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A Linear-Time Construction of Reuleaux Polygons

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Abstract. The relation between planar geometric graphs which are full equi-intersectors and Reuleaux polygons is studied. It is shown that there is a one-to-one correspondence between these objects. This is used to present exhaustive constructions of Reuleaux polygons of arbitrary (odd) order n. The obvious construction is running in quadratic time, but more careful investigations lead to a refined version, running in $0(n)$ time.

0. Introduction

Let $C \subset \mathbf{R}^2$ be a compact, convex set. The set C is said to be of *constant width* if its width function, i.e., the support function of $C + (-C)$, is constant (see [15, p. 158] or [5, p. 122 etc.]). Equivalently, it can be shown that C has constant width if and only if every boundary point of C is incident with (at least one) diameter (= chord of maximal length) of C. Basic surveys on this interesting class of convex bodies are [4] and [7], §5.

Assume that $C \subset \mathbf{R}^2$ has contant width and that $V \subset \partial C$, where ∂C denotes the boundary of C. The set V is a *pinching set* of C if every diameter of C is incident with (at least) one point of V; e.g., ∂C is a pinching set of C. The set C is a *Reuleaux Polygon* (RP, for brevity) if it admits a *finite* pinching set. (A different definition, adapted in [5, p. 128], turns out to be equivalent to the present one, see Remark 1.6 (iii) below). It can be shown that an RP has a unique minimal (finite) pinching set (see Remark 1.6 (i) below) called the set of *vertices* of the RP. The number of these vertices is the *order* of the RP. It is well known that the order of an RP is necessarily odd (see Theorem 1.5 (i) below). On the other hand, for any odd integer $n > 3$ there is an RP of order n. (Take the regular n-gon ($n > 3$ odd) whose principal diagonals have length 1, say, and join its vertices consecutively by circular arcs of radii 1.)

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It is known that the family of RPs of width 1 is dense in the family of planar sets of constant width 1 with respect to the Hausdorff distance. E.g., [1] uses this fact in his proof of the area minimizing property of the RP of order 3 and width 1 (which is unique up to isometry and usually called the *Reuleaux triangle*) among the sets of constant width 1, see also [17, pp. 76-80] or [5, p. 128, lines -9, -8]. In these references, the proofs of the density of RPs among the sets of contant width may be viewed, in a very implicit sense, as a construction of RPs. This is implicit because the construction pertains to a *given* set of constant width. Intentions to more explicit constructions of RPs can be found in [5, p. 128] and in [13, pp. 165-166], but these cannot be regarded as more than vague recipes. The problem seems to deserve a more serious treatment, and this paper intends to give an explicit detailed solution.

A principal merit of our construction, apart from its simplicity, is its exhaustiveness, where "exhaustive" has a two-fold meaning: (i) any concrete execution of the construction cannot produce a set which is not an RP, and (ii) any RP can be (potentially) produced by this construction. We present the construction as an algorithm, called *Algorithm RP* (see §2). For its main part, the algorithm has a procedure called *ProcVRP*, whose output is the set of vertices of the intended RP. The ProcVRP consumes most of the running time of the Algorithm RP; once its output is given nothing remains for the algorithm but to form the boundary of the RP by joining the given vertices consecutively by circular arcs of equal radii (= the width of the intended RP).

The running time of the obvious version of ProcVRP is quadratic in the cardinality of its output, i.e., it is quadratic in the order of the intended RP (the order being a part of the input). A refined, linear time version of ProcVRP, called *ProcVRPref*, is presented in §3, and the version of the algorithm based on this linear time procedure is naturally called *Algorithm RPref*. Clearly, ProcVRP and ProcVRPref cannot be distinguished by their outputs (both are exhausitive), but ProcVRPref will run in a considerably shorter time.

In order to describe the simple idea behind ProcVRP, we fix some notation:
For $V \subset \mathbf{R}^2$, put $diam\, V := \sup\{\| u - v \| : u, v \in V\}$ (= the numeric diameter of V) and $D(V) := \{[u, v] : u, v \in V, \| u - v \| = diam\, V\}$ (= the set of geometric diameters of V), $[u, v]$ denoting the closed straight line segment bewteen the points u and v. (The choice between the numeric and geometric meaning of the term "diameter" always will be clear from the context.) The geometric graph $GD(V) := \langle V, D(V)\rangle$ is the *diameter graph* of V.

For $n \geq 3$, put

$VRP(n) := \{V \subset \mathbf{R}^2 : V$ is the vertex set of some RP of order n and width $1\}$,

where $VRP(n) = \emptyset$ for $n \geq 3$ an even number, by Proposition 1.5 below. The following Theorem, proved in §1 below (see Remark 1.8), gives the principal tool for constructing a set $V \in VRP(n)$.

Theorem 0.1. *Assume that $V \subset \mathbf{R}^2$, $\#V = n \geq 3$, and $diam\, V = 1$. The following three conditions are equivalent:*

(i) *The diameter graph $GD(V) = \langle V, D(V)\rangle$ is a cycle (= n–gon);*

(ii) *$GD(V)$ is hamiltonian;*

(iii) *$V \in RP(n)$.*

The point here is the equivalence (i) $\Longleftrightarrow$ (iii) ((ii) plays just an intermediate role; note the obvious implication (i) $\Longrightarrow$ (ii)). A more explicit presentation of condition (i) of Theorem 0.1 runs as follows.

Condition 0.2. *All the (Euclidean) distances among the points of* V *are* < 1 $(= \operatorname{diam} V)$, *with the exception of* $\#V = n$ *distances of the form* $\| v_i - v_{i-1} \| = 1$ *for an appropriate cyclic ordering* $V = (v_1, \ldots, v_n)$ *of* V *(the indices are taken modulo* n*; thus* $v_{n+1} := v_1$*).*

Scheme 0.3. Condition 0.2 suggests a straightforward construction of the points $v_1, \ldots, v_n$ of a set $V \in VRP(n)$, sequentially according to their cyclic order as follows: $v_1 \in \mathbf{R}^2$ is chosen arbitrarily; assuming that $v_1, \ldots, v_{i-1}$ were chosen for $2 \leq i \leq n-1$, keep (in selecting v_i) just the $i-2$ inequalities $\| v_i - v_j \| < 1$ for $1 \leq j \leq i-2$, and the equality $\| v_i - v_{i-1} \| = 1$ ($i-1$ conditions in choosing v_i). This is essentially what ProcVRP performs.

Note that such a procedure is going to be quadratic in n, since the total number of conditions taken into account is $\sum_{i=1}^{n}(i-1) = 0(n^2)$. One point deserves attention in this description: The last step, that of selecting v_n, was not described (note the assumption $2 \leq i \leq n-1$ in the description of how to select v_i). This was intentionally omitted because there is some delicacy here: Since v_n must be at distance 1 from both v_1 and v_{n-1}, the choice here is *a-priori* between only two points. It turns out, however, that one and only one of these points is appropriate, and *ProcVRP* is instructed to make the right choice here.

Historic details. The construction belongs to a circle of ideas which seem to be scattered in the literature from as early as 1935 (see [8]), but whose real start was [6] (in 1946), where it is shown that $\#D(V) \leq \#V$ for any finite set $V \subset \mathbf{R}^2$. The simple and fruitful idea behind this inequality is that any two diameters of a planar set must intersect (Proposition 1.1 below), and it was further observed in [16] and in [10] that the inequality $\#E \leq \#V$ holds, in fact, for any geometric graph $G = \langle V, E \rangle$ any two edges of which intersect. Call such a graph an *intersector*, call an intersector $G = \langle V, E \rangle$ with $\#E = \#V$ a *full intersector*, and call a full intersector all of whose edges have the same (Euclidean) length a *full equi-intersector*. Full intersectors were studied in [16] in connection with Conway's Thrackle Conjecture, and in a more lucid form in [10, pp. 16-25]. Full equi-intersectors were studied in [3, pp. 209-211] in connection with Borsuk's Conjecture (which was recently refuted), and some ingredients of their structure can be found in [12, pp. 206-207]. The main contribution of the present paper to this circle of ideas is the manifestation of the close connection between full equi-intersectors and RPs, in particular with respect to their construction. This connection is exploited further in [11] where it is shown that the space of full equi-intersectors is compact with respect to the Hausdorff distance. Combining this with an interesting area formula for RPs appearing in [1], one gets the result that the regular RPs of order n and width 1 (equivalently: diameter 1) are area maximizing among all RPs of order n and width 1. This easily implies the isoperimetric inequality for sets of constant width.

1. Proof of Theorem 0.1

In this section we prove Theorem 0.1, thereby introducing some useful results and notation.

Proposition 1.1. *If $C \subset \mathbf{R}^2$ is compact and convex, then any two diameters (=chords of maximal length) of C intersect.*

This is a corollary of the following statement which is easy to prove.

Lemma 1.2. *If $e, e' \subset \mathbf{R}^2$ are two segments of the same length $\| e \| = \| e' \| = l$, without a common vertex, and if* diam $(e \cup e') \leq l$, *then* (relint (e))$\cap$ (relint (e')) $\neq \emptyset$.

Proposition 1.3. *Let $C \subset \mathbf{R}^n$ be a convex set. The set C has constant width if and only if every point on the relative boundary* relbd C *of C is incident with (at least) one diameter of C.*

The proof of this proposition essentially is covered by the arguments given in [2, p.128 ff.]. For a related characterization of sets of constant width see [12].

Corollary. *If $C \subset \mathbf{R}^n$ is a set of constant width, then the diameter graph $GD($*relbd C*) of its relative boundary has no isolated vertices.*

Remark. The diameter graph $GD($relbd $C)$ of a convex set $C \subset \mathbf{R}^n$ of constant width may be disconnected. E.g., the diameter graph of the unit sphere S^{n-1} is an infinite, uncountable, perfect matching.

Assume now that $C \subset \mathbf{R}^2$ is a set of constant width 1. For $v \in \partial C$ put $N_C(v) = \{u \in \partial C : [u, v] \in D(C)\}$, the set of neighbours of v in $GD(\partial C)$.

Proposition 1.4. *For $u, v \in \partial C$ the following conditions hold:*

(i) *$N_C(v)$ is a (possibly degenerated) connected circular arc (on ∂C) of radius 1 and of length $\leq \frac{\pi}{3}$, and it is a closed set.*

(ii) *For $p \in$ relint$(N_C(v))$ we have $N_C(p) = \{v\}$.*

(iii) (relint$(N_C(v))$) $\cap$ (relint$(N_C(u))$) $= \emptyset$, *unless $u = v$.*

Proof: (i) & (ii): By the Corollary above, $N_C(v) \neq \emptyset$. If $N_C(v)$ consists of only one point there is nothing to prove. Assume that $p', p'' \in N_C(v)$, $p' \neq p''$. Let $p \in (\partial C)\cap$ relint ($\sphericalangle\, p'vp''$) and $q \in N_C(p) (\neq \emptyset)$. By Proposition 1.1, $[p, q]$ intersects both $[v, p']$ and $[v, p'']$, which is impossible unless $q = v$, i.e., $N_C(p) = \{v\}$ (this proves (ii)). It follows that $p \in N_C(v)$, hence $N_C(v)$ is a (connected circular) arc.

If $\sphericalangle\, p'vp'' > \frac{\pi}{3}$, then $\| p'' - p' \| > 1$, a contradiction. Thus the length of $N_C(v)$ is $\leq \frac{\pi}{3}$. Clearly, the endpoints of $N_C(v)$ belong to $N_C(v)$, hence it is a closed set.

(iii): This statement follows from (ii). □

Proposition 1.4 implies that the valence of a point of ∂C in $GD(\partial C)$ is either 1 or ∞ (a continuum, in fact), and its set of neighbours in $GD(\partial C)$ is a (connected) closed set in ∂C.

Denote by $V_\infty(C)$ the set of vertices of $GD(\partial C)$ having valence ∞, and assume now that C is an RP. Let $\tilde{V} \subset \partial C$ be a finite pinching set of C. If $v \notin \tilde{V}$, then relint $(N_C(v)) \subset \tilde{V}$, by Proposition 1.4 (ii). If $v \in V_\infty(C)$, then relint $(N_C(v))$ is infinite (Proposition 1.4 (i)), hence relint $(N_C(v)) \not\subset \tilde{V}$. Thus $v \in V_\infty(C) \Longrightarrow v \in \tilde{V}$, i.e.,

$$V_\infty(C) \subset \tilde{V} \tag{1.1}$$

(hence $V_\infty(C)$ is finite). Next we show that

$$\partial C = \bigcup \{N_C(v) : v \in V_\infty(C)\}\,. \tag{1.2}$$

Denote the right hand side of (1.2) by N_C^∞. Clearly, $\partial C \supset N_C^\infty$. Hence it suffices to show that $(\partial C) \setminus (N_C^\infty) = \emptyset$. Assume, by r.a.a., that $(\partial C) \setminus (N_C^\infty) \neq \emptyset$. Since $N_C(v)$ is a closed set for $v \in \partial C$ (Proposition 1.4 (i)) and since $V_\infty(C)$ is finite (by (1.1)), N_C^∞ is a closed set (as a finite union of closed sets). It follows that $(\partial C) \setminus (N_C^\infty)$ is a relatively open non-empty set in ∂C, hence it is infinite. But according to the above alternative, $(\partial C) \setminus (N_C^\infty) \subset \{N(v)\,|v \in \tilde{V}, v \text{ has valence } 1\}$ which is finite. This gives a contradiction. □

It follows from (1.1) and (1.2) that we have the following

Corollary. *The set $V_\infty(C)$ is the unique minimal pinching set of ∂C.*

Next we show that $GD(V_\infty(C))$ is an odd cycle. The right hand side of (1.2) is a finite union of non-degenerate circular arcs of radii 1, each of length $\leq \frac{\pi}{3}$ (Proposition 1.4 (i)), whose relative interiors are mutually disjoint (Proposition 1.4 (iii)). By (1.2) this union is homeomorphic to S^1, and this happens (if and) only if, the arcs $N_C(v)(v \in V_\infty(C))$ form a closed concatenation, i.e., an endpoint of each arc is a common endpoint of exactly two arcs of the union. It follows that the endpoints of the arcs $N_C(v), v \in V_\infty(C)$, are in $V_\infty(C)$ as well, and that $GD(\partial C)|_{V_\infty(C)}$ is a (finite) 2-regular graph. Let $(v_1, \ldots, v_n, v_1)$ be a cycle in $GD(\partial C)\,|_{V_\infty(C)}$ (clearly, $n \geq 3$). At this point it is useful to introduce the following

Notation. For $v \in \mathbf{R}^2$ denote by $S(v, 1)$ the *boundary of the unit circle around* v, and for $u, w \in S(v, 1)$ put $S(v; u, w) = (S(v, 1)) \bigcap (\sphericalangle\, uvw)$, the *arc of length $\leq \pi$ on $S(v, 1)$, whose endpoints are u and w.*

Since the length of $N_C(v)$ is $\leq \frac{\pi}{3} < \pi$ for $v \in V_\infty(C)$, $N_C(v_i) = S(v_i; v_{i-1}, v_{i+1})$ for $1 \leq i \leq n$ (all indices are taken modulo n). By Proposition 1.4 (iii), $\bigcup_{i=1}^n N_C(v_i) (= \bigcup_{i=1}^n S(v_i; v_{i-1}, v_{i+1}) \subset \partial C)$ forms a closed concatenation on $\partial C \approx S^1$, hence $\bigcup_{i=1}^n N_C(v_i) = \partial C$ (since S^1 is not homeomorphic to any proper subset). This can be written as

$$\partial C = \bigcup_{i=1}^{n} S(v_i; v_{i-1}, v_{i+1}), \tag{1.3}$$

and it follows that $V_\infty(C) = \{v_1, \ldots, v_n\}$, i.e., $\#(V_\infty(C)) = n$, and that $GD(\partial C)\,|_{V_\infty(C)}$ is a cycle. It remains to show that n is odd. Since $n \geq 3$, it suffices to consider the case $n \geq 4$.

Since $[v_i, v_{i+1}] \in D(C)$ for $1 \leq i \leq n$, any two edges of the cycle $(v_1, \ldots, v_n, v_1)$ intersect, and if they have no common endpoint they intersect in their relative interiors (Proposition 1.1 and Lemma 1.2). In particular, every edge $[v_i, v_{i+1}], 3 \leq i \leq n-1$, intersects the line $L = v_1v_2$ in its relative interior. It follows that all vertices v_i with odd i and $3 \leq i \leq n$ are on the same open side, say H_L^+, of the line L, and those with $3 \leq i \leq n$ are on the other open side of L. Since $n \geq 3$, the edges $[v_2, v_3]$ and $[v_n, v_1]$ intersect in their relative interiors, and since $v_3 \in H_L^+$, the intersection belongs to H_L^+. Hence $v_n \in H_L^+$ and n is odd. The foregoing considerations may be summarized as follows.

Theorem 1.5. *Let $C \in \mathbf{R}^2$ be an RP, and put $n := \#V_\infty(C)$. Then we have the following statements:*

(i) *The number n is odd with $3 \leq n < \infty$.*
(ii) *The set C has a unique minimal pinching set which is identical with $V_\infty(C)$, where $V_\infty(C) \in VRP(n)$.*
(iii) *The graph $GD(\partial C)$ contains exactly one cycle, which is $GD(\partial C)\mid_{V_\infty(C)}$ (which is the same as $GD(V_\infty(C))$).*
(iv) *Put $GD(V_\infty(C))) = (v_1, \ldots, v_n, v_1)$; then (1.3) holds.*

Remark 1.6. (i) The set of vertices of a Reuleaux polygon C was defined in the introduction as the minimal pinching set of C. Theorem (1.5) (ii) establishes the correctness of this definition by identifying this set with $V_\infty(C)$.
(ii) Theorem 1.5 (iii) establishes (iii) $\Leftrightarrow$ (i) in Theorem 0.1, since $V_\infty(C) \in VRP(n)$ (by Theorem 1.5 (ii)).
(iii) It follows from Theorem 1.5 (iv) that the boundary of a Reuleaux polygon C is the finite union of circular arcs of radii 1, the centers of which are the vertices of C. This goes vice versa: If $C \in \mathbf{R}^2$ is a set of contant width 1, and if

$$\partial C = \bigcup_{i=1}^{n} A_i, \tag{1.4}$$

where $A_1, \ldots, A_n$ are circular arcs of radii 1 whose centers are $v_1, \ldots, v_n$, respectively, then C is an RP with $\{v_1, \ldots, v_n\}$ as its set of vertices.

Proof: By (1.4), ∂C is a piecewise smooth curve. Let $p \in \mathrm{relint} A_i$ for some i, $1 \leq i \leq n$, and let T_p be the tangent line to A_i at p. T_p supports C, and there is a diameter D_p of C incident with p (Proposition 1.3). Since C admits no supporting line through p, except for T_p, C admits no diameter through p, except for D_p and $D_p \perp T_p$. It follows that the line spanned by D_p passes through v_i (=the center of A_i), and since $\| D_p \| = 1$ and $\| v_i - p \| = 1$, D_p is incident with v_i. This shows that $\{v_1, \ldots, v_n\}$ is a finite pinching set of C, and that C is an RP. □

Thus, one can rephrase the definition of an RP of diameter 1 as a (planar) set of constant width 1 whose boundary is a finite union of circular arcs of equal radii 1. Sometimes, this short cut property is taken as the defining property of an RP (see, e.g., [5, p. 128]), but it is less adapted to our construction. The reader may readily check that a set $C \in \mathbf{R}^2$

of constant width is an RP iff its diameter graph $GD(\partial C)$ contains no more then a finite matching, and this may serve as yet another defining property of an RP. Planar sets of constant width, whose diameter graph contains at most a denumerable matching, seem to form a subclass which is worth of a separate investigation.

Proof of (ii) $\Leftrightarrow$ (iii) in Theorem 0.1: Assume that $V \in \mathbf{R}^2$ is a finite set of diameter 1 whose diameter graph $GD(V) = < V, D(V) >$ is hamiltonian, say $(v_1, \ldots, v_n, v_1)$ is a hamiltonian cycle in $GD(V)(\| v_{i+1} - v_i \| = 1$ for $1 \leq i \leq n)$. Let $C \subset \mathbf{R}^2$ be a set of constant width 1 such that $C \supset V$ (every planar set of diameter 1 is contained in a planar set of constant width 1; see [5, Theorem 54, p. 126]). By Proposition 1.4 (i), $S(v_i; v_{i-1}, v_{i+1}) \subset N_C(v_i) \subset \partial C$ for $1 \leq i \leq n$ and $V \subset V_\infty(C) \subset \partial C$. Thus

$$\bigcup_{i=1}^{n} S(v_i; v_{i-1}, v_{i+1}) \subset \partial C. \tag{1.5}$$

By Proposition 1.4 (iii), the left hand side of (1.5) is a simple closed curve, homeomorphic to S^1, which is homeomorphic to ∂C. It follows that (1.5) is an equality ($S^1 \approx \partial C$ is not homeomorphic to a proper subset) and that $V = \{v_1, \ldots, v_n\}$ is a (finite) pinching set of $C, V = V_\infty(C)$, and C is an RP whose vertex set is V (Theorem 1.5 (ii)). Thus n is odd and $V \in VRP(n)$. Moreover, (1.5) implies that C is the unique set of constant width 1 containing V. □

The following statements summarize these considerations.

Theorem 1.7. *Let $V \subset \mathbf{R}^2$ be a finite set of order n and diameter 1. Assume that the diameter graph $GD(V) = \langle V, D(V) \rangle$ is hamiltonian, and let $(v_1, \ldots, v_n, v_1)$ be a hamiltonian cycle in $GD(V)$. Then*

(i) *V is contained in a unique set $C \subset \mathbf{R}^2$ of constant width 1;*
(ii) *C is an RP, $V = V_\infty(C)$ is the set of vertices of C and $\partial C = \bigcup_{i=1}^{n} S(v_i; v_{i-1}, v_{i+1})$;*
(iii) *$n \geq 3$ is odd and $V \in VRP(n)$.*

Remark 1.8. This finishes the proof of Theorem 0.1. Namely, (i) $\Longrightarrow$ (ii) is trivial; (ii) $\Longrightarrow$ (iii) is Proposition 1.7 (iii), and (iii) $\Longrightarrow$ (i) was proved above (Remark 1.6 (ii)).

2. Straightforward Construction of an RP

For the sake of completeness and later comparison we give a short description of the obvious way to construct an RP. This turns out to be quadratic in time. The first part is a formality only.

Algorithm RP
(1) **read** n; **if** n is even **then write** "There is no RP of order n"; **stop**;
    {* now n is assumed to be odd *}
(2) **else** call ProcVRP(n);
(3) $C :=$ conv $(\bigcup_{i=1}^{n} S(v_i; v_{i-1}, v_{i+1}))$;
(4) **end**.

Comments. Obviously "quadratic time" refers essentially to the procedure ProcVRP(n), called for in line (2); this is presented and explained below. The output of this procedure is a set $V = \{v_1, \ldots, v_n\}$ of diameter 1 whose diameter graph $GD(V)$ is the cycle $(v_1, \ldots, v_n, v_1)$. Theorem 1.7 above ensures that $V \in VRP(n)$, that there is a unique RP C whose vertex set is V ($V = V_\infty(C)$), and that the boundary of C is given by $\partial C = \bigcup_{i=1}^{n} S(v_i; v_{i-1}, v_{i+1})$ (see in line (3) of the algorithm). Theorem 1.5 (iii) and (iv) above says that every RP C of order n and diameter 1 is of this form (for some $V \subset \mathbf{R}^2$ such that $GD(V)$ is a cycle). Hence also the correctness and exhaustiveness of the Algorithm RP entirely depends on that of its "backbone", ProcVRP.

We shall not discuss the correctness and exhaustiveness of ProcVRP in detail, because according to the goals of this paper, this will be only of interest for its linear time companion ProcVRPref. presented in the next section. Clearly the algorithm is obtained from the Algorithm RP by merely substituting ProcVRPref(n) for ProcVRP(n) (in line (2)), leading to the Algorithm RPref.

In the formulation of ProcVRP(n) we use the following

Notation. For $v \in \mathbf{R}^2$, denote by $B^0(v, 1)$ the *open disc* of radius 1 around v. For $u, v, w \in \mathbf{R}^2$ non-collinear, denote by $H(u, v; w)$ the *open half plane* bounded by uv which contains w.

Procedure ProcVRP(n) (n; odd integer ≥ 3)

(1) choose $v_1 :\in \mathbf{R}^2$ and $v_2 :\in S(v_1, 1)$;
(2) **if** $n = 3$ **then** choose $v_3 :\in (S(v_1, 1) \cap (S(v_2, 1))$; **stop**;
  {* the choice here is between two points *}
(3) **else** $m := 2$;
(4) **repeat**
(5) choose $v_{m+1} :\in \left(\bigcap_{i=1}^{m-1} B^0(v_i, 1)\right) \bigcap S(v_m, 1)$;
(6) $m := m + 1$;
(7) **until** $m := n - 1$;
(8) $v_n := (S(v_1, 1)) \bigcap (S(v_{m-1}, 1) \bigcap (H(v_1, v_{n-1}; v_2))$;
(9) **end**;

Remark. The procedure is a faithful realization of Scheme 0.3 to construct a set of points $V = \{v_1, \ldots, v_n\}$ whose diameter graph $GD(V)$ is the cycle $(v_1, \ldots, v_n, v_1)$. As suggested in Scheme 0.3, this is straightforwardly done as follows: v_1 and v_2 are initially chosen such that $\| v_2 - v_1 \| = 1$ (line (1)). If $n = 3$, then v_3 is chosen such that $\| v_3 - v_1 \| = \| v_3 - v_2 \| = 1$ and the procedure stops (line (2)). Note that there are two possible choices in this case, both leading to an equilateral triangle, resp. to a Reuleaux triangle.

Else, if $n > 3$ then v_3 is to be chosen at distance 1 from v_2 and at distance < 1 from v_1. This is performed in the first iteration of the loop $(4) - (7)$. Subsequent iterations of this loop pick up the points $v_4, v_5, \ldots, v_{n-1}$ sequentially such that, for $4 \leq m \leq n - 1$, v_m is at distance < 1 from $v_1, \ldots, v_{m-2}$ and at distance 1 from v_{m-1}. Line (5) is an inner loop within the main loop $(4) - (7)$. This is responsible for the quadratic time performance of the procedure. The choice of v_n in line (8) is unique now. It can be easily seen that this is the only choice from two possibilities which leads to a closure of the figure as required.

It is not obvious that the algorithm will not run into any deadlock. Hence for the sake of correctness one has to prove the following statements:

(a) *For* $2 \leq m \leq n-1$, *the intersection* $\bigcap_{i=1}^{m-2}(B^o(v_i,1)) \cap (S(v_{m-1},1))$ *in line* (5) *is non-empty, and*

(b) *not depending on the history of the process, the 'no other choice' determination of* v_n *in line* (8) *terminates the process gracefully, i.e., without violation of any of the conditions* $\| v_n - v_i \| < 1\,(2 \leq i \leq n-2)$.

This is a little tedious, but a rigorous proof of these statements can be given. Having in mind to present a linear time algorithm, this proof is left out here.

3. Linear-Time Construction of an RP

The linear-time construction of an RP reflects the relations developed above between RPs and full equi-intersectors. As indicated in the previous section the essential part of the corresponding algorithm will be the refined procedure ProcVRPref(n) for the construction of the vertex set. Because this is only of interest for odd n we set $n = 2k+1$.

Procedure ProcVRPref(n) ($k;\ k \geq 1,\ n = 2k+1$)
(1) choose $v_0 :\in \mathbf{R}^2$ and $v_1 :\in S(v_0, 1)$;
(2) **if** $k = 1$ **then** choose $v_2 :\in (S(v_0,1) \cap (S(v_1,1))$;
(3) **stop**;
{* the choice here is between two points *}
(4) **else**
(5) choose $v_2 :\in (B^0(v_0,1) \cap (S(v_1,1)) \setminus \{v_0\}$,
$p_1 := (S(v_0,1) \cap (S(v_1,1)) \cap H(v_0,v_1;v_2)$, $p_0 := (S(v_0,1) \cap (S(v_1,1)) \setminus \{p_1\}$,
$m := 1$;
(6) **repeat**
(7) $p_{2m} := S(v_{2m},1) \cap S(v_1; v_0, p_{2m-2})$,
choose $v_{2m+1} :\in S(v_{2m}; v_{2m-1}, p_{2m}) \setminus \{v_{2m-1}, p_{2m}\}$
(8) $p_{2m+1} := S(v_{2m+1},1) \cap S(v_0; v_1, p_{2m-1})$,
choose $v_{2m+2} :\in S(v_{2m+1}; v_{2m}, p_{2m+1}) \setminus \{v_{2m}, p_{2m+1}\}$
(9) $m := m+1$;
(10) **until** $m := k-1$;
(11) $p_{2k-2} := S(v_{2k-2},1) \cap S(v_1; v_0, p_{2k-4})$,
choose $v_{2k-1} :\in S(v_{2k-2}; v_{2k-3}, p_{2k-2}) \setminus \{v_{2k-3}, p_{2k-2}\}$
(12) $p_{2k-1} := S(v_{2k-1},1) \cap S(v_0; v_1, p_{2k-3})$, $v_{2k} := p_{2k-1}$
(13) **end**;

Justification. We have to check that there are no deadlocks in the construction and that the required distance conditions hold. Obviously v_0, v_1 and v_2 exist and we have according to our Procedure $\|v_0 - v_1\| = \|v_1 - v_2\| = \|v_2 - v_0\| = 1$ for $n = 3$ and $\|v_0 - v_1\| = \|v_1 - v_2\| = 1$, $\|v_2 - v_0\| < 1$ for $n > 3$. Also the existence of p_0 and p_1 is obvious. Furthermore, for a point x in the convex domain $C(p_0, p_1, p_0)$ bounded by the circular arcs $S(v_0; p_1, p_0)$ and $S(v_1; p_0, p_1)$ we have the inequalities $\|x - v_i\| \le 1$ for $i = 0, 1$ where $\|x - v_0\| = 1$ implies $x \in S(v_0; p_1, p_0)$ and $\|x - v_1\| = 1$ implies $x \in S(v_1; p_0, p_1)$.

Now we want to proceed by induction, but before describing the general step it will be instructive to consider the construction of v_3 and v_4 for $n \ge 5$: The choice of v_2 in the set $H^0(v_0, v_1; v_2) \cap B^0(v_0, 1) \cap B(v_1, 1)$ implies that $S(v_2, 1)$ enters $C(p_0, p_1, p_0)$ at v_1 in $H(v_0, v_1; p_0)$, and by $\|v_2 - p_0\| > 1 > \|v_2 - v_0\|$ its intersection with $C(p_0, p_1, p_0)$ remains totally in $C(p_0, p_1, p_0) \cap H(v_0, v_1; p_0)$, leaving this set again at the interior point p_2 of the circular arc $S(v_1; p_0, v_0)$. Hence it is obvious that $S(v_2, 1) \cap C(p_0, p_1, p_0) = S(v_2; v_1, p_2)$ and the intersection of this set with the boundary of $C(p_0, p_1, p_0)$ exactly happens at v_1 and p_2. For a point x in the convex domain $C(p_1, v_1, p_2, p_1) = B(v_2, 1) \cap C(p_0, p_1, p_0)$ we have the inequalities $\|x - v_i\| \le 1$ for $i = 0, 1, 2$ where equality is attained at the boundary arcs: $\|x - v_0\| = 1 \Leftrightarrow x \in S(v_0; p_1, v_1)$, $\|x - v_1\| = 1 \Leftrightarrow x \in S(v_1; p_2, v_1)$, $\|x - v_2\| = 1 \Leftrightarrow x \in S(v_2; v_1, p_2)$. Hence choosing v_3 as in line (7) of our Procedure we get $\|v_3 - v_i\| < 1$, $i = 0, 1$, and $\|v_3 - v_2\| = 1$.

The choice of v_3 in the set $H^0(v_1, v_2; v_3) \cap B^0(v_1, 1) \cap B(v_2, 1)$ implies that $S(v_3, 1)$ enters $C(p_1, v_1, p_2, p_1)$ at v_2 in $H(v_1, v_2; p_1)$, and by $\|v_3 - p_1\| > 1 > \|v_3 - v_1\|$ its intersection with $C(p_1, v_1, p_2, p_1)$ remains totally in $C(p_1, v_1, p_2, p_1) \cap H(v_1, v_2; p_1)$, leaving this set again at the interior point p_3 of the circular arc $S(v_0; p_1, v_1)$. Hence it is obvious that $S(v_3, 1) \cap C(p_1, v_1, p_2, p_1) = S(v_3; v_2, p_3)$ and the intersection of this set with the boundary of $C(p_1, v_1, p_2, p_1)$ exactly happens at v_2 and p_3. For a point x in the convex domain $C(p_2, v_2, p_3, v_1, p_2) = B(v_3, 1) \cap C(p_1, v_1, p_2, p_1)$ we have the inequalities $\|x - v_i\| \le 1$ for $i = 0, 1, 2, 3$ where equality is attained at the circular boundary arcs. Hence choosing v_4 as in line (8) of our Procedure we get $\|v_4 - v_i\| < 1$, $i = 0, 1, 2$, and $\|v_4 - v_3\| = 1$. There is only one exception in the case $n = 5$. Here we have to set $v_4 = p_3$ according to line (12) of our Procedure, and this implies $\|v_4 - v_i\| < 1$, $i = 1, 2$, and $\|v_4 - v_3\| = \|v_4 - v_0\| = 1$, as requested for $n = 5$.

Now we are ready to indicate the argument for the general step: Let us assume that the construction has proceeded successfully until v_{2m} which has been chosen in the interior of $S(v_{2m-1}; v_{2m-2}, p_{2m-1})$. The convex domain

$$C_{2m} := C(v_1, v_3, \ldots, v_{2m-3}, p_{2m-2}, v_2, v_4, \ldots, v_{2m-2}, p_{2m-1}, v_1)$$

bounded by the circular arcs

$S(v_2; v_1, v_3)$, $S(v_4; v_3, v_5)$, …, $S(v_{2m-2}; v_{2m-3}, p_{2m-2})$,
$S(v_1; p_{2m-2}, v_2)$, $S(v_3; v_2, v_4)$, …, $S(v_{2m-1}; v_{2m-2}, p_{2m-1})$, $S(v_0; p_{2m-1}, v_1)$

has the property that for $x \in C_{2m}$ $\|x - v_i\| \le 1$ for all $i = 0, \ldots, 2m-1$ such that equality is attained at the boundary arcs with respect to their corresponding centers. Again the choice of v_{2m} had implied $\|v_{2m} - v_i\| < 1$ for $i = 0, \ldots, 2m-2$, and $\|v_{2m} - v_{2m-1}\| = 1$ unless the construction of the RP has been finished by $2m + 1 = n$.

Then the location of v_{2m} in $H^0(v_{2m-2}, v_{2m-1}; v_{2m}) \cap C_{2m}$ again implies that the circle $S(v_{2m}, 1)$ enters the interior of C_{2m} at v_{2m-1} in $H^0(v_{2m-2}, v_{2m-1}; v_0)$, implying $\|p_{2m-2} - v_{2m}\| > 1$. Together with $\|v_0 - v_{2m}\| < 1$ we get that $C_{2m} \cap S(v_{2m}, 1)$ is the circular arc $S(v_{2m}; v_{2m-1}, p_{2m})$ with intersection point p_{2m} at $S(v_1; v_0, p_{2m-2})$ located in the interior of that arc. For any point x in the convex domain

$$C_{2m+1} := C_{2m} \cap B(v_{2m}, 1) = C(v_1, v_3, \ldots, v_{2m-3}, v_{2m-1}, p_{2m}, v_2, v_4, \ldots, v_{2m-2}, p_{2m-1}, v_1)$$

with the obvious circular arcs composing its boundary we get $\|x - v_i\| \leq 1$ for all $i = 0, \dots, 2m$ such that equality is attained only at the circular boundary arcs with respect to their corresponding centers. The choice of v_{2m+1} in the interior of the circular arc $S(v_{2m}; v_{2m-1}, p_{2m})$ according to line (7) of our Procedure leads to an additional vertex of the RP, because we have $\|v_{2m+1} - v_i\| < 1$ for $i = 0, \dots, 2m-1$, and $\|v_{2m+1} - v_{2m}\| = 1$.

The next step uses the same argument and therefore we do not need to write this down explicitly. Obviously no problems will occur with the optional final choice of v_{2m+2} according to line (12) of our Procedure in the case $2m + 3 = n$. This has been sufficiently explained in the case $n = 5$ above. Hence we have established the correctness of our Procedure. Its exhaustiveness is obvious.

4. Complete intersections characterize RPs

We finish this paper with a further interesting characterization of a set $V \in VRP(n)$. One of the known characterizations is that $V \in VRP(n)$ if and only if the diameter graph $GD(V) = \langle V, D(V)\rangle$ is a cycle (Theorem 0.1). Considering that every two edges of the diameter graph must intersect (in their relative interiors, unless they have a common vertex, by Lemma 1.2) one may ask for some sort of a reverse implication, namely: Is a cycle, all of whose edges have equal lenghts and any two of them intersect, necessarily the diameter graph of its vertices. An affirmative answer is given by

Theorem 4.1. *Let $G = \langle V, E\rangle$ be a geometric cycle of order $n \geq 3$ with edges having equal (Euclidean) lengths, such that any two edges intersect. Then*
(i) $\mathit{diam}\, V = 1$;
(ii) $V \in VRP(n)$;
(iii) $D(V) = E$;
(iv) $G = GD(V)$.

Remark: The theorem may be rephrased, using the notation of the introduction as follows: If $G = \langle V, E\rangle$ is a geometric cycle which is a full equi-intersector, then $G = GD(V)$. It can be shown that, in fact, for any full equi-intersector $G = \langle V, E\rangle$, $G = GD(V)$.

Notation for the subsequent proof: Let p, q, r be three non-collinear points in the plane. The straight line connecting p and q divides the plane into two open half-planes: one of them is $H(p, q; r)$ according to our previous notation, and the other will be called $\tilde{H}(p, q; r)$. Now we write $\mathrm{int}(\sphericalangle\, pqr) := H(p, q; r) \cap H(q, r; p)$ and $\mathrm{int}(\not\sphericalangle\, pqr) := \tilde{H}(p, q; r) \cap H(q, r; p)$, and in addition we introduce the point set $\Pi(p, r; q) := \mathrm{int}(\sphericalangle\, pqr) \cap \tilde{H}(p, r; q)$.

Proof: It suffices to prove (i) because (i) $\Rightarrow$ (ii) (by the implication (ii) $\Rightarrow$ (iii) in Theorem 0.1), (ii) $\Rightarrow$ (iii) (by the implication (iii) $\Rightarrow$ (i) in Theorem 0.1), and (iii) $\Rightarrow$ (iv) is obvious.

The case $n = 3$ of (i) is trivial. Assume that $n \geq 4$ (it follows easily that $n \geq 5$, since n must be odd, but we will not elaborate this point further). Assume that $G = (v_1, \dots, v_n, v_1)$, where $\| v_{i+1} - v_i \| = 1$ for $1 \leq i \leq n$ $(v_{n+1} := v_1)$ and $[v_i, v_{i+1}] \cap [v_j, v_{j+1}] \neq \emptyset$ for $1 \leq i, j \leq n$. Put $\triangle := \mathrm{diam}\, V (= \sup\{\| v_i - v_j \| : 1 \leq i, j \leq n\})$ and assume that $\triangle > 1$. This implies that there are two *non-adjacent* vertices in G whose (Euclidean) distance apart is $\triangle$, say, w.l.o.g., $\| v_3 - v_{i_0} \| = \triangle (> 1)$ (i.e., $[v_3, v_{i_0}] \in D(V)$) for some $i_0 \in \{1, \ldots, n\} \setminus \{2, 3, 4\}$.

If $v_{i_0} \in \mathrm{int}(\sphericalangle\, v_2 v_3 v_4)$, then $[v_{i_0}, v_{i_0+1}]$ would not intersect either $[v_2, v_3]$ or $[v_4, v_3]$, a contradiction. Hence $v_{i_0} \notin \mathrm{int}(\sphericalangle\, v_2 v_3 v_4)$, and similar arguments show that

$$v_i \in (\mathrm{int}(\sphericalangle\, v_2 v_3 v_4)) \cup (\mathrm{int}(\rotatebox[origin=c]{180}{$\sphericalangle$}\, v_2 v_3 v_4)), \tag{4.1}$$

$$v_i \notin (\mathrm{int}(\sphericalangle\, v_1 v_2 v_3)) \cup (\mathrm{int}(\rotatebox[origin=c]{180}{$\sphericalangle$}\, v_1 v_2 v_3)) \cup (\mathrm{int}(\sphericalangle\, v_3 v_4 v_5)) \cup (\mathrm{int}(\rotatebox[origin=c]{180}{$\sphericalangle$}\, v_3 v_4 v_5)) \tag{4.2}$$

and

$$\begin{aligned} v_i \notin & ((\mathrm{aff}(v_1, v_2)) \setminus [v_1, v_2]) \cup ((\mathrm{aff}(v_2, v_3)) \setminus [v_2, v_3]) \cup \\ & ((\mathrm{aff}(v_4, v_5)) \setminus [v_4, v_5]) \cup ((\mathrm{aff}(v_3, v_4)) \setminus [v_3, v_4]). \end{aligned} \tag{4.3}$$

Put $\{z\} := [v_1, v_2] \cap [v_4, v_5]$, $\{p\} := [v_3, v_4] \cap [v_1, v_2]$, and $\{q\} := [v_4, v_5] \cap [v_2, v_3]$ (Figure 4.1). (Possibly $q \in \{v_2, v_5\}$ or $p \in \{v_1, v_4\}$.) If $q = v_2$, then $z = q = v_2$, and if $p = v_4$, then $z = p = v_4$. Since $\| v_{i_0} - v_3 \| = \triangle > 1$, clearly

$$v_{i_0} \notin [v_2, v_3] \cup [v_4, v_3]. \tag{4.4}$$

Careful inspection of the regions blocked out for v_{i_0} by (4.1), (4.2), (4.3), and (4.4) ends up with *either* $v_{i_0} \in S_2 \cup [v_5, q[$, where we define $S_2 := \Pi(v_2, q; z)$ (if $v_2 = q (= z)$, then $S_2 = \mathrm{int}(\rotatebox[origin=c]{180}{$\sphericalangle$}\, v_1 z v_4)$), *or* $v_{i_0} \in S_4 \cup [v_4, p[$, where $S_4 := \Pi(v_4, p; z)$ (if $v_4 = p (= z)$, then $S_4 = \mathrm{int}(\rotatebox[origin=c]{180}{$\sphericalangle$}\, v_5 z v_2)$).

Figure 4.1

Since Figure 4.1 is combinatorially symmetric with respect to the line $v_3 z$, we may assume, w.l.o.g., that $v_{i_0} \in S_2 \cup [v_5, q[\subset \Pi(v_2, v_3; v_1)$. We have $\triangle = \| v_{i_0} - v_3 \| < \| v_{i_0} - v_1 \|$, a contradiction (since $\triangle = \mathrm{diam}\, V$). $\square$

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