Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 2, 391-398.

Arrangements of Arcs and Pseudocircles

Johann Linhart

Mathematical Institute
University of Salzburg
Austria

Yanling Yang*

Beijing Institute
of Light Industry
P.R. China

Abstract. Arrangements of (pseudo-)circles have already been studied in connection with algorithms in computational geometry. Thereby information on the numbers v_k of intersection points contained in k circles seems to be particularly interesting. On each circle, there is an induced arrangement of arcs. This is why we begin by studying arrangements of arcs, and we arrive at a complete characterization of the "v-vectors" $(v_0, \ldots, v_{n-1})$ in this case. For arrangements of pseudocircles, a sharp upper bound on $\sum_{i \geq k} v_i$ is derived, which leads to interesting "extremal" arrangements.

Introduction

By an *arrangement of circles* we mean a finite set of circles in the plane such that no two of them touch each other. For combinatorial investigations it seems to be natural to consider more generally a finite set of simple closed Jordan curves, such that any two of them either are disjoint or intersect in exactly two points where they cross each other. This is what we call an *arrangement of pseudocircles*, and thus the simple closed Jordan curves themselves are also called *pseudocircles.* (Two pseudocircles C_1, C_2 are said to *cross* each other in a point $P \in C_1 \cap C_2$ if every neighborhood of P contains points of C_1 lying in the interior of C_2 as well as such lying in the exterior of C_2.)

Let A be an arrangement of pseudocircles. The points of intersection of the pseudocircles are called the *vertices* of A. Each vertex is contained in the interior of a certain number of pseudocircles which is called its *weight* or *level.* Now let $v_{\leq k}$ be the number of vertices with weight at most k. These numbers are of interest in computational geometry, and the

*This paper was prepared partially while the second author stayed at the Mathematical Institute of the University of Salzburg, Austria.

0138-4821/93 \$ 2.50

following results have been obtained so far:

$$\begin{aligned} v_{\le 0} &\le 6(n-2) && \text{for } n \ge 3 \quad [3], \\ v_{\le k} &\le 26\,kn && \text{for } k \ge 1 \quad [10]. \end{aligned}$$

The first inequality is sharp. There are even arrangements of (proper) circles where equality is attained (for each $n \geq 3$). The second inequality is probably not sharp. Maybe $6(k+1)n$ is a correct upper bound for arbitrary k and n. If this is true, it might be relatively close to the supremum. P. Brass [1] and the first author have constructed arrangements of circles with $v_{\leq k} \approx 5.5\,(k+1)\,n$ for $k = 1$ and $n \to \infty$.

Since improvements on the upper bound of $v_{\leq k}$ seem to be difficult to obtain, we directed our attention to a very similar problem which turned out to be easier to solve. We considered the number $v_{\geq k}$ of vertices having weight *at least* k and could prove $v_{\geq k} \leq (n+k)(n-k-1)$. This is sharp in the following strong sense: for every $n \geq 2$ there is an "extremal" arrangement of n circles such that equality is attained simultaneously for each $k \in \{0, \ldots, n-2\}$.

Similar problems for arrangements of lines or hyperplanes have been investigated already ([5] – [9], [12]).

In an arrangement of pseudocircles, there is an induced *arrangement of arcs* on each pseudocircle. This is why we begin by studying arrangements of arcs. Here the corresponding problems are of course easier, but they are not trivial. We shall arrive at a complete characterization of the possible numbers $v_{\leq k}$ (and thus also $v_{\geq k}$) for arrangements of arcs.

1. Arrangements of Arcs

An *arrangement of arcs* is a finite set of open arcs on the circle S^1. Such arrangements have already been studied from a different point of view. V. Klee [4] posed the problem of characterizing their intersection graphs, and A. Tucker [11] provided an answer. But this does not seem to be of relevance for the problems discussed here.

In a more general fashion, we might define an *arc* to be a relatively open, connected, proper subset of a pseudocircle, and an *arrangement of arcs* to be a finite set of such arcs on a fixed pseudocircle. Since any pseudocircle is homeomorphic to S^1, this is not really more general than the previous definition.

We consider only *simple* arrangements, which means that no two arcs have a common endpoint. The empty set and the whole circle S^1 are not considered as arcs. In a simple arrangement of n arcs there are $2n$ different endpoints, which are called *vertices*. The vertices dissect S^1 into $2n$ (open) circular segments, which are called the *edges* of the arrangement. In general an arc is made up of several, but not all, edges.

The number of arcs containing a given vertex or edge is called its *weight*. The number of vertices or edges having weight k is denoted by v_k or e_k, respectively. $(v_0, \ldots, v_{n-1})$ is called the *v-vector,* and $(e_0, \ldots, e_n)$ the *e-vector* of the arrangement.

Let V_i and E_i be the vertices and edges, respectively, numbered according to an orientation of S^1, such that V_i and V_{i+1} are the endpoints of the edge E_i. The weight of V_i is denoted by $w(V_i)$, and accordingly that of E_i by $w(E_i)$. The finite sequences $(w(V_1), \ldots, w(V_{2n}))$ and $(w(E_1), \ldots, w(E_{2n}))$ are called the *weight sequences* of the vertices and edges, respectively. (The indices are understood modulo $2n$.) We shall first characterize the weight sequences of the edges, and subsequently the v-vectors.

Lemma 1.1. *In a simple arrangement of n arcs, the following relations hold:*

a) $0 \leq w(V_i) \leq n-1$,

b) $|w(E_{i-1}) - w(E_i)| = 1$;

c) $|w(V_{i-1}) - w(V_i)| \leq 1$,

d) $w(V_i) = \min(w(E_{i-1}), w(E_i))$.

Proof.

a) Since no arc contains its endpoints, the weight of a vertex can be at most $n-1$.

b) Consider two adjacent edges E_{i-1} and E_i. Their common endpoint V_i is an endpoint of an arc which covers exactly one of these two edges. All other arcs contain either both edges or none of them. Thus $w(E_{i-1})$ and $w(E_i)$ differ exactly by one.

c) Now consider the vertices V_{i-1} and V_i. If $w(E_{i-2}) < w(E_{i-1}) > w(E_i)$ or $w(E_{i-2}) > w(E_{i-1}) < w(E_i)$, then $w(V_{i-1}) = w(V_i)$. Otherwise, we have $|w(V_{i-1}) - w(V_i)| = 1$.

d) This follows from the fact that an arc contains V_i if and only if it contains both E_{i-1} and E_i. □

Theorem 1.2. *A $2n$-tuple of integers $(w_1, \ldots, w_{2n})$ is the weight sequence of the edges of a simple arrangement of n arcs if and only if the following conditions hold (for all indices i modulo $2n$):*

a) $0 \leq w_i \leq n$;

b) $|w_{i-1} - w_i| = 1$.

Proof. Since the necessity follows directly from Lemma 1.1, we have to prove only the sufficiency. This we do by induction on n. For $n = 1$, the assertion is obviously true. Now consider a $2(n+1)$-tuple satisfying a) and b) with $n+1$ instead of n. We distinguish two cases.

Case 1: There is at most one w_i equal to $n+1$.

Let w_m be the maximal element of the sequence. Deleting w_m and w_{m+1}, we get a $2n$-tuple satisfying a) and b). By induction there is a simple arrangement of n arcs having this $2n$-tuple as weight sequence of the edges. Now we just have to add an arc which is contained in the edge E_{m-1}.

Case 2: There are two or more w_i equal to $n+1$.

In this case b) implies that there is no w_i equal to zero. Let r be the first index $\in \{1, \ldots, 2n\}$ such that $w_r = n+1$, and s the last one. Note that $(r, s) = (1, 2n+2)$ is not possible because of b). Delete w_r and w_s, and decrement all elements in between by one. The resulting $2n$-tuple satisfies the conditions and thus corresponds to a simple arrangement of n arcs. If we add an arc with starting point in E_{r-1} and end point in E_{s-1}, we get an arrangement as desired. □

Remark. There does not seem to be an equally simple characterization of the weight sequences of the vertices. On the other hand, there is a very close connection between the v-vector and the e-vector:

Observation 1.3. *In a simple arrangement of n arcs, the following equation holds for each $k \in \{0, \ldots, n\}$, with $v_{-1} := v_n := 0$:*

$$2e_k = v_k + v_{k-1}.$$

Proof. Consider all edges of weight k and their endpoints. By Lemma 1.1, they form a 1-regular graph, the nodes of which are exactly the vertices having weight k or $k-1$. □

U. Brehm [2] has found a rather complicated characterization of the e-vectors of arrangements of half-lines which might be modified to get one for the case of arcs. But in view of the above Observation it seems to be more natural to study the v-vectors, since this implicitly also yields a result for the e-vectors.

So we want to characterize the v-vectors of arrangements of arcs. To do this, we introduce the following notion. An n-tuple of numbers $(v_0, \ldots, v_{n-1})$ has the *positive consecutive property*, if the set $\{ k \mid v_k > 0 \}$ consists of consecutive integers (compare [11]).

Theorem 1.4. *An n-tuple of non-negative integers $(v_0, \ldots, v_{n-1})$ is the v-vector of a simple arrangement of n arcs if and only if the following conditions hold:*

a) $\sum_{k=0}^{n-1} v_k = 2n$,

b) each v_k is even,

c) $(v_0, \ldots, v_{n-1})$ *has the positive consecutive property.*

Proof.

Necessity.

a) This is just the fact that there are $2n$ vertices.

b) This follows from Observation 1.3 by induction on k.

c) Let $0 \le i < j < k \le n-1$ and $v_i > 0,\ v_k > 0$. When we walk along the circle from a vertex of weight i to a vertex of weight k, we have to pass by at least one vertex of weight j, since the weight can change at most by one from one vertex to the next (Lemma 1.1, c). Thus v_j is also positive.

Sufficiency.

It suffices to construct a weight sequence of the edges of a simple arrangement, whose v-vector is the given n-tuple. Suppose that v_r is the leftmost positive element in the n-tuple, and v_s the rightmost one. Consider the following concatenation of tuples:

$$W := B_r \vee B_{r+1} \vee \ldots \vee B_s \vee M,$$

where $B_k = (b_{k2}, \ldots, b_{kv_k})$ with $b_{kj} = k + \left(1 + (-1)^j\right)/2$, and $M = (s, s-1, \ldots, r)$. From $0 \le r \le s \le n-1$ we conclude that all elements of W are non-negative integers $\le n$. Since v_k is even, $B_k = (k+1, k, \ldots, k+1)$, and thus any two consecutive elements of W differ by one. According to Theorem 1.2, W is the weight sequence of the edges of a simple arrangement of arcs. Now, for given k, we count the entries of W equal to k. For $r \le k \le s$, B_k contains $v_k/2 - 1$ elements equal to k, and $v_k/2$ ones equal to $k+1$. Finally, M contains exactly one element equal to k. Therefore, in this arrangement,

$$e_k = \begin{cases} v_r/2 & \text{for } k = r, \\ (v_k + v_{k-1})/2 & \text{for } r+1 \le k \le s, \\ v_s/2 & \text{for } k = s+1, \\ 0 & \text{for } k < r \text{ or } k > s+1. \end{cases}$$

By Observation 1.3 we see that the v-vector of the arrangement is just the given n-tuple. □

2. Arrangements of pseudocircles

Let $A = \{C_1, \ldots, C_n\}$ be an arrangement of pseudocircles as defined in the introduction. It is called *simple*, if no three pseudocircles of A have a point in common.

For each $i_0 \in \{1, \ldots, n\}$, the set

$$A_{i_0} := \{C_{i_0} \cap \operatorname{int} C_j \mid j \in \{1, \ldots, n\}\} \setminus \{\emptyset, C_{i_0}\}$$

is a (possibly empty) arrangement of at most $n-1$ arcs on C_{i_0}. It is called the *induced arrangement* on C_{i_0}. If A is simple, then each induced arrangement is also simple. In this case each vertex of A is a vertex of exactly two induced arrangements, and its weight is the same with respect to each of the three concerned arrangements. Conversely, each vertex of A_{i_0} is also a vertex of A having the same weight. This enables us to prove the following inequality by applying Theorem 1.4 to a suitable induced arrangement. (Actually, only the necessity of the conditions is needed.) v_k denotes the number of vertices of A having weight k, and $v_{\geq k} = \sum_{i=k}^{n-2} v_i$.

Theorem 2.1. *In an arrangement of $n \geq 2$ pseudocircles,*

$$v_{\geq k} \leq (n+k)\,(n-k-1)$$

for $0 \leq k \leq n-2$.

Proof. First we observe that we may restrict our attention to simple arrangements, since in a non-simple arrangement the pseudocircles may be "perturbed" a little bit to get a simple one, such that the numbers $v_{\geq k}$ do not decrease. Now we proceed by induction, the case $n = 2$ being trivial.

Let A be a simple arrangement of $n+1$ pseudocircles. There is at least one pseudocircle $C_1 \in A$ which contains points not lying inside of any other pseudocircle of A. We distinguish two cases.

Case 1. The induced arrangement A_1 on C_1 is not empty.
Let m be the number of arcs of A_1, $1 \leq m \leq n$, and $(v_0^1, \ldots, v_{m-1}^1)$ the v-vector of A_1. By the choice of C_1, we have $v_0^1 \geq 2$. If $\sum_{i \geq k} v_i^1 > 0$, the positive consecutive property (Theorem 1.4, c) implies $v_j^1 > 0$ for all $j \in \{0, \ldots, k\}$. Together with the properties a) and b) of Theorem 1.4 we get $\sum_{i \geq k} v_i^1 \leq 2n - 2k$. If $\sum_{i \geq k} v_i^1 = 0$, this is trivial.

Now consider the arrangement A' obtained by taking away C_1 from A. Each vertex of A belongs to A_1 or A', and its weight with regard to A' is the same as or one less than the weight with regard to A.

Denoting by $v'_{\geq k}$ the number of vertices of A' having weight at least k, we get

$$v_{\geq k} \leq v'_{\geq k-1} + 2n - 2k.$$

By the inductive hypothesis we have

$$v'_{\geq k-1} \leq (n+k-1)(n-k),$$

and thus

$$v_{\geq k} \leq (n+k-1)(n-k) + 2n - 2k = (n+1+k)(n-k).$$

This is the assertion with $n+1$ instead of n.

Case 2. The induced arrangement A_1 on C_1 is empty.
Here we get immediately $v_{\geq k} \leq v'_{\geq k-1}$ and thus

$$v_{\geq k} \leq (n+k-1)(n-k) < (n+1+k)(n-k). \qquad \square$$

To discuss the question of equality in Theorem 2.1, we call an arrangement of (pseudo-)circles *extremal* if $v_k = 2(k+1)$ and thus $v_{\geq k} = (n+k)\,(n-k-1)$ for each $k \in \{0,\ldots,n-2\}$.

Theorem 2.2. *For every $n \geq 2$, there are extremal arrangements of n circles.*

Proof. We proceed again by induction on n. The case $n = 2$ is trivial. Fig. 1 shows a (simple) extremal arrangement of 3 circles. Note that there is a region D_3 in the interior of all three circles. The boundary of D_3 consists of 4 edges. One of them belongs to C_1, one to C_2, and the remaining two belong to C_3.

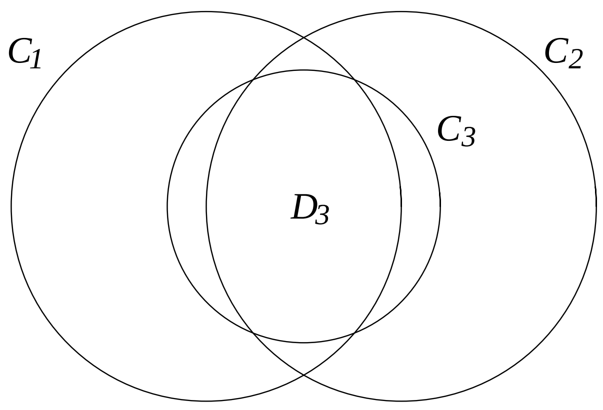

Fig.1. An extremal arrangement of three circles.

Suppose we have constructed a simple extremal arrangement of n circles $\{C_1,\ldots,C_n\}$, and denote the center of C_i by O_i. Then there is a region D_n in the interior of all circles, and the boundary of D_n consists of $2(n-1)$ edges. Suppose $n-1$ of them belong to $C_1,\ldots,C_{n-1}$ (one to each of these circles), and the others to C_n. Choose one of these edges of C_n, call it E, and denote its middle point by M. Now draw a new circle in the following way. Put the center O_{n+1} on the line segment O_nM such that $|O_n - O_{n+1}| = \alpha$, where $\alpha > 0$ is a small number which will be determined immediately. With the radius $r_{n+1} := r_n - \alpha$ we get a circle C_{n+1} touching C_n in the point M. If we choose α small enough, C_{n+1} will intersect each of the $n-1$ edges belonging to $C_1,\ldots,C_{n-1}$ in two points. Now we may increase r_{n+1} by a (very) small amount, such that C_{n+1} retains this property, but also intersects the edge E of C_n in two points. By this procedure the numbers v_i remain unchanged for $1 \leq i \leq n-2$, and there are $2n$ new vertices having weight $n-1$, all on the boundary of the region D_{n+1} lying within all circles. D_{n+1} has all of the supposed properties of D_n with n replaced by $n+1$. $\square$

Remarks.

1. We have also found an alternative way to construct extremal arrangements. Figure 2 shows an example for $n = 6$.

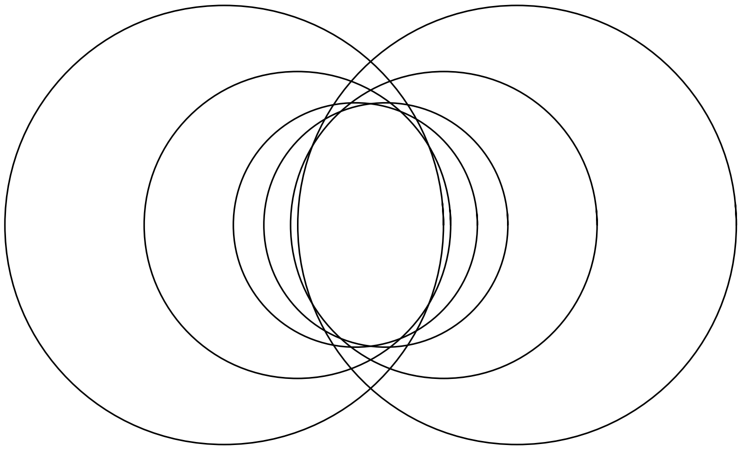

Fig. 2. An extremal arrangement of six circles.

2. If we sum up the inequality of Theorem 2.1 for $k = 1, \ldots, n-2$, we get

$$\sum_{k=1}^{n-2} k v_k = \sum_{k=1}^{n-2} \sum_{i=k}^{n-2} v_i \leq \sum_{k=1}^{n-2} (n+k)(n-k-1) = 4\binom{n}{3}$$

with equality for extremal arrangements. The expression on the left hand side is the sum of the weights of all vertices, also called their *total weight* (compare [6]).

References

[1] Brass, P. (University of Greifswald, Germany): *(private communication)*, September 1995.

[2] Brehm, U.: *A characterization of the e-vectors of arrangements of half-lines.* (Manuscript, 1992).

[3] Kedem, K.; Livne, R.; Pach, J.; Sharir, M.: *On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles.* Discrete Comput. Geom. **1** (1986), 59–71.

[4] Klee, V.: *What are the intersection graphs of arcs in a circle?* Amer. Math. Monthly **76** (1969), 810–813.

[5] Linhart, J.: *Arrangements of oriented hyperplanes.* Discrete Comput. Geom. **10** (1993), 435–446.

[6] Linhart, J.: *On the total weight of arrangements of halfplanes.* Geom. Dedicata **46** (1993), 165–172.

[7] Linhart, J.: *The upper bound conjecture for arrangements of halfspaces.* Contributions to Algebra and Geometry **35** (1994), 29–35.

[8] Linhart, J.: *The weights of the vertices in an arrangement of hemispheres.* Manuscript, 1993.

[9] Philipp, M.: *Combinatorial properties of arrangements of halfspaces.* Ph.D. thesis, University of Salzburg, Austria, 1994.

[10] Sharir, M.: *On k-sets in arrangements of curves and surfaces.* Discrete Comput. Geom. **6** (1991), 593–613.

[11] Tucker, A.: *Characterizing circular-arc graphs.* Bull. Amer. Math. Soc. **76** (1970), 1257–1260.

[12] Yang, Y. L.: *Arrangements of half-lines and half-planes and two conjectures of Linhart.* Manuscript, 1994.

Received October 10, 1995