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# Centrally-symmetric Tight Surfaces and Graph Embeddings

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**Abstract.** We prove a sharp upper bound for the substantial codimension of a centrally-symmetric tight polyhedral surface in Euclidean space. This is related to embeddings of the edge graph of the $m$-octahedron into surfaces.

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## 1. Introduction

Generalizing the concept of convexity, a compact connected surface $M$ (or a 2-dimensional manifold) without boundary, embedded into Euclidean space $\mathbf{E}^d$, is called *tight* if $M \cap h$ is connected for any open or closed half space $h \subset \mathbf{E}^d$. It is called *substantial* if it is not contained in any hyperplane. It is called *centrally-symmetric* if $M$ coincides with $(-M)$, the reflection of $M$ at the origin $0 \in \mathbf{E}^d$.

A smooth surface in $\mathbf{E}^d$ is tight if and only if it has the *minimum total absolute curvature* $TA(M) = 4 - \chi(M)$, and a smooth tight surface in $\mathbf{E}^d$ cannot be substantial if $d \geq 6$. For $d = 5$ the only possibility is the Veronese surface (up to projective transformations of $\mathbf{E}^5$), see [14], [15]. A conjecture of W. Pohl says that for $d \geq 6$ the convex hull of any tight and substantial surface $M \subset \mathbf{E}^d$ is a polyhedron. Polyhedral tight surfaces exist in arbitrarily high codimension, as originally found in [1]. This is closely related with the embeddability of the complete graph into $M$ which in turn is essentially the Heawood map color problem, see [3], [11].

For centrally-symmetric surfaces the situation is similar: Since the Veronese surface is not centrally-symmetric any smooth tight and centrally-symmetric surface lies in some 4-space. Examples are the Clifford torus $S^1 \times S^1 \subset S^3 \subset \mathbf{E}^4$ and certain modified versions with arbitrarily many handles. In this paper we analyze the situation for centrally-symmetric polyhedral surfaces. It turns out that – at least in the simplicial case – this is closely related with the embeddability of the edge graph of the $d$-dimensional octahedron (the dual of the cube). This is nothing but the *complete graph* $K_{2d}$ on $2d$ vertices with $d$ pairwise disjoint edges (= the diagonals) removed.

## 2. Tight polyhedral surfaces in higher codimension

This section is a brief review about results on tight polyhedral surfaces, in particular the connection with the Heawood map color problem. One important aspect is the relationship between a polyhedral surface and the edge graph of its convex hull as follows:

**Lemma 2.1.** (i) ([1]) *Let $M \subset \mathbf{E}^d$ be a tight polyhedral surface. Then $M$ contains the 1-skeleton of its convex hull.*

(ii) *If a polyhedral surface $M$ is a subcomplex of the 2-skeleton of a convex polytope $P$, then $M$ is tight if and only if $M$ contains the 1-skeleton of $P$.*

(iii) ([6, 11.1]) *The 1-skeleton of any convex $d$-polytope contains the complete graph $K_{d+1}$ as a subset (not necessarily as a subgraph).*

**Corollary 2.2.** ([2]) *Let $f : S^2 \to \mathbf{E}^d$ be a tight and substantial polyhedral embedding. Then $d = 3$, and $f(S^2)$ is the boundary of a convex 3-polytope.*

The proof uses essentially the fact that the complete graph $K_5$ is not planar, so it does not embed into $S^2$. Therefore $f$ cannot be substantial in $\mathbf{E}^k$, $k \geq 4$.

**Theorem 2.3.** ([11]) *For an abstract surface $M$ and a number $n \geq 6$, the following conditions are equivalent:*

(i) *There exists a tight and substantial polyhedral embedding $M \to \mathbf{E}^{n-1}$.*

(ii) *There exists an embedding $K_n \to M$.*

2.3 together with the Map Color Theorem [19] implies the following:

**Corollary 2.4.** ([11, Theorem A]) *Let $M$ be an abstract surface which is not topologically equivalent to the Klein bottle. Then there exists a tight substantial polyhedral embedding $M \to \mathbf{E}^d$ if and only if*

(i) *$d \geq 3$ (if $M$ is orientable) or $d \geq 4$ (if $M$ is nonorientable), and*

(ii) *$\binom{n-3}{2} \leq 3(2 - \chi(M))$ or, equivalently, $d \leq \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)})$.*

*The Klein bottle can be embedded tightly and substantially into $\mathbf{E}^4$ and $\mathbf{E}^5$ but not into $\mathbf{E}^k$, $k \geq 6$.* (ii) *is known as Heawood's inequality.*

A *triangulation* of a surface is a simplicial decomposition into triangles. In particular, for any two vertices there can be at most one edge joining them.

**Theorem 2.5.** (Minimal triangulations of surfaces, [17], [10]) *Let $M$ be an abstract surface which is distinct from the Klein bottle, from the orientable surface of genus 2, and from the surface with $\chi = -1$. Then the following conditions are equivalent:*

(i) *There exists a triangulation of $M$ with $n$ vertices.*

(ii) $\binom{n-3}{2} \geq 3(2-\chi(M))$.

In the three exceptional cases the left hand side of (ii) has to be replaced by $\binom{n-4}{2}$, see [7].

A triangulated surface (or any simplicial complex) with $n$ vertices can always be regarded as a subcomplex of an $(n-1)$-dimensional simplex $\triangle^{n-1}$. We call this the *canonical embedding* of the triangulation. An abstract triangulation of a surface is called *tight* if its canonical embedding is tight.

**Corollary 2.6.** (Regular cases, triangular embeddings of $K_n$, tight triangulations) *Let $M$ be a triangulated surface with $n$ vertices. Then the following conditions are equivalent:*

(i) *The triangulation is tight.*

(ii) *The triangulation is 2-neighborly, i.e., its edge graph is a complete graph $K_n$.*

(iii) $\binom{n-3}{2} = 3(2-\chi(M)) = 6g \quad (g = \textit{genus of } M)$.

*Conversely, given an abstract surface (not topologically equivalent to the Klein bottle) and a number $n$ satisfying* (iii)*, then there is a tight triangulation of $M$ with $n$ vertices.*

**Example 2.7.** The following series of examples is given in [19, Fig. 2.15] as a triangular embedding of the graph $K_{12s+7}$ into the orientable surface of genus $\frac{1}{6}\binom{12s+4}{2}$. The vertices are the elements of $\mathbf{Z}_{12s+7}$, the triangles are the orbits under the action of this group of the following triangles ($\overline{x}$ denotes the element $-x$ in $\mathbf{Z}_{12s+7}$):

$$
\begin{array}{cc}
\multicolumn{2}{c}{0(2s+1)(5s+3)} \\
0(5s+3)(5s+4) & 0(\overline{3s+2})(\overline{3s+3}) \\
0(5s+4)(5s+2) & 0(\overline{3s+3})(\overline{3s+1}) \\
0(5s+2)(5s+5) & 0(\overline{3s+1})(\overline{3s+4}) \\
0(5s+5)(5s+1) & 0(\overline{3s+4})(\overline{3s}) \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
0(4s+4)(6s+3) & 0(\overline{2s+3})(\overline{4s+2}) \\
0(6s+3)(4s+3) & 0(\overline{4s+2})(\overline{2s+2}) \\
\multicolumn{2}{c}{0(4s+3)(2s+2)}
\end{array}
$$

In the case $s=0$ we get the unique 7-vertex triangulation of the torus as the union of the $\mathbf{Z}_7$-orbits of the triangles 013 and 032. (The middle part of the list above does not exist in this case.)

**Theorem 2.8.** ([12]) *Let $P$ be a convex $d$-polytope containing a surface $M$ in its 2-skeleton such that $M$ contains the 1-skeleton of $P$. Then the following hold:*

(i) $\binom{d-2}{2} \leq 3(2-\chi(M))$.

(ii) *If $d \geq 4$ then equality in* (i) *holds if and only if $P$ is a simplex.*

The same assertion holds for any tight polyhedral and substantial embedding by the Banchoff-Pohl theorem. The smallest $d$-polytope (= the $d$-simplex) contains tight surfaces of maximal genus. For analogues of 2.8 in the case of higher dimensional manifolds see the generalized Heawood inequalities in [13].

## 3. The centrally-symmetric case

The *d-octahedron* or *d-dimensional cross-polytope* is defined as the convex hull of the $2d$ points $(0,\dots,\pm 1,\dots,0)$ in $\mathbf{E}^d$. It is the dual of the $d$-cube which is the convex hull of the $2^d$ points $(\pm 1,\dots,\pm 1)$.

**Examples 3.1.** 1. The square torus $\{4\}\times\{4\} = \{4,4\}_{4,0}$ (Coxeter's regular map) with 16 vertices and 16 squares is a subcomplex of the 4-cube, containing all edges of the cube. It is therefore tight in $\mathbf{E}^4$ by 2.1 (ii). This torus decomposes the boundary of the 4-cube into two solid tori, each tightly embedded.

2. The $\mathbf{Z}_8$-orbit of the two triangles 013 and 023 can be regarded as a subcomplex of the 4-octahedron with diagonals 04, 15, 26, 37. It is tight by the same argument as in 1. It decomposes the boundary complex into two solid tori: the $\mathbf{Z}_8$-orbit of 0123 and the $\mathbf{Z}_8$-orbit of 0136. This decomposition is preserved by the 'rotation' $x \mapsto 3x \pmod 8$. The central symmetry is the automorphism $x \mapsto x+4 \pmod 8$.

3. If we remove two antipodal triangles from the boundary of an octahedron, we obtain a centrally-symmetric tight cylinder in $\mathbf{E}^3$. Now take two copies in two parallel hyperplanes of $\mathbf{E}^4$ and join their boundaries by a straight cylinder. This leads to a centrally-symmetric tight torus in $\mathbf{E}^4$ as a subcomplex of the prism over an octahedron. Again the tightness follows from 2.1 (ii). For other examples as subcomplexes of prisms see [20].

4. Surfaces of genus $g = 2^{d-3}(d-4)+1$ in the 2-skeleton of the $d$-cube, containing the entire 1-skeleton of the $d$-cube, tight and centrally-symmetric, have been described by several authors independently, see [5], [1], [4], [18]. For polyhedral realizations of these examples in $\mathbf{E}^3$ see [16].

**Theorem 3.2.** *Assume that $M \subset \mathbf{E}^d$ is a tight polyhedral surface whose convex hull is a centrally-symmetric simplicial d-polytope. Then the following hold:*

(i) $2(d-1)(d-3) \le 3(2-\chi(M))$.

(ii) *For $d \ge 4$ equality in (i) holds if and only if $M$ is a subcomplex of the d-octahedron with $n = 2d$ vertices (up to affine transformations).*

(ii) does not hold for $d = 3$ as can be seen from the icosahedron.

*Proof:* (i) By assumption the convex hull $\mathcal{H} = \mathcal{H}(M)$ is a centrally-symmetric simplicial polytope with $f_0 \ge 2d$ vertices. For the number $f_1$ of edges the following inequality is known as part of R. Stanley's Lower Bound Theorem [22]: $f_1 \ge d(f_0-2) \ge 4\binom{d}{2}$. By the tightness we have $Sk_1(\mathcal{H}) \subset M$ according to 2.1. Let $c_i$ denote the number of components in the complement $M \setminus Sk_1(\mathcal{H})$ which are abstract $i$-gons. Then we conclude

$$\begin{aligned}
2-\chi(M) &= 2 - f_0 + f_1 - \chi(M \setminus Sk_1(\mathcal{H})) \ge 2 - f_0 + f_1 - \textstyle\sum_{i\ge 3} c_i \\
&\ge 2 - f_0 + f_1 - \tfrac{2}{3} f_1 \ge 2 - f_0 + \tfrac{1}{3} d(f_0 - 2) \\
&\ge 2 + 2d(\tfrac{d}{3} - 1) - \tfrac{2d}{3} \\
&= \tfrac{2}{3}(d-1)(d-3).
\end{aligned}$$

(ii) If $d \geq 4$ then equality in (i) implies $f_0 = 2d, f_1 = d(f_0 - 2) = 4\binom{d}{2}$ and that each component of $M \setminus Sk_1(\mathcal{H})$ is homeomorphic to a disc, bounded by three edges of $Sk_1(\mathcal{H})$. An elementary argument shows that any centrally-symmetric $d$-polytope with $2d$ vertices is affinely equivalent to the $d$-dimensional cross-polytope because it is the convex hull of two antipodal facets with $d$ vertices each. By the tightness each component of $M \setminus Sk_1(\mathcal{H})$ is a planar triangle. Thus $M$ is a subcomplex of the 2-skeleton of its convex hull.

For $d = 3$ the boundary of any centrally-symmetric simplicial polytope is an example, e.g., the icosahedron.

For generalizations of 3.2 to the case of higher dimensions see [21].

The following theorem is a centrally-symmetric counter-part of 2.5.

**Theorem 3.3.** *Let $M$ be a surface, triangulated with $n$ vertices and with a fixed-point free involution $\sigma$ acting on the triangulation. Then $n = 2m$ is even and the inequality $2(m-1)(m-3) \geq 3(2 - \chi(M))$ holds with equality if and only if the edge graph of the triangulation is isomorphic to the edge graph of the $m$-octahedron. Moreover, in the case of equality the triangulation can be regarded as a subcomplex of the $m$-octahedron.*

*Proof:* By assumption $\sigma$ acts as a permutation of the vertices. On the one hand it follows that $\sigma(x) \neq x$ for any vertex $x$, on the other hand $x$ is not joined with $\sigma(x)$ by an edge because otherwise the centre of the edge would be a fixed point of $\sigma$. In particular $n = 2m$ is even and the number of edges satisfies $f_1 \leq \binom{n}{2} - m$. This implies

$$2 - \chi(M) = 2 - f_0 + f_1 - f_2 = 2 - 2m + \tfrac{1}{3} f_1 \leq 2 - 2m + \tfrac{1}{3}(\tbinom{2m}{2} - m) = \tfrac{2}{3}(m-1)(m-3).$$

Equality implies that $f_1 = \binom{2m}{2} - m$, i.e., every possible edge occurs except for the diagonals $x_1\sigma(x_1), \ldots, x_m\sigma(x_m)$. This is precisely the edge graph of the $m$-dimensional octahedron. The embeddability into the 2-skeleton of the $m$-octahedron follows because the $m$-octahedron contains all possible triangles except those containing diagonals.

For an analogue in the case of 4-dimensional manifolds see [21].

**Corollary 3.4.** *For $m \geq 5$ any triangulation of a surface with a fixed-point free involution $\sigma$ satisfying equality in 3.3 has the minimum number of vertices among all triangulations of $M$.*

*Proof:* Equality in 3.3 implies that $M$ is triangulated with $n = 2m$ vertices and $f_1 = 4\binom{m}{2}$ edges. There are $m$ missing edges (= the diagonals) in $M$ by the equation $\binom{2m}{2} = f_1 + m$. Let us verify the inequality in 2.5 which necessarily has to be satisfied for $n = 2m$:

$$\binom{2m-3}{2} - 3(2 - \chi(M)) = \binom{2m-3}{2} - 2(m-1)(m-3) = m > 0.$$

Now let us try whether a triangulation with $n - 1 = 2m - 1$ vertices is possible:

$$\binom{2m-4}{2} - 3(2 - \chi(M)) = \binom{2m-4}{2} - 2(m-1)(m-3) = 4 - m,$$

which is negative for $m \geq 5$. This contradicts 2.5 and thus proves the assertion. For the exceptional case $m = 4$ and $\chi = 0$ compare 3.1 and 3.6. In the orientable case this equality can always be satisfied whenever the left hand side is divisible by 3:

**Theorem 3.5.** (Regular cases [9]) *For any orientable surface $M$ and any $m \not\equiv 2$ (mod 3) satisfying the equality $2(m-1)(m-3) = 3(2-\chi(M))$ there exists a triangulation of $M$ whose edge graph is the edge graph of the m-octahedron.*

The proof in [9] is given by current graphs (possibly with 'broken edges') separately in the cases $m \equiv 0, 1, 3, 4$ (mod 6). The simplest case $m \equiv 4$ (mod 6) can be solved by the same current graph as for a triangular embedding of $K_{2m-1}$ into the orientable surface of genus $g = \frac{1}{6}\binom{2m-4}{2}$ ([19, Fig.2.15], compare 2.7):

**Example 3.6.** Let us regard the vertices as the elements of $\mathbf{Z}_{12s+8}$. The collection of $\mathbf{Z}_{12s+8}$-orbits of the same triangles as in 2.7 above defines a triangulation of the orientable surface of genus $g = \frac{1}{3}(6s+3)(6s+1)$ as a subcomplex of the $m$-octahedron for $m = 6s+4$. Now $\overline{x}$ has to be interpreted as the element $-x$ in $\mathbf{Z}_{12s+8}$). The diagonals (missing edges) are the edges $x(x+6s+4)$. By 3.7 this can be regarded as a centrally-symmetric tight surface in $\mathbf{E}^m$. The case $s = 0$ is the tight centrally-symmetric torus in the skeleton of the 4-octahedron in 3.1.2.

**Corollary 3.7.** *For any orientable surface $M$ and any $d \not\equiv 2$ (mod 3) satisfying the equality $2(d-1)(d-3) = 3(2-\chi(M))$ there exists a centrally-symmetric tight polyhedral embedding $M \to \mathbf{E}^d$ as a subcomplex of the d-octahedron. For $d \geq 5$ the induced triangulation of $M$ has the minimum number of vertices.*

In this context we mention the following observation:

**Proposition 3.8.** ([8]) *The graph $K(2,2,2,2)$ (= the edge graph of the 4-octahedron) cannot be embedded into the Klein bottle although equality in 3.3 is satisfied for $m = 4, \chi = 0$.*

*Proof:* In [8] a combinatorial proof is given. We give the following simpler topological proof: Assume that there is an embedding of $K(2,2,2,2)$ into the Klein bottle. By standard counting arguments the complement consists of abstract triangles only, i.e., the embedding is triangular. This induces an embedding of the Klein bottle into the 2-skeleton of the 4-octahedron, just by choosing the diagonals of the graph as the diagonals of the 4-octahedron. Hence the Klein bottle is embedded into the boundary of a convex 4-polytope which is a polyhedral 3-sphere. This is impossible by the non-orientability.

**Question 3.9.** *Is there any tight polyhedral Klein bottle in $\mathbf{E}^4$ which is centrally-symmetric?* We conjecture that the answer will be 'no' because neither the edge graph of the 4-cube nor the edge graph of the 4-octahedron can be embedded into the Klein bottle. The edge graph of the prism over an octahedron can be embedded but apparently not in an appropriate way.

**Conjecture 3.10.** *For any centrally-symmetric tight and substantial $M \subset \mathbf{E}^d$ the inequality $2(d-1)(d-3) \leq 3(2-\chi(M))$ holds with equality for $d \geq 5$ only for subcomplexes of the d-octahedron.*

The restriction of this conjecture to the polyhedral case would carry over the Banchoff-Pohl theorem to the centrally-symmetric case. It seems to be plausible because the $d$-octahedron is the smallest centrally-symmetric $d$-polytope. Note that the given upper bound for $d$ in terms of $\chi(M)$ is sharp by Corollary 3.7.

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