Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 2, 329-336.

On the Generalized Stereographic Projection

Baki Karlığa

Department of Mathematics, Arts and Sciences Faculty,
Gazi University, 06500, Teknikokullar Ankara, Turkey

1. Introduction

Stereographic projection, originally, was discovered by Hipparch. When Claudius Ptolemy was describing the instrument for measuring coordinates of stars on the celestial sphere, so called astrolabe, it was utilized. In 1613, the astrolabe projection was called *stereographic projection* by d'Aguillon in [1]. Stereographic projection was applied to geographic maps and surfaces by Lambert([2]), Euler ([3], [4]), Lagrange ([5]), Gauss ([6]). The conformal disc model for hyperbolic space is obtained by using stereographic projection by Killing ([7], [8]) and Poincare ([9]). From this study, we may obtain the model of hyperbolic space in special case of $\nu = 1$. In addition to, it is well-known that stereographic projection is necessary to determine the inversion of sphere. *Inversion of sphere* was found by L. J. Magnus in 1831. It is also studied in detail by F. Vieta, R. Simson, J. Steiner, Kelvin. Stereographic projection has played a very important role in Weiner ([10]), Akutagawa and Nishikawa ([11]), Magid ([12]), Kobayashi ([13]).

The main aim of this paper is to give a possible generalization for stereographic projection of n-dimensional pseudosphere with ν-index, inversion of $(n-1)$-dimensional pseudosphere with ν-index and to obtain some geometrical results. We see that stereographic projection in [10], [11], [12], [13] is a special case of $\nu = 1, 2$ in our generalized stereographic projection. The basic definitions and background material required here may be found in O'Neill ([14]), Kreyszing ([15]), Schwerdtfeger ([16]), Smart ([17]), Reynolds ([18]).

2. The Generalized Stereographic Projection

The sets

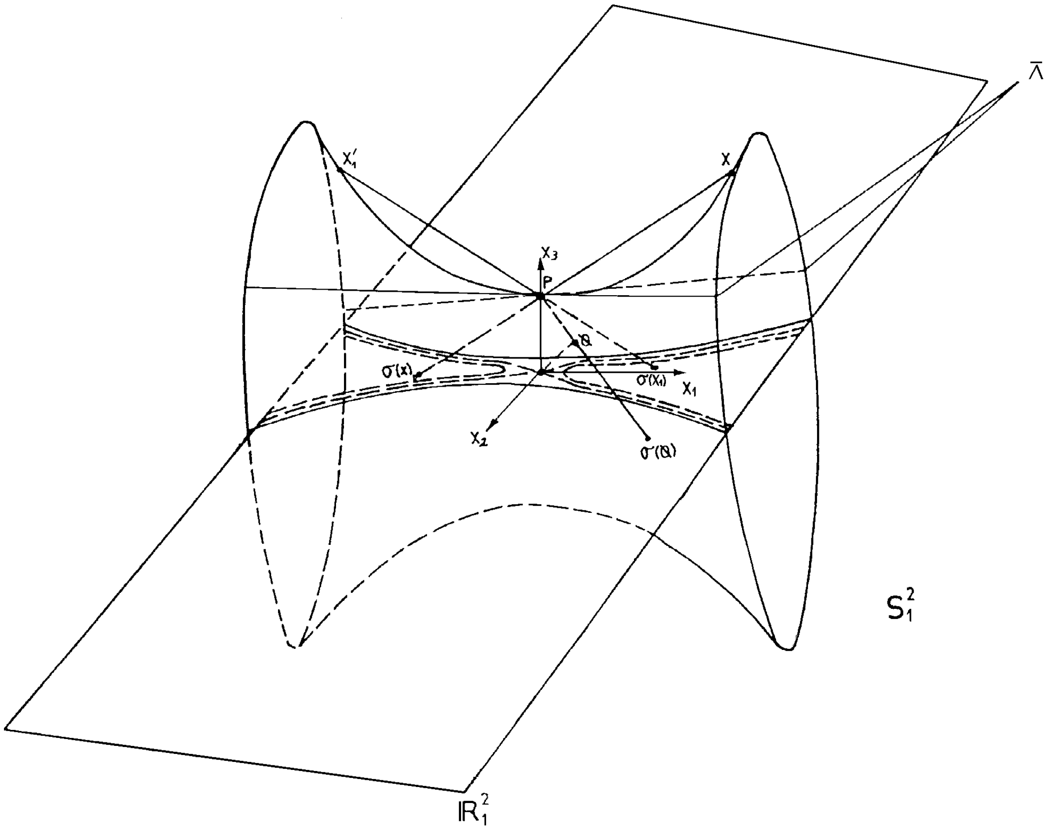
$$S^n_\nu(r) = \{x \in {R_\nu}^{n+1} \mid \langle x, x\rangle = r^2\}$$

and

$$H^n_\nu(r) = \{x \in {R_{\nu+1}}^{n+1} \mid \langle x, x\rangle = -r^2\}$$

are called *n-dimensional pseudosphere, pseudohyperbolic space of ν-index and radius* r, respectively [14].

Now, by using the following figure we shall try to define a map which is important to understand rest of the paper in special case $\nu = 1$. Of course it is true for the other cases.



For the general case, it can be easily seen that $p = (0, \ldots, 0, r) \in S_\nu{}^n(r)\overline{\Lambda} = \{x \in S_\nu{}^n(r) \mid x_{n+1} = r\}$, $x \in S_\nu{}^n(r) \setminus \overline{\Lambda}$, $\overline{x} = \sigma(x) \in \{\overline{x} \in R_\nu{}^{n+1} \mid \overline{x}_{n+1} = 0\}$. Thus we can write $\overrightarrow{o\overline{x}}=\overrightarrow{op} + \overrightarrow{p\overline{x}}$, $\overrightarrow{o\overline{x}}=\overrightarrow{op} +\lambda\ \overrightarrow{px}$. From the last equality we obtain that $\lambda = \frac{r}{r-x_{n+1}}$ and $\overline{x}_i = \frac{rx_i}{r-x_{n+1}}$, $1 \leq i \leq n$. Consequently, we can give a map

$$\sigma : S_\nu{}^n(r) \setminus \overline{\Lambda} \longrightarrow R_\nu{}^n \setminus H_{\nu-1}{}^{n-1}(r)$$

defined by

$$\sigma(x) = \frac{r}{r - x_{n+1}}(x_1, \ldots, x_n).$$

Theorem 2.1 *The map σ is one to one and onto.*

Proof. Let x, $y \in S_\nu{}^n(r) \setminus \overline{\Lambda}$. If $\sigma(x) = \sigma(y)$ then

$$\frac{r}{r - x_{n+1}}(x_1, \ldots, x_n) = \frac{r}{r - y_{n+1}}(y_1, \ldots, y_n)$$

and so

$$x_i = \frac{r - x_{n+1}}{r - y_{n+1}} y_i, \ (1 \le i \le n). \tag{1}$$

Since $x \in {S_\nu}^n(r) \setminus \overline{\Lambda}$, we have

$$\langle x, x \rangle = -\sum_{i=1}^{\nu} {x_i}^2 + \sum_{i=\nu+1}^{n} {x_i}^2 + {x_{n+1}}^2 = r^2$$

By (1), we have

$$\frac{r - x_{n+1}}{(r - y_{n+1})^2} \Big(-\sum_{i=1}^{\nu} {y_i}^2 + \sum_{i=\nu+1}^{n} {y_i}^2\Big) = r + x_{n+1}. \tag{2}$$

On the other hand, for $y \in {S_\nu}^n(r) \setminus \overline{\Lambda}$, we write

$$< y, y >= r^2$$

and

$$-\sum_{i=1}^{\nu} {y_i}^2 + \sum_{i=\nu+1}^{n} {y_i}^2 = r^2 - {y_{n+1}}^2. \tag{3}$$

If we substitute (3) in (2), we see that $x_{n+1} = y_{n+1}$ and by (1) $x_i = y_i, \ 1 \le i \le n+1$. This implies that σ is one to one. It is routine to check that σ is onto. This completes the proof. □

Theorem 2.2 *The map σ is a conformal map.*

Proof. By the definition of σ, it is easy to show that for $\overline{x} \in {R_\nu}^n \setminus {H_{\nu-1}}^{n-1}(r)$

$$\sigma^{-1}(\overline{x}) = \frac{r}{r^2 + \langle \overline{x}, \overline{x} \rangle} (2r\overline{x}_1, \ldots, 2r\overline{x}_n, \langle \overline{x}, \overline{x} \rangle - r^2).$$

Thus,

$$\frac{\partial {\sigma_\ell}^{-1}}{\partial \overline{x}_k} = \begin{cases} (2r^2(r^2 + \langle \overline{x}, \overline{x} \rangle) - 4\varepsilon_k {\overline{x}_k}^2 r^2)/(r^2 + \langle \overline{x}, \overline{x} \rangle)^2 & k = \ell \\ (-4\varepsilon_k \overline{x}_k \overline{x}_\ell r^2)/(r^2 + \langle \overline{x}, \ \overline{x} \rangle)^2 & k \ne \ell \end{cases} \tag{4}$$

and

$$\frac{\partial {\sigma_{n+1}}^{-1}}{\partial \overline{x}_k} = \frac{4\varepsilon_k \overline{x}_k r^3}{(r^2 + \langle \overline{x}, \overline{x} \rangle)^2}, \tag{5}$$

where $\varepsilon_i = \langle e_i, e_i \rangle, \ \sum_{i=1}^n \varepsilon_k {\overline{x}_i}^2 = \langle \overline{x}, \overline{x} \rangle, \ 1 \le k, \ \ell \le n$ and $\{e_1, \ldots, e_n\}$ is the standard frame of ${R_\nu}^n$. Let $\overline{x} \in {R_\nu}^n \setminus {H_{\nu-1}}^{n-1}(r)$ and $V_{\overline{x}}$ be a tangent vector of $T_{\overline{x}}{R_\nu}^n$. For the standard frame $\{\overline{e}_1, \ldots, \overline{e}_{n+1}\}$ of ${R_\nu}^{n+1}$, we have

$$\sigma_*^{-1}(V_{\overline{x}}) = \sum_{\ell=1}^{n+1} \varepsilon_\ell V_{\overline{x}}[\sigma_\ell^{-1}] \overline{e}_\ell. \tag{6}$$

From (4) and (5), we obtain

$$V_{\overline{x}}[\sigma_\ell^{-1}] = \frac{-4r^2 \overline{x}_\ell (\sum_{k=1}^n \overline{x}_k V_k(\overline{x}))}{(r^2 + \langle \overline{x}, \ \overline{x} \rangle)^2} + \frac{2r^2 \varepsilon_\ell V_\ell(\overline{x})}{(r^2 + \langle \overline{x}, \ \overline{x} \rangle)} \tag{7}$$

and

$$V_{\overline{x}}[\sigma_{n+1}^{-1}] = \frac{4r^3(\sum_{k=1}^{n} \overline{x}_k V_k(\overline{x}))}{(r^2 + \langle \overline{x},\ \overline{x}\rangle)^2}. \tag{8}$$

Substituting (7), (8) in (6), we have

$$\sigma_*{}^{-1}(V_{\overline{x}}) = \sum_{\ell=1}^{n} \overline{\varepsilon}_\ell \Big[\frac{-4r^2\overline{x}_\ell(\sum_{k=1}^{n} \overline{x}_k V_k(\overline{x}))}{(r^2 + \langle \overline{x}, \overline{x}\rangle)^2} + \frac{2r^2\varepsilon_\ell V_\ell(\overline{x})}{(r^2 + \langle \overline{x}, \overline{x}\rangle)}\Big]\overline{e}_\ell + \overline{\varepsilon}_{n+1}\Big[\frac{4r^3(\sum_{k=1}^{n} \overline{x}_k V_k(\overline{x}))}{(r^2 + \langle \overline{x}, \overline{x}\rangle)^2}\Big]\overline{e}_{n+1}. \tag{9}$$

Taking into account the fact that $\overline{\varepsilon}_\ell = \varepsilon_\ell$ for $1 \leq \ell \leq n$, by (9), we obtain

$$\sigma_*{}^{-1}(V_{\overline{x}}) = \Big[\frac{-4r^2(\sum_{k=1}^{n} \overline{x}_k V_k(\overline{x}))}{(r^2 + \langle \overline{x}, \overline{x}\rangle)^2}\Big] \sum_{\ell=1}^{n} \varepsilon_\ell \overline{x}_\ell \overline{e}_\ell +$$

$$+ \frac{2r^2}{(r^2 + \langle \overline{x}, \overline{x}\rangle)} \sum_{\ell=1}^{n} V_\ell(\overline{x})\overline{e}_\ell + \overline{\varepsilon}_{n+1}\Big[\frac{4r^3(\sum_{k=1}^{n} \overline{x}_k V_k(\overline{x}))}{(r^2 + \langle \overline{x}, \overline{x}\rangle)^2}\Big]\overline{e}_{n+1}. \tag{10}$$

If we apply the same arguments to $W_{\overline{x}} \in T_{\overline{x}}R_\nu{}^n$, we see that

$$\sigma_*{}^{-1}(W_{\overline{x}}) = \Big[\frac{-4r^2(\sum_{k=1}^{n} \overline{x}_k W_k(\overline{x}))}{(r^2 + \langle \overline{x}, \overline{x}\rangle)^2}\Big] \sum_{\ell=1}^{n} \varepsilon_\ell \overline{x}_\ell \overline{e}_\ell +$$

$$+ \frac{2r^2}{(r^2 + \langle \overline{x}, \overline{x}\rangle)} \sum_{\ell=1}^{n} W_\ell(\overline{x})\overline{e}_\ell + \overline{\varepsilon}_{n+1}\Big[\frac{4r^3(\sum_{k=1}^{n} \overline{x}_k W_k(\overline{x}))}{(r^2 + \langle \overline{x}, \overline{x}\rangle)^2}\Big]\overline{e}_{n+1}. \tag{11}$$

Using (10) and (11) we have

$$\langle \sigma_*{}^{-1}(V_{\overline{x}}), \sigma_*{}^{-1}(W_{\overline{x}})\rangle = \frac{4r^4}{(r^2 + \langle \overline{x}, \overline{x}\rangle)^2}\langle V_{\overline{x}},\ W_{\overline{x}}\rangle. \tag{12}$$

This implies that σ is a conformal map. □

Corollary 2.3 *If $\nu = 0$, σ reduces to usual stereographic projection of $S^n(r)$.*

Proof. Since $S_0{}^n(r) = S^n(r)$, $\overline{\Lambda} = \{(0, \ldots, 0, r)\}$ and $H_{-1}^{n-1}(r) = \emptyset$, we show that

$$\sigma : S^n(r) \setminus \{(0, \ldots, 0, r)\} \longrightarrow R_0{}^n = E^n$$

$$\sigma(x) = \frac{r}{r - x_{n+1}}(x_1, \ldots, x_n),$$

$$\sigma^{-1}(\overline{x}) = \frac{r}{r^2 + \parallel \overline{x} \parallel^2}(2r\overline{x}_1, \ldots, 2r\overline{x}_n,\ \parallel \overline{x} \parallel^2 - r^2).$$ □

Consequently, we may give the following definition.

Definition 2.4 *The map*

$$\sigma : S_\nu{}^n(r) \setminus \overline{\Lambda} \longrightarrow R_\nu{}^n \setminus H_{\nu-1}{}^{n-1}(r)$$

is called the generalized stereographic projection of $S_\nu{}^n(r)$, $0 \leq \nu \leq n$.

Corollary 2.5 *For $\nu = n$, the form of σ is*

$$\sigma : {S_n}^n(r) \setminus \{(0,0,\ldots,r)\} \longrightarrow {R_n}^n \setminus S^{n-1}(r).$$

Corollary 2.6 *The following statements are valid for the generalized stereographic projection:*

(i) $x_{n+1} > r$ *iff* $\sigma(x) \in \{\overline{x} \in {R_\nu}^n \mid \langle \overline{x}, \overline{x} \rangle < -r^2\},\ x \in {S_\nu}^n(r) \setminus \overline{\Lambda}$,

(ii) $x_{n+1} < r$ *iff* $\sigma(x) \in \{\overline{x} \in {R_\nu}^n \mid \langle \overline{x}, \overline{x} \rangle > -r^2\},\ x \in {S_\nu}^n(r) \setminus \overline{\Lambda}$.

Proof.
(i) If $x_{n+1} > r$, $r(\langle \overline{x}, \overline{x} \rangle - r^2)/(\langle \overline{x}, \overline{x} \rangle + r^2) > r$, $-2r/(\langle \overline{x}, \overline{x} \rangle + r^2) > 0$, and so $\langle \overline{x}, \overline{x} \rangle + r^2 < 0$. It follows that $\langle \sigma(x), \sigma(x) \rangle < -r^2$. Conversely, if $\langle \sigma(x), \sigma(x) \rangle < -r^2$, then

$$x_{n+1} = \frac{r\ (\langle \sigma(x), \sigma(x) \rangle - r^2)}{\langle \sigma(x), \sigma(x) \rangle + r^2} > r.$$

(ii) Similarly, if $x_{n+1} < r$, we find $\langle \sigma(x), \sigma(x) \rangle > -r^2$. Conversely, if $\langle \sigma(x), \sigma(x) \rangle > -r^2$, then

$$x_{n+1} = \frac{r\ (\langle \sigma(x), \sigma(x) \rangle - r^2)}{\langle \sigma(x), \sigma(x) \rangle + r^2} < r. \qquad \square$$

Corollary 2.7 *Let $\nu = n$, $x \in {S_n}^n(r) \setminus \{(0,0,\ldots,r)\}$ and B be the open ball $\{x \in R^n \mid \| x \| < r\}$. Then*

(i) $x_{n+1} > r$ *iff* $\sigma(x) \in R^n \setminus B$,

(ii) $x_{n+1} < r$ *iff* $\sigma(x) \in B$.

Proof. It follows from Corallary 2.5 and Corollary 2.6. $\square$

Corollary 2.8 *Let x be any element of ${S_\nu}^n(r)$. Then $x_{n+1} = -r$ iff $\sigma(x)$ is a null point of ${R_\nu}^n \setminus {H_{\nu-1}}^{n-1}(r)$.*

Proof. If $x \in {R_\nu}^n \setminus \overline{\Lambda}$ and $x_{n+1} = -r$, then we have

$$\langle x, x \rangle = r^2, \qquad -\sum_{i=1}^{\nu} {x_i}^2 + \sum_{i=\nu+1}^{n} x_i^2 = 0. \tag{13}$$

By definition of σ, $\sigma(x) = (x_1, x_2, \ldots, x_n)/2$ and it follows from (13) that $\langle \sigma(x), \sigma(x) \rangle = 0$. Thus $\sigma(x)$ is a null point of ${R_\nu}^n \setminus {H_{\nu-1}}^{n-1}(r)$.

Conversely let $\sigma(x)$ be a null point of ${R_\nu}^n \setminus {H_{\nu-1}}^{n-1}(r)$, then $\langle \sigma(x), \sigma(x) \rangle = 0$ and it is not difficult to show that $x_{n+1} = -r$. $\square$

3. The Generalized Inversion

Let σ be the generalized stereographic projection, ϕ be a *semi-Euclidean reflection of ${R_\nu}^{n+1}$ with respect to the plane* $x_{n+1} = 0$ defined in [19] and Λ be the null cone of ${R_\nu}^n$. The following theorem enables us to define inversion of ${S_\nu}^{n-1}(r)$.

Theorem 3.1 *For $0 \leq \nu \leq n-1$, let Ψ be the map*

$$\Psi = \sigma \circ \phi \circ \sigma^{-1} : {R_\nu}^n \setminus ({H_{\nu-1}}^{n-1}(r) \cup \Lambda) \longrightarrow {R_\nu}^n \setminus ({H_{\nu-1}}^{n-1}(r) \cup \Lambda)$$

defined by

$$\Psi(\overline{x}) = \frac{r^2 \overline{x}}{\langle \overline{x}, \overline{x} \rangle}.$$

Then $S_\nu^{n-1}(r)$ is invariant under the mapping Ψ.

Proof. If $\overline{x} \in {S_\nu}^{n-1}(r)$ then we have $\langle \overline{x}, \overline{x} \rangle = r^2$ and by definition of Ψ, $\Psi(\overline{x}) = \overline{x}$. Thus

$$\langle \Psi(\overline{x}), \Psi(\overline{x}) \rangle = \langle \overline{x}, \overline{x} \rangle = r^2,$$

as required. □

Now we give the following definition

Definition 3.2 *The map in Theorem 3.1 is called Generalized Inversion with respect to pseudosphere ${S_\nu}^{n-1}(r)$.*

Corollary 3.3 *If $\nu = 0$, then the mapping Ψ coincides with the inversion in E^n with respect to sphere $S^{n-1}(r)$.*

Corollary 3.4 *The spacelike and timelike points of ${R_\nu}^n \setminus ({H_{\nu-1}}^{n-1}(r) \bigcup \Lambda)$ are invariant under the mapping Ψ.*

Corollary 3.5 *Ψ maps the sets $\{\overline{x} \in {R_\nu}^n \mid \langle \overline{x}, \overline{x} \rangle < -r^2\}$, $\{\overline{x} \in {R_\nu}^n \mid -r^2 < \langle \overline{x}, \overline{x} \rangle < 0\}$ onto $\{\overline{x} \in {R_\nu}^n \mid \langle \overline{x}, \overline{x} \rangle > -r^2\}$ and $\{\overline{x} \in {R_\nu}^n \mid \langle \overline{x}, \overline{x} \rangle < -r^2\}$, respectively.*

Proof. If $\langle \overline{x}, \overline{x} \rangle < -r^2$, we obtain $\langle \Psi(\overline{x}), \Psi(\overline{x}) \rangle > -r^2$. Since $-r^2 < \langle \overline{x}, \overline{x} \rangle < 0$, we find $\langle \Psi(\overline{x}), \Psi(\overline{x}) \rangle < -r^2$. □

Corollary 3.6 *Ψ maps the sets $\{\overline{x} \in {R_\nu}^n \mid 0 < \langle \overline{x}, \overline{x} \rangle < r^2\}$, $\{\overline{x} \in {R_\nu}^n \mid r^2 < \langle \overline{x}, \overline{x} \rangle\}$ onto $\{\overline{x} \in {R_\nu}^n \mid \langle \overline{x}, \overline{x} \rangle > r^2\}$, $\{\overline{x} \in {R_\nu}^n \mid \langle \overline{x}, \overline{x} \rangle < r^2\}$, respectively.*

Proof. This follows from Corollary 3.5. □

Theorem 3.7 *Let Ψ be the Generalized Inversion of ${S_\nu}^{n-1}(r)$. Then $\Psi = \Psi^{-1}$.*

Proof. Since $\phi = \phi^{-1}$ and $\Psi = \sigma \circ \phi \circ \sigma^{-1}$, we have $\Psi = \Psi^{-1}$. □

4. The Generalized Stereographic Projection for Hyperbolic Spaces

In this section we show that stereographic projection for hyperbolic spaces will be obtained by using anti-isometries defined in [14].

Lemma 4.1 *The mapping $\mu : {R_\nu}^{n+1} \longrightarrow {R_{n-\nu+1}}^{n+1}$ given by*

$$\mu(x_1, x_2, \ldots, x_\nu, x_{\nu+1}, \ldots, x_{n+1}) = (x_{\nu+1}, \ldots, x_{n+1}, x_1, \ldots, x_\nu)$$

is an anti-isometry that carries each ${S_\nu}^n(r)$ anti-isometrically onto ${H_{n-\nu}}^n(r)$ and vice versa.

Proof. It may be seen in [14] p. 110. □

Theorem 4.2 *Let σ, μ be as defined earlier and $\overset{\sim}{\mu}^{-1}$ be the restriction of μ^{-1} on $H_{n-\nu}{}^n(r)$. Then the mapping*

$$\sigma \circ \overset{\sim}{\mu}^{-1} : H_{n-\nu}{}^n(r) \setminus \overline{\overline{\Lambda}} \longrightarrow R_\nu{}^n \setminus S_\nu{}^{n-1}(r)$$

defined by

$$\sigma \circ \overset{\sim}{\mu}^{-1}(x) = \frac{r}{r - x_{n-\nu+1}} \left(x_{n-\nu+2}, \ldots, x_{n+1}, x_1, \ldots, x_{n-\nu} \right)$$

is a conformal mapping, where $\overline{\overline{\Lambda}} = \{x \in H_{n-\nu}{}^n(r) \mid x_{n-\nu+1} = r\}$.

Proof. By Lemma 4.1 and Theorem 2.2, $\overset{\sim}{\mu}^{-1}$ and σ are conformal mappings. Thus $\sigma \circ \overset{\sim}{\mu}^{-1}$ is a conformal mapping. □

Let be $\sigma \circ \overset{\sim}{\mu}^{-1} = \tau$, $\overset{\sim}{\phi}$ a *Semi-Euclidean Reflection* with respect to $x_{n-\nu+1} = 0$ and $\overline{\Psi} = \tau \circ \overset{\sim}{\phi} \circ \tau^{-1}$. Then we see that

$$\overline{\Psi} : R_\nu{}^n \setminus (S_\nu{}^{n-1}(r) \cup \Lambda) \longrightarrow R_\nu{}^n \setminus (S_\nu{}^{n-1}(r) \cup \Lambda)$$

and

$$\overline{\Psi}(\overline{x}) = \frac{-r^2 \overline{x}}{\langle \overline{x}, \overline{x} \rangle}.$$

The maps τ, $\overline{\Psi}$ are called *generalized stereographic projection of $H_{n-\nu}{}^n(r)$* and *generalized inversion of $H_{\nu-1}{}^{n-1}(r)$*. Thus, we can give an analogue of all theorems and corollaries in section 2 and section 3.

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Received January 9, 1995