\ a_p\}$ and $B = \{b_1,\ b_2,\ \ldots,\ b_i\}$, and consider the corresponding Garnir element $\mathcal{G}_{A,B} = \sum_\sigma (sgn\ \sigma)\sigma$. By Proposition 5.5, we have $\mathcal{G}_{A,B}e_t = 0$ so that

$$e_t = -\sum_{\sigma \neq e} (sgn\ \sigma) e_{\sigma t}.$$

But $b_1 < b_2 < \ldots < b_i < a_i < \ldots < a_p$ implies that $[\sigma t] \trianglerighteq [t]$ for $\sigma \neq e$ by the column analogue of the dominance lemma for tabloids (Lemma 4.4). Since

$$e_t = -\sum_{\sigma \neq e} (sgn\ \sigma) e_{\sigma t},$$

the result follows from our induction hypothesis for the $k$-th constituent of $t$ associated with $e_t$. Hence, if we repeat the above process $(r-1)$-times then the result can be deduced. q.e.d.

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