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# A Family of Minkowski Planes over Half-ordered Fields

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**Abstract.** This paper concerns a construction of J. Jakóbowski [7] of Minkowski planes over half-ordered fields. A more group theoretic definition of such planes is given. Isomorphisms between and automorphisms of these Minkowski planes are determined.

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## 1. Introduction and notation

A *Minkowski plane* $\mathcal{M} = (P, \mathcal{K}, \{\|_+, \|_-\})$ consists of a set of points $P$, a set of at least two circles $\mathcal{K}$ (considered as subsets of $P$) and two equivalence relations $\|_+$ and $\|_-$ on $P$ (parallelisms) such that three mutually non-parallel points (that is, neither (+)-parallel nor (−)-parallel) can be joined by a unique circle, such that the circles which touch a fixed circle $K$ at $p \in K$ partition $P \setminus |p|$ (where $|p| = |p|_+ \cup |p|_-$ denotes the union of the two parallel classes of $p$), such that each parallel class meets each circle in a unique point (parallel projection), such that each (+)-parallel class and each (−)-parallel class intersect in a unique point, and such that there is a circle that contains at least three points (compare [2], [4], [9] or [10]).

Associated with every point $p$ of $\mathcal{M}$ there is an incidence structure, called the *derived affine plane* or *residual plane* $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ *at* $p$, whose point set $A_p$ consists of all points of $\mathcal{M}$ that are not parallel to $p$ and whose set of lines $\mathcal{L}_p$ consists of all restrictions to $A_p$ of circles of $\mathcal{M}$ passing through $p$ and of all parallel classes not passing through $p$. In fact, $\mathcal{M}$ is a Minkowski plane if and only if all incidence structures $\mathcal{A}_p$ are affine planes.

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The *Miquelian Minkowski plane* over a field $\mathbb{F}$ is obtained as the geometry of non-trivial plane sections of a ruled quadric in 3-dimensional projective space over $\mathbb{F}$. Another description of this model is as follows. The point set is $\overline{\mathbb{F}} \times \overline{\mathbb{F}}$, where $\overline{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$ and $\infty$ is an element not contained in $\mathbb{F}$. Two points $(x, y)$ and $(x', y')$ are (+)- or (–)-parallel if and only if $x = x'$ or $y = y'$ respectively. Each circle is the graph of a fractional linear map $x \mapsto \frac{ax+b}{cx+d}$. This generalizes to the following description of an arbitrary Minkowski plane $\mathcal{M}$. The point set of $\mathcal{M}$ is $\overline{\mathbb{F}} \times \overline{\mathbb{F}}$, where $\overline{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$ and $\mathbb{F}$ is a coordinatizing ternary field of a derived affine plane, parallel classes are of the form $\{x_0\} \times \overline{\mathbb{F}}$ and $\overline{\mathbb{F}} \times \{y_0\}$ for $x_0, y_0 \in \overline{\mathbb{F}}$. Each circle $K$ of $\mathcal{M}$ can be described by a function $f_K : \overline{\mathbb{F}} \to \overline{\mathbb{F}}$ as

$$K = \{(x, f_K(x)) |\ x \in \overline{\mathbb{F}}\}.$$

The axiom of parallel projection shows that each function $f_K$ is a permutation of $\overline{\mathbb{F}}$. The axiom of joining implies that the collection of all those permutations $f_K$ is a sharply 3-transitive set of permutations of $\overline{\mathbb{F}}$. Conversely, each such incidence structure constructed from a sharply 3-transitive set of permutations of $\overline{\mathbb{F}}$ is equivalent to a more general hyperbola structure or (B*)-geometry, that is, all axioms of a Minkowski plane are satisfied except the axiom of touching.

A *half-* (or *pseudo-*) *ordered field* $\mathbb{F}$ is a field with a multiplicative subgroup $\mathbb{P}$ of index two. In particular, $\mathbb{P}$ contains all non-zero squares of $\mathbb{F}$ so that a finite half-ordered field cannot have characteristic two. Elements of $\mathbb{P}$ and of the other coset of non-zero elements are called positive and negative respectively. We write $x > 0$ for $x \in \mathbb{P}$ and $x < 0$, if $x$ is negative.

We generalize the notion of order-preserving, order-reversing and monotonic permutations of a half-ordered field $\mathbb{F}$ to permutations of $\overline{\mathbb{F}}$. To that end, let

$$\varepsilon(x_1, x_2, x_3) = \begin{cases} (x_1 - x_2)(x_2 - x_3)(x_3 - x_1), & \text{if } x_1, x_2, x_3 \neq \infty \\ x_3 - x_2, & \text{if } x_1 = \infty \\ x_1 - x_3, & \text{if } x_2 = \infty \\ x_2 - x_1, & \text{if } x_3 = \infty \end{cases}$$

for mutually disctinct $x_1, x_2, x_3 \in \overline{\mathbb{F}}$. A permutation $f$ of $\overline{\mathbb{F}}$ is called *order-preserving* or *order-reversing* if and only if $\varepsilon(f(x_1), f(x_2), f(x_3)) \cdot \varepsilon(x_1, x_2, x_3)^{-1} > 0$ or $\varepsilon(f(x_1), f(x_2), f(x_3)) \cdot \varepsilon(x_1, x_2, x_3)^{-1} < 0$, respectively, for all mutually disctinct $x_1, x_2, x_3 \in \overline{\mathbb{F}}$. We call $f$ *monotonic* if $f$ is order-preserving or order-reversing. When $x_3 = \infty$ and $f$ fixes that point one obtains the familiar definition of an order-preserving or order-reversing permutation of $\mathbb{F}$; cf. [6]. In the respective cases $(f(x) - f(y))(x - y)^{-1} > 0$ or $(f(x) - f(y))(x - y)^{-1} < 0$ for all distinct $x, y \in \mathbb{F}$.

We define $\Pi^+(\overline{\mathbb{F}})$ and $\Pi^-(\overline{\mathbb{F}})$ to be the collection of all order-preserving and all order-reversing permutations of $\overline{\mathbb{F}}$ respectively. Finally let $\Pi(\overline{\mathbb{F}}) = \Pi^+(\overline{\mathbb{F}}) \cup \Pi^-(\overline{\mathbb{F}})$ be the collection of all monotonic permutations of $\overline{\mathbb{F}}$.

One readily verifies that $\Pi(\overline{\mathbb{F}})$ and $\Pi^+(\overline{\mathbb{F}})$ are groups with respect to composition of permutations. $\Pi^+(\overline{\mathbb{F}})$ is a normal subgroup of $\Pi(\overline{\mathbb{F}})$ of index 2, and $\Pi^-(\overline{\mathbb{F}})$ is the other coset of $\Pi^+(\overline{\mathbb{F}})$ in $\Pi(\overline{\mathbb{F}})$.

In this note we investigate a family of Minkowski planes over half-ordered fields. These planes were first constructed by J. Jakóbowski in a slightly different form. Those planes over the real numbers $\mathbb{R}$ have been studied in [12]. We determine all isomorphisms between and automorphisms of these Minkowski planes.

## 2. The Minkowski planes $\mathcal{M}(\mathbb{F}; f, g)$

We denote the projective linear group over the field $\mathbb{F}$ by PGL(2,$\mathbb{F}$), that is, the quotient group formed by the general linear group GL(2,$\mathbb{F}$) of invertible $2 \times 2$ matrices with entries in $\mathbb{F}$ modulo the non-zero scalar matrices. Each element of PGL$(2,\mathbb{F})$ can be represented by a $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$. This matrix operates on the set of 1-dimensional subspaces of $\mathbb{F}^2$ like the fractional linear map $x \mapsto \frac{ax+b}{cx+d}$ on $\overline{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$. We define

$$\mathrm{PGL}^+(2,\mathbb{F}) = \Pi^+(\overline{\mathbb{F}}) \cap \mathrm{PGL}(2,\mathbb{F}); \quad \mathrm{PGL}^-(2,\mathbb{F}) = \Pi^-(\overline{\mathbb{F}}) \cap \mathrm{PGL}(2,\mathbb{F}).$$

PGL$^+(2,\mathbb{F})$ is a normal subgroup of index 2 in PGL$(2,\mathbb{F})$. An easy calculation shows that a permutation $\gamma : x \mapsto \frac{ax+b}{cx+d}$ belongs to PGL$^+(2,\mathbb{F})$ or PGL$^-(2,\mathbb{F})$ if and only if $ad - bc > 0$ and $ad - bc < 0$ respectively. (In the generic case $\varepsilon(\gamma(x_1), \gamma(x_2), \gamma(x_3)) = \frac{(ad-bc)^3}{(cx_1+d)^2(cx_2+d)^2(cx_3+d)^2}\varepsilon(x_1, x_2, x_3)$.) Note that if one chooses a different representing matrix the determinant multiplies by a square in $\mathbb{F}$ which is positive; hence the above condition is independent of the representing matrix. It is well known that PGL$(2,\mathbb{F})$ is a sharply 3-transitive permutation group of $\overline{\mathbb{F}}$.

Let Aut$(\mathbb{F})$ be the collection of all automorphisms of the field $\mathbb{F}$ and let

$$\mathrm{P\Gamma L}(2,\mathbb{F}) = \mathrm{PGL}(2,\mathbb{F})\mathrm{Aut}(\mathbb{F})$$

be the collection of all semi-linear fractional permutations of $\overline{\mathbb{F}}$. Moreover, Aut$^+(\mathbb{F})$ denotes the order-preserving automorphisms and P$\Gamma$L$^\pm(2,\mathbb{F})$ denotes the order-preserving and order-reversing transformations respectively, that is,

$$\mathrm{P\Gamma L}^\pm(2,\mathbb{F}) = \Pi^\pm(\overline{\mathbb{F}}) \cap \mathrm{P\Gamma L}(2,\mathbb{F}) = \mathrm{PGL}^\pm(2,\mathbb{F})\mathrm{Aut}^+(\mathbb{F}).$$

Note that, in general, not every automorphism of a field is order-preserving although squares are mapped to squares. (E.g., the automorphism $x + y\sqrt{2} = x - y\sqrt{2}$, $x, y \in \mathbb{Q}$, of $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$ is not order-preserving with respect to the Euclidean ordering.)

### 2.1. The incidence structures $\mathcal{M}(\mathbb{F}; f, g)$

Let $f, g \in \Pi^+(\overline{\mathbb{F}})$. We define

$$\mathcal{F}_{f,g} = \mathrm{PGL}^+(2,\mathbb{F}) \cup g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f \subseteq \Pi(\overline{\mathbb{F}}).$$

$\mathcal{M}(\mathbb{F}; f, g)$ is the following incidence structure. The point set of $\mathcal{M}(\mathbb{F}; f, g)$ is $\overline{\mathbb{F}} \times \overline{\mathbb{F}}$, parallel classes are of the form $\{x_0\} \times \overline{\mathbb{F}}$ and $\overline{\mathbb{F}} \times \{y_0\}$ for $x_0, y_0 \in \overline{\mathbb{F}}$. The set of circles $\mathcal{K}_{f,g}$ of $\mathcal{M}(\mathbb{F}; f, g)$ is the collection of all graphs of permutations in $\mathcal{F}_{f,g}$, that is, circles are of the form

$$K_\gamma = \{(x, \gamma(x)) | \ x \in \overline{\mathbb{F}}\}$$

for $\gamma \in \mathrm{PGL}^+(2,\mathbb{F}) \cup g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f$. We call $\mathrm{PGL}^+(2,\mathbb{F})$ and $g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f$ the *components* of $\mathcal{F}_{f,g}$. We also call the collection of all circles whose describing permutations are in one component of $\mathcal{F}_{f,g}$ a *component* of the circle set $\mathcal{K}_{f,g}$. (For $\mathbb{F} = \mathbb{R}$, $\mathrm{PGL}^+(2,\mathbb{F}) = \mathrm{PSL}(2,\mathbb{R})$ is the connected component of the identity of the topological group PGL(2,$\mathbb{R}$) and $\mathcal{K}_{f,g}$ has two connected components.)

Note that the Miquelian Minkowski plane over $\mathbb{F}$ occurs for $f = g = id$ (but not only for these permutations). Furthermore, every incidence structure $\mathcal{M}(\mathbb{F}; f, g)$ shares one component with the Miquelian plane and the other component can be transformed into a component of the Miquelian Minkowski plane in a simple way. Therefore many calculations can be done in the corresponding Miquelian plane.

**Remark 2.2.** Substituting $f$ and $g$ by $\delta f$ and $\delta' g$ respectively for two permutations $\delta, \delta' \in \mathrm{PGL}^+(2,\mathbb{F})\alpha$ with $\alpha \in \mathrm{Aut}^+(\mathbb{F})$, does not alter the circle set, that is, $\mathcal{M}(\mathbb{F}; f, g) = \mathcal{M}(\mathbb{F}; \delta f, \delta' g)$. Since $\mathrm{PGL}^+(2,\mathbb{F})$ is still 2-transitive and because the stabilizer $\mathrm{PGL}^+(2,\mathbb{F})_{\infty,0}$ of $\infty$ and 0 is transitive on $\mathbb{P}$, we may assume, if necessary, that $f$ and $g$ both fix $\infty$, 0 and 1. We denote the stabilizers of $\infty$, 0 and 1 in $\Pi(\overline{\mathbb{F}})$ and $\Pi^\pm(\overline{\mathbb{F}})$ by $\Pi_{\infty,0,1}(\overline{\mathbb{F}})$ and $\Pi^\pm_{\infty,0,1}(\overline{\mathbb{F}})$ respectively.

In order to prove that $\mathcal{M}(\mathbb{F}; f, g)$ is a hyperbola structure we need a well-known lemma that allows us to recognize order-preserving permutations among monotonic permutations of $\overline{\mathbb{F}}$ by the number of their fixed points.

**Lemma 2.3.** *Suppose that $\phi \in \Pi(\overline{\mathbb{F}})$ fixes at least three points. Then $\phi$ is order-preserving.*

*Proof.* Suppose that $\phi$ fixes three mutually distinct points $x_1, x_2, x_3 \in \overline{\mathbb{F}}$. Then $\frac{\varepsilon(\phi(x_1),\phi(x_2),\phi(x_3))}{\varepsilon(x_1,x_2,x_3)} = 1 > 0$. So, $\phi$ must be order-preserving. □

**Corollary 2.4.** *$\mathcal{F}_{f,g}$, $f, g \in \Pi^+(\overline{\mathbb{F}})$, is a sharply 3-transitive set of permutations of $\overline{\mathbb{F}}$. Equivalently, $\mathcal{M}(\mathbb{F}; f, g)$ is a hyperbola structure.*

*Proof.* Let $x_i, y_i \in \mathbb{F}$, $i = 1, 2, 3$, such that the $x_i$'s and also the $y_i$'s are mutually distinct. By the 3-transitivity of $\mathrm{PGL}(2,\mathbb{F})$ on $\overline{\mathbb{F}}$ we find $\sigma, \sigma' \in \mathrm{PGL}(2,\mathbb{F})$ with $\sigma(x_i) = y_i$ and $\sigma'(f(x_i)) = g(y_i)$ for $i = 1, 2, 3$. Since $\sigma^{-1}g^{-1}\sigma' f \in \Pi(\overline{\mathbb{F}})$ fixes the three points $x_1$, $x_2$, $x_3$, this map must be order-preserving by Lemma 2.3. Hence $\sigma$ and $g^{-1}\sigma' f$ are either both order-preserving or both order-reversing. Since $f$ and $g$ are order-preserving, the orientations of $g^{-1}\sigma' f$ and $\sigma'$ agree. Hence, $\sigma$ and $\sigma'$ are either both in $\mathrm{PGL}^+(2,\mathbb{F})$ or both in $\mathrm{PGL}^-(2,\mathbb{F})$. This shows that there is precisely one element in $\mathrm{PGL}^+(2,\mathbb{F}) \cup g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f$ that maps $x_i$ to $y_i$ for $i = 1, 2, 3$ – namely, $\sigma$, if it is in $\mathrm{PGL}^+(2,\mathbb{F})$, or $g^{-1}\sigma' f$, if $\sigma \in \mathrm{PGL}^-(2,\mathbb{F})$. □

**Lemma 2.5.** *If two circles $K_\gamma$ and $K_\delta$, $\gamma, \delta \in \mathrm{PGL}(2,\mathbb{F})$, of the Miquelian Minkowski plane over $\mathbb{F}$ touch each other, then $\gamma^{-1}\delta \in \mathrm{PGL}^+(2,\mathbb{F})$.*

*Proof.* $K_\gamma$ and $K_\delta$ touch each other if and only if $\gamma = \delta$ or $\sigma = \gamma^{-1}\delta$ has precisely one fixed point. Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{F})$ have precisely one fixed point. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has precisely one eigenvalue in $\mathbb{F}$. Hence the characteristic polynomial $X^2 - (a+d)X + (ad - bc)$ of this matrix has a double root. When $\mathbb{F}$ has characteristic $\neq 2$, this occurs if and only if the discriminant $(a+d)^2 - 4(ad - bc)$ equals 0. Thus $\det(\sigma) = ad - bc = (\frac{a+d}{2})^2 > 0$ and

$\sigma \in \mathrm{PGL}^+(2,\mathbb{F})$. When $\mathbb{F}$ has characteristic 2, then $a+d=0$ and $ad-bc$ must be a square of $\mathbb{F}$. In particular, $\det(\sigma) > 0$. □

**Corollary 2.6.** *For every circle $K_\gamma$, $\gamma \in \mathcal{F}_{f,g}$, $f, g \in \Pi^+(\overline{\mathbb{F}})$, and any two non-parallel points $z_0, z_1$ such that $z_0 \in K_\gamma$ there is precisely one circle $K'_\gamma$, $\gamma'$ in the same component of $\gamma$, through $z_0, z_1$ that touches $K_\gamma$ at $z_0$.*

*Proof.* We consider the two cases $\gamma \in \mathrm{PGL}^+(2,\mathbb{F})$ and $\gamma \in g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f$. In the former case there is precisely one circle $K'_\gamma$ in the Miquelian plane through $z_0, z_1$ that touches $K_\gamma$ at $z_0$. By Lemma 2.5 we have $\gamma' \in \mathrm{PGL}^+(2,\mathbb{F}) \subseteq \mathcal{F}_{f,g}$. In the latter case let $\gamma = g^{-1}\sigma f$ with $\sigma \in \mathrm{PGL}^-(2,\mathbb{F})$. Let $z'_i$ be the points $(f(x_i), g(y_i))$ for $i = 0, 1$ where $z_i = (x_i, y_i)$. In the Miquelian plane we find precisely one circle $K'_\sigma$ through $z'_0, z'_1$ that touches $K_\sigma$ at $z'_0$. By Lemma 2.5 we have $\sigma' \in \mathrm{PGL}^-(2,\mathbb{F})$. Hence, $\gamma' = g^{-1}\sigma' f \in g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f) \subseteq \mathcal{F}_{f,g}$. Furthermore, $K'_\gamma$ passes through $z_0, z_1$ and touches $K_\gamma$ at $z_0$. □

We say that two permutations $f, g \in \Pi^+(\overline{\mathbb{F}})$ have the *fixed point property* (FP) if and only if

$$\text{(FP)} \qquad |\mathrm{Fix}(\gamma)| \neq 1 \text{ for all } \gamma \in \mathrm{PGL}^+(2,\mathbb{F})g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f$$

where $\mathrm{Fix}(\gamma)$ is the collection of all points fixed by $\gamma$.

**Theorem 2.7.** *$\mathcal{M}(\mathbb{F}; f, g)$, $f, g \in \Pi^+(\overline{\mathbb{F}})$, is a Minkowski plane if and only if $f$ and $g$ have the fixed point property (FP).*

*Proof.* By Corollary 2.4 the incidence structure $\mathcal{M}(\mathbb{F}; f, g)$ is a hyperbola structure. To verify that $\mathcal{M}(\mathbb{F}; f, g)$ is a Minkowski plane we only have to verify the axiom of touching. Given a circle $K_\gamma$, $\gamma \in \mathcal{F}_{f,g}$, we know from Corollary 2.6 that there is precisely one circle $K'_\gamma$ through $(x_1, y_1)$ that touches $K_\gamma$ at $(x_0, y_0)$ such that $\gamma'$ is in the same component as $\gamma$. If $\gamma \in \mathrm{PGL}^+(2,\mathbb{F})$ then $\gamma' \in \mathrm{PGL}^+(2,\mathbb{F})$ too and $\gamma'$ can uniquely be found in the Miquelian Minkowski plane over $\mathbb{F}$. Similarly, if $\gamma \in g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f$, i.e. $\gamma = g^{-1}\sigma f$ for $\sigma \in \mathrm{PGL}^-(2,\mathbb{F})$, then $\sigma' \in \mathrm{PGL}^-(2,\mathbb{F})$ can be uniquely found in the Miquelian Minkowski plane over $\mathbb{F}$ as the describing permutation of the circle through $(f(x_1), g(y_1))$ that touches $K_\sigma$ at $(f(x_0), g(y_0))$. Then $\gamma' = g^{-1}\sigma' f$ describes the circle through $(x_1, y_1)$ that touches $K_\gamma$ at $(x_0, y_0)$.

Now suppose that there is a touching circle described by $\gamma'$ in the other component of the circle set. Then $\gamma^{-1}\gamma'$ or $(\gamma')^{-1}\gamma$ is in $\mathrm{PGL}^+(2,\mathbb{F})g^{-1}\mathrm{PGL}^-(2,\mathbb{F})f$ and has precisely one fixed point. Assuming (FP) this is not possible. This shows that there is a unique circle through $(x_1, y_1)$ that touches $K_\gamma$ at $(x_0, y_0)$. □

**Remark 2.8.**

(1) Any two order-preserving permutations of $\mathbb{R}$ with respect to the Euclidean ordering, that is, any two orientation preserving homeomorphisms of $\mathbb{R}$, have the fixed-point property (FP); cf. [12, Lemma 2.3 and Theorem 2.6].

(2) Similarly, any two order-preserving permutations of a finite field of odd order have the fixed-point property (FP); cf. [6, Proposition 2.2]. This follows from the fact that a finite hyperbola structure is already a Minkowski plane, if circles and parallel classes contain the same number of points.

(3) A special case of (FP) is condition (3) in [6, Theorem 1]. Suppose that $f$ and $g$ both fix $\infty$. If we let $a, b, c, d \in \mathbb{F}$ such that $a > 0 > c$, then the permutations $\alpha(x) = a^{-1}(x - b)$ and $\beta(x) = cx + d$ belong to $\mathrm{PGL}^+(2, \mathbb{F})$ and $\mathrm{PGL}^-(2, \mathbb{F})$ respectively. Furthermore, $\alpha$ and $\beta$ both fix $\infty$. Hence $\gamma = \alpha g^{-1} \beta f$ fixes $\infty$. Therefore, $\gamma$ must have a fixed point in $\mathbb{F}$, that is, there is an $x \in \mathbb{F}$ such that $\beta f(x) = g\alpha^{-1}(x)$. Thus $g(ax + b) - cf(x) = d$. Since we can choose $d$ arbitrarily, this shows that the map $x \mapsto g(ax + b) - cf(x)$ must be surjective. However, this also follows from the fact that the residual plane at $(\infty, \infty)$ is isomorphic to the affine plane $\mathcal{C}_{f,g}$ constructed in [6].

**Theorem 2.9.** *$\mathcal{M}(\mathbb{F}; f, g)$, $f, g \in \Pi^+(\overline{\mathbb{F}})$ satisfying (FP), is Miquelian if and only if $f, g \in \mathrm{PGL}^+(2, \mathbb{F})\alpha$ with the same $\alpha \in \mathrm{Aut}^+(\mathbb{F})$.*

*Proof.* If $f, g \in \mathrm{PGL}^+(2, \mathbb{F})\alpha$ with $\alpha \in \mathrm{Aut}^+(\mathbb{F})$, then $\mathcal{M}(\mathbb{F}; f, g) = \mathcal{M}(\mathbb{F}; id, id)$ by Remark 2.2. Conversely, suppose that $\mathcal{M}(\mathbb{F}; f, g)$ is Miquelian. Then the residual plane at $(\infty, \infty)$ is Desarguesian. According to Remark 2.2 we can assume that $f, g \in \Pi^+_{\infty,0,1}(\overline{\mathbb{F}})$. The dualisation of [13, Theorem 3.9] yields $f = g \in \mathrm{Aut}^+(\mathbb{F})$. □

## 3. Isomorphisms between the planes $\mathcal{M}(\mathbb{F}; f, g)$

An isomorphism between Minkowski planes is a bijection of the point sets that maps circles onto circles. In this section we investigate isomorphisms between Minkowski planes of the form $\mathcal{M}(\mathbb{F}; f, g)$ where we always make the assumption that $f, g \in \Pi^+(\overline{\mathbb{F}})$ and that $f$ and $g$ satisfy the fixed point property (FP). There are four fundamental types of isomorphisms between such Minkowski planes.

**3.1.** Isomorphisms induced by linear fractional maps:

$$(x, y) \mapsto (\alpha(x), \beta(y))$$

where $\alpha, \beta \in \mathrm{PGL}^+(2, \mathbb{F})$. A circle $K_\tau$, $\tau \in \mathcal{F}_{f,g}$, is mapped to $K_{\beta\tau\alpha^{-1}}$. This map yields an isomorphism from $\mathcal{M}(\mathbb{F}; f, g)$ to $\mathcal{M}(\mathbb{F}; f\alpha^{-1}, g\beta^{-1})$.

**3.2.** Isomorphisms induced by isomorphisms from a half-ordered field $\mathbb{F}$ to a half-ordered field $\mathbb{E}$:

$$(x, y) \mapsto (\phi(x), \phi(y))$$

where $\phi$ is an order-preserving isomorphism from $\mathbb{F}$ to $\mathbb{E}$. A circle $K_\tau$, $\tau \in \mathcal{F}_{f,g}$, is mapped to $K_{\phi\tau\phi^{-1}}$. This map yields an isomorphism from $\mathcal{M}(\mathbb{F}; f, g)$ to the plane $\mathcal{M}(\mathbb{E}; \phi f\phi^{-1}, \phi g\phi^{-1})$.

**3.3.** An isomorphism that interchanges the components of the circle sets:

$$(x, y) \mapsto (f(x), g(y)).$$

A circle $K_\tau$, $\tau \in \mathcal{F}_{f,g}$, is mapped to $K_{g\tau f^{-1}}$. This map yields an isomorphism from $\mathcal{M}(\mathbb{F}; f, g)$ to $\mathcal{M}(\mathbb{F}; f^{-1}, g^{-1})$.

**3.4.** An isomorphism that interchanges (+)- and (−)-parallel classes:

$$(x, y) \mapsto (y, x).$$

A circle $K_\tau$, $\tau \in \mathcal{F}_{f,g}$, is mapped to $K_{\tau^{-1}}$. This map yields an isomorphism from $\mathcal{M}(\mathbb{F}; f, g)$ to $\mathcal{M}(\mathbb{F}; g, f)$.

**Lemma 3.5.** *$\mathcal{M}(\mathbb{F}; f, g)$ is Miquelian if and only if at least one residual plane is Desarguesian.*

*Proof.* Using an isomorphism from $\mathcal{M}(\mathbb{F}; f, g)$ to $\mathcal{M}(\mathbb{F}; f', g')$ (where $f' = \alpha' f \alpha^{-1}$, $g' = \beta' g \beta^{-1}$ for suitable $\alpha, \alpha', \beta, \beta' \in \mathrm{PGL}^+(2, \mathbb{F})$, so that $f', g' \in \Pi_{\infty,0,1}(\overline{\mathbb{F}})$, see Remark 2.2) of type 3.1, if necessary, one can assume that the residual plane of $\mathcal{M}(\mathbb{F}; f', g')$ at $(\infty, \infty)$ is Desarguesian. As in the proof of Theorem 2.9 we see that $f' = g' \in \mathrm{Aut}^+(\mathbb{F})$. Hence $\mathcal{M}(\mathbb{F}; f', g')$ is Miquelian. Therefore $\mathcal{M}(\mathbb{F}; f, g)$ is also Miquelian. □

**Theorem 3.6.** *Let $\gamma$ be an isomorphism from $\mathcal{M}(\mathbb{F}; f, g)$ to $\mathcal{M}(\mathbb{E}; f', g')$. If $\mathbb{F} \not\simeq GF(9)$, the Galois field of order nine, and if $\mathcal{M}(\mathbb{F}; f, g)$ is non-Miquelian, then $\gamma$ is a composition of isomorphisms of types 3.1 to 3.4.*

*Proof.* Using an isomorphism of $\mathcal{M}(\mathbb{F}; f, g)$ of type 3.4, if necessary, one can assume that (+)-parallel classes are mapped to (+)-parallel classes and (–)-parallel classes are mapped to (–)-parallel classes. Using an isomorphism of $\mathcal{M}(\mathbb{F}; f, g)$ of type 3.3, if necessary, one can further assume that the circle $K_{id}$ is mapped to a circle described by an order-preserving permutation. Applying an isomorphism of type 3.1, if necessary, we then can ensure that the points $(\infty, \infty), (0, 0), (1, 1)$ in $\mathcal{M}(\mathbb{F}; f, g)$ are mapped to the points $(\infty, \infty), (0, 0), (1, 1)$ in $\mathcal{M}(\mathbb{E}; f', g')$.

We now consider the residual planes at $(\infty, \infty)$. $\gamma$ induces an isomorphism between the projective extensions of these planes. Furthermore, points of the frame $u$, $v$, $(0, 0)$, $(1, 1)$ in one plane where $u$ and $v$ denote the infinite points of (+)- and (–)-parallel classes, respectively, are mapped to the corresponding points in the other plane. According to the dualisation of [13, Prop. 2.6], this isomorphism is induced by an order-preserving isomorphism from $\mathbb{F}$ to $\mathbb{E}$ unless $\mathbb{F} \simeq GF(9)$ or the residual plane is Desarguesian. This isomorphism of projective planes extends to an isomorphism of Minkowski planes of the the form 3.2. By Lemma 3.5 no residual plane is Desarguesian unless $\mathcal{M}(\mathbb{F}; f, g)$ is Miquelian. □

An immediate consequence of Theorem 3.6 and the form of the fundamental isomorphisms 3.1 to 3.4 is the following.

**Corollary 3.7.** *Assume that $\mathbb{F} \not\simeq GF(9)$ and that $\mathcal{M}(\mathbb{F}; f, g)$ is non-Miquelian. An isomorphism from $\mathcal{M}(\mathbb{F}; f, g)$ to $\mathcal{M}(\mathbb{E}; f', g')$ maps each component of the circle set of $\mathcal{M}(\mathbb{F}; f, g)$ to a component of the circle set of $\mathcal{M}(\mathbb{E}; f', g')$.*

**Remark 3.8.**

(1) If $\mathcal{M}(\mathbb{F}; f, g)$ and $\mathcal{M}(\mathbb{E}; f', g')$ are Miquelian, an isomorphism between the two planes still is a composition of isomorphisms of types 3.1 to 3.4 where, however, 3.2 is defined for an arbitrary isomorphism from $\mathbb{F}$ to $\mathbb{E}$ (not necessarily order-preserving).

(2) Miquelian Minkowski planes can be isomorphic even when the corresponding fields are not order-isomorphic. For example, consider the field $\mathbb{F} = \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$ with the Euclidean ordering and the proper half-ordering defined by $x + y\sqrt{2} \in \mathbb{P}$ if and only if

its norm $x^2 - 2y^2$ is positive in the Euclidean ordering of $\mathbb{Q}$. Then $\mathcal{M}(\mathbb{F}; id, id)$ is the Miquelian Minkowski plane over $\mathbb{F}$ irrespective of the chosen half-ordering. However, $\mathbb{F}$ with the Euclidean ordering and $\mathbb{F}$ with the proper half-ordering defined above are not isomorphic as half-ordered fields.

(3) There are precisely two Minkowski planes of order 9; see [11, Theorem C]. One is the Miquelian plane over the Galois field $GF(9)$ and the other is the Minkowski plane over the Dickson near-field of type {3,2}; see also [5] for the latter plane. The non-Miquelian plane is isomorphic to $\mathcal{M}(GF(9); \phi, id)$ where $\phi$ is the Frobenius automorphism $x \mapsto x^3$ of $GF(9)$.

(4) Every known finite Minkowski plane of odd order is isomorphic to a plane $\mathcal{M}(\mathbb{F}; \phi, id)$ for some automorphism $\phi$ of $\mathbb{F}$.

## 4. Automorphisms of $\mathcal{M}(\mathbb{F}; f, g)$

**4.1.** The collection $\Gamma = \mathrm{Aut}(\mathcal{M})$ of all automorphisms of $\mathcal{M}$ (i.e. isomorphisms of $\mathcal{M}$ onto itself) is a group with respect to composition. Let $\Gamma_{f,g}$ denote the automorphism group of $\mathcal{M}(\mathbb{F}; f, g)$, $f, g \in \Pi^+(\overline{\mathbb{F}})$ satisfying (FP). An automorphism $\gamma \in \Gamma_{f,g}$ has the form $(x, y) \mapsto (\alpha(x), \beta(y))$ or the form $(x, y) \mapsto (\alpha(y), \beta(x))$ for two permutations $\alpha$, $\beta$ of $\overline{\mathbb{F}}$ depending on whether $\gamma$ maps (+)-parallel classes to (+)-parallel classes and (–)-parallel classes to (–)-parallel classes or exchanges (+)- and (–)-parallel classes. In the former case, $\gamma$ maps a circle $K_\tau$, $\tau \in \mathcal{F}_{f,g}$, to $K_{\beta\tau\alpha^{-1}}$. In the latter case, $K_\tau$, $\tau \in \mathcal{F}_{f,g}$, is taken to $K_{\beta\tau^{-1}\alpha^{-1}}$.

We assume that the Minkowski plane $\mathcal{M}(\mathbb{F}; f, g)$ is not Miquelian and that $\mathbb{F}$ is not the field of order nine. Since the circle set has two components (namely, all graphs of permutations in $\mathrm{PGL}^+(2, \mathbb{F})$ and all graphs of permutations in $g^{-1}\mathrm{PGL}^-(2, \mathbb{F})f$), $\gamma$ can either fix both components or interchange these components; see Corollary 3.7. In summary we obtain four possible cases:

(1) $(x, y) \mapsto (\alpha(x), \beta(y))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that preserves (+)- and (–)-parallel classes and fixes each of the two components of the circle set if and only if

$$\begin{aligned} \beta\mathrm{PGL}^+(2, \mathbb{F})\alpha^{-1} &\subseteq \mathrm{PGL}^+(2, \mathbb{F}) \quad \text{and} \\ g\beta g^{-1}\mathrm{PGL}^-(2, \mathbb{F})(f\alpha f^{-1})^{-1} &\subseteq \mathrm{PGL}^-(2, \mathbb{F}); \end{aligned}$$

(2) $(x, y) \mapsto (\alpha(x), \beta(y))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that preserves (+)- and (–)-parallel classes and exchanges the two components of the circle set if and only if

$$\begin{aligned} g\beta\mathrm{PGL}^+(2, \mathbb{F})(f\alpha)^{-1} &\subseteq \mathrm{PGL}^-(2, \mathbb{F}) \quad \text{and} \\ \beta g^{-1}\mathrm{PGL}^-(2, \mathbb{F})(\alpha f^{-1})^{-1} &\subseteq \mathrm{PGL}^+(2, \mathbb{F}); \end{aligned}$$

(3) $(x, y) \mapsto (\alpha(y), \beta(x))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that exchanges (+)- and (–)-parallel classes and leaves each component of the circle set invariant if and only if

$$\begin{aligned} \beta\mathrm{PGL}^+(2, \mathbb{F})\alpha^{-1} &\subseteq \mathrm{PGL}^+(2, \mathbb{F}) \quad \text{and} \\ g\beta f^{-1}\mathrm{PGL}^-(2, \mathbb{F})(f\alpha g^{-1})^{-1} &\subseteq \mathrm{PGL}^-(2, \mathbb{F}); \end{aligned}$$

(Note that $\mathrm{PGL}^+(2, \mathbb{F})^{-1} = \mathrm{PGL}^+(2, \mathbb{F})$ and $\mathrm{PGL}^-(2, \mathbb{F})^{-1} = \mathrm{PGL}^-(2, \mathbb{F})$.)

(4) $(x, y) \mapsto (\alpha(y), \beta(x))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that exchanges the two components of the circle set and also exchanges (+)- and (–)-parallel classes if and only if

$$\begin{aligned} g\beta \mathrm{PGL}^+(2,\mathbb{F})(f\alpha)^{-1} &\subseteq \mathrm{PGL}^-(2,\mathbb{F}) \quad \text{and} \\ \beta f^{-1}\mathrm{PGL}^-(2,\mathbb{F})(\alpha g^{-1})^{-1} &\subseteq \mathrm{PGL}^+(2,\mathbb{F}). \end{aligned}$$

The following Lemma shows that these conditions severely restrict the possible forms of $\alpha$ and $\beta$.

**Lemma 4.2.** *Let $L_2^+(\mathbb{F}) = \{x \mapsto rx + t \mid t \in \mathbb{F}, r \in \mathbb{P}\} \leq \mathrm{PGL}^+(2,\mathbb{F})$. We assume that $\phi L_2^+(\mathbb{F})\psi^{-1} \subseteq \mathrm{PGL}^+(2,\mathbb{F})$ for two permutations $\phi$, $\psi$ of $\overline{\mathbb{F}}$. Then $\phi, \psi \in \mathrm{PGL}(2,\mathbb{F})\alpha$ for some $\alpha \in \mathrm{Aut}^+(\mathbb{F})$. (The same automorphism $\alpha$ for both permutations.)*

*Proof.* Replacing $\phi$ and $\psi$ by $\sigma\phi$ and $\tau\psi$, respectively, for two suitable permutations $\sigma, \tau \in \mathrm{PGL}(2,\mathbb{F})$, we may assume that $\phi$ and $\psi$ both fix $\infty$, 1 and 0. We consider the subgroup $\{x \mapsto x + t \mid t \in \mathbb{F}\} \leq L_2^+(\mathbb{F})$. For each $t \in \mathbb{F}$ there then exist $a_t, b_t, c_t, d_t \in \mathbb{F}$, $a_t d_t - b_t c_t \neq 0$, such that

$$\phi(x+t) = \frac{a_t\psi(x) + b_t}{c_t\psi(x) + d_t}$$

for all $x \in \overline{\mathbb{F}}$. Evaluating both sides at $x = \infty$ and $x = 0$ gives us $c_t = 0$ and $\frac{b_t}{d_t} = \phi(t)$. For $t = 0$ we then obtain $\phi(x) = \frac{a_0}{d_0}\psi(x)$. Evaluating at $x = 1$ yields $\frac{a_0}{d_0} = 1$ and thus $\phi = \psi$. Let $\alpha_t = \frac{a_t}{d_t}$. Then

$$\phi(x+t) = \alpha_t\phi(x) + \phi(t)$$

for all $x, t \in \mathbb{F}$. Since the left-hand side is symmetrical in $x$ and $t$, we find that

$$\alpha_t = \gamma\phi(t) + 1$$

for some constant $\gamma \in \mathbb{F}$. Since $\alpha_t \neq 0$ for all $t \in \mathbb{F}$, we must have $\gamma = 0$. (Otherwise $t_0 = \phi^{-1}(-\frac{1}{\gamma})$ is defined and $\alpha_{t_0} = 0$.) Hence $\alpha_t = 1$ is constant. Therefore the restriction of $\phi$ to $\mathbb{F}$ is additive.

We now consider the subgroup $\{\rho_r : x \mapsto rx \mid r \in \mathbb{P}\} \leq L_2^+(\mathbb{F})$. For each $r \in \mathbb{P}$ there then exist $a_r, b_r, c_r, d_r \in \mathbb{F}$, $a_r d_r - b_r c_r \neq 0$, such that

$$\phi(xr) = \frac{a_r\phi(x) + b_r}{c_r\phi(x) + d_r}$$

Evaluating both sides at $x = \infty$ and $x = 0$ gives us $c_r = b_r = 0$. Evaluating at $x = 1$ yields $\phi(r) = \frac{a_r}{d_r}$. Thus $\phi(rx) = \phi(r)\phi(x)$ for all $x \in \mathbb{F}$ and $r \in \mathbb{P}$. According to [13, Lemma 2.5], $\phi$ is an automorphism of $\mathbb{F}$. (This is also true when $\mathbb{F}$ has order nine, since we have the multiplicative rule for all positive elements.) Hence $\phi = \psi \in \mathrm{Aut}(\mathbb{F})$. Then $\phi\rho_r\phi^{-1} = \rho_{\phi(r)} \in \mathrm{PGL}^+(2,\mathbb{F})$ for all $r > 0$. Therefore, $\phi(r) > 0$ for all $r > 0$ and $\phi$ is order-preserving. Hence $\phi = \psi \in \mathrm{Aut}^+(\mathbb{F})$. □

Since $\sigma L_2^+(\mathbb{F})$ is a coset of $L_2^+(\mathbb{F})$ entirely contained in $\mathrm{PGL}^-(2,\mathbb{F})$ for $\sigma \in \mathrm{PGL}^-(2,\mathbb{F})$ one readily obtains

**Corollary 4.3.** *Let $\mathcal{M}(\mathbb{F}; f, g)$ be non-Miquelian and assume that $\mathbb{F} \not\cong GF(9)$. Then the following holds:*

*(1) $(x, y) \mapsto (\alpha(x), \beta(y))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that preserves (+)- and (–)-parallel classes and fixes each of the two components of the circle set if and only if*

$$\begin{aligned} \alpha &\in \mathrm{PGL}(2, \mathbb{F})\phi \cap f^{-1}\mathrm{PGL}(2, \mathbb{F})\psi f \\ \beta &\in \mathrm{PGL}(2, \mathbb{F})\phi \cap g^{-1}\mathrm{PGL}(2, \mathbb{F})\psi g \\ \beta\alpha^{-1} &\in \mathrm{PGL}^+(2, \mathbb{F}) \end{aligned}$$

*for $\phi, \psi \in \mathrm{Aut}^+(\mathbb{F})$.*

*(2) $(x, y) \mapsto (\alpha(x), \beta(y))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that preserves (+)- and (–)-parallel classes and exchanges the two components of the circle set if and only if*

$$\begin{aligned} \alpha &\in f^{-1}\mathrm{PGL}(2, \mathbb{F})\phi \cap \mathrm{PGL}(2, \mathbb{F})\psi f \\ \beta &\in g^{-1}\mathrm{PGL}(2, \mathbb{F})\phi \cap \mathrm{PGL}(2, \mathbb{F})\psi g \\ \beta\alpha^{-1} &\in g^{-1}\mathrm{PGL}^-(2, \mathbb{F}) f \end{aligned}$$

*for $\phi, \psi \in \mathrm{Aut}^+(\mathbb{F})$.*

*(3) $(x, y) \mapsto (\alpha(y), \beta(x))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that exchanges (+)- and (–)-parallel classes and leaves each component of the circle set invariant if and only if*

$$\begin{aligned} \alpha &\in \mathrm{PGL}(2, \mathbb{F})\phi \cap f^{-1}\mathrm{PGL}(2, \mathbb{F})\psi g \\ \beta &\in \mathrm{PGL}(2, \mathbb{F})\phi \cap g^{-1}\mathrm{PGL}(2, \mathbb{F})\psi f \\ \beta\alpha^{-1} &\in \mathrm{PGL}^+(2, \mathbb{F}) \end{aligned}$$

*for $\phi, \psi \in \mathrm{Aut}^+(\mathbb{F})$.*

*(4) $(x, y) \mapsto (\alpha(y), \beta(x))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$ that exchanges the two components of the circle set and also exchanges (+)- and (–)-parallel classes if and only if*

$$\begin{aligned} \alpha &\in f^{-1}\mathrm{PGL}(2, \mathbb{F})\phi \cap \mathrm{PGL}(2, \mathbb{F})\psi g \\ \beta &\in g^{-1}\mathrm{PGL}(2, \mathbb{F})\phi \cap \mathrm{PGL}(2, \mathbb{F})\psi f \\ \beta\alpha^{-1} &\in g^{-1}\mathrm{PGL}^-(2, \mathbb{F}) f \end{aligned}$$

*for $\phi, \psi \in \mathrm{Aut}^+(\mathbb{F})$.*

**Remark 4.4.**

(1) The automorphism group of the Miquelian Minkowski plane $\mathcal{M}(\mathbb{F}; id, id)$ over $\mathbb{F}$ is the semi-direct product of $\mathrm{PGL}(2, \mathbb{F}) \times \mathrm{PGL}(2, \mathbb{F})$ and the subgroup generated by $\mathrm{Aut}(\mathbb{F})$ and the inversion $(x, y) \mapsto (y, x)$; cf. Remark 3.8.(1).

(2) The automorphism group of the non-Miquelian Minkowski plane of order 9, i.e. the plane $\mathcal{M}(GF(9); \phi, id)$, cf. Remark 3.8.(3), is generated by $\mathrm{PGL}^+(2, \mathbb{F}) \times \mathrm{PGL}^+(2, \mathbb{F})$ and the automorphisms $(x, y) \mapsto (y, x)$, $(x, y) \mapsto (\phi(x), y)$ and $(x, y) \mapsto (nx, ny)$ where $n \in \mathbb{F}$, $n < 0$. Hence, every automorphism of $\mathcal{M}(GF(9); \phi, id)$ maps each component of the circle set to a component. Furthermore, the automorphism group of this plane has order $2^9 \cdot 3^4 \cdot 5^2 = 1,036,800$.

(3) Over some fields $\mathbb{F}$ one can achieve by choosing $f$ and $g$ suitably that $\mathcal{M}(\mathbb{F}; f, g)$ admits no automorphism except the identity; see [12, Theorem 3.12] for $\mathbb{F} = \mathbb{R}$.

An involutory automorphism $\alpha$ of a Minkowski plane is called a *circle-symmetry at a circle $K$* or simply a *symmetry at $K$* if and only if $\alpha$ fixes precisely the points of $K$. A circle-symmetry exchanges (+)- and (−)-parallel classes; more precisely, a point $p$ is mapped to the unique point that is (+)-parallel to $K \cap |p|_-$ and (−)-parallel to $K \cap |p|_+$. In particular, a symmetry at a circle $K$ is unique, if it exists.

**Corollary 4.5.** *A Minkowski plane $\mathcal{M}(\mathbb{F}; f, g)$ admits a symmetry at a circle $K_\sigma$ if and only if*

– $\sigma \in \mathrm{PGL}^+(2, \mathbb{F}) \cap g^{-1}\mathrm{P\Gamma L}^+(2, \mathbb{F})f$ *or*

– $\sigma \in g^{-1}\mathrm{PGL}^-(2, \mathbb{F})f \cap \mathrm{P\Gamma L}^-(2, \mathbb{F})$ *and* $\sigma^2 \in \mathrm{PGL}^+(2, \mathbb{F})$.

*If $-1 < 0$ with respect to the half-ordering of $\mathbb{F}$, then the latter condition is equivalent to*

– $\sigma \in g^{-1}\mathrm{PGL}^-(2, \mathbb{F})f \cap \mathrm{PGL}^-(2, \mathbb{F})$.

*Proof.* Suppose that $\mathcal{M}(\mathbb{F}; f, g)$ admits a symmetry at a circle $K_\sigma$, that is,

$$(x, y) \mapsto (\sigma^{-1}(y), \sigma(x))$$

is an automorphism of the Minkowski plane.
If $\mathcal{M}(\mathbb{F}; f, g)$ is Miquelian, then $f, g \in \mathrm{PGL}^+(2, \mathbb{F})\phi$ with $\phi \in \mathrm{Aut}^+(\mathbb{F})$. In this case $\mathrm{PGL}^+(2, \mathbb{F}) \cap g^{-1}\mathrm{P\Gamma L}^+(2, \mathbb{F})f = \mathrm{PGL}^+(2, \mathbb{F})$ and $g^{-1}\mathrm{PGL}^-(2, \mathbb{F})f \cap \mathrm{P\Gamma L}^-(2, \mathbb{F}) = \mathrm{PGL}^-(2, \mathbb{F})$. Indeed, a Miquelian plane admits a symmetry at every circle.

In the non-Miquelian Minkowski plane of order nine, that is, the Minkowski plane $\mathcal{M}(GF(9); \phi, id)$ where $\phi$ is the Frobenius automorphism $x \mapsto x^3$ of $GF(9)$, cf. Remark 3.8.(3), $\mathrm{PGL}^+(2, \mathbb{F}) \cap g^{-1}\mathrm{P\Gamma L}^+(2, \mathbb{F})f = \mathrm{PGL}^+(2, \mathbb{F})$ and $g^{-1}\mathrm{PGL}^-(2, \mathbb{F})f \cap \mathrm{P\Gamma L}^-(2, \mathbb{F}) = \mathrm{PGL}^-(2, \mathbb{F})\phi$. Hence every permutation in $\mathcal{F}_{\phi,id}$ occurs. Indeed, this plane admits a symmetry at every circle, cf. [3, Satz 1].

We now assume that $\mathcal{M}(\mathbb{F}; f, g)$ is non-Miquelian and that $\mathbb{F} \not\cong GF(9)$. Since $\sigma^2$ is order-preserving, the circle-symmetry at $K_\sigma$ is an automorphism of the form 4.3.(3). Hence we must have that $\sigma^2 \in \mathrm{PGL}^+(2, \mathbb{F})$ and that $\sigma \in \mathrm{P\Gamma L}(2, \mathbb{F}) \cap g^{-1}\mathrm{P\Gamma L}(2, \mathbb{F})f$. Furthermore, $\sigma \in \mathrm{PGL}^+(2, \mathbb{F}) \cup g^{-1}\mathrm{PGL}^-(2, \mathbb{F})f$. If $\sigma$ is order-preserving, we obtain the first condition; if $\sigma$ is order-reversing, we obtain the second condition. Conversely, it readily follows from Corollary 4.3 that each of the two conditions implies that $(x, y) \mapsto (\sigma^{-1}(y), \sigma(x))$ is an automorphism of $\mathcal{M}(\mathbb{F}; f, g)$.

Finally, let $-1 < 0$ and let $\sigma = \pi\phi$ with $\pi \in \mathrm{PGL}^-(2, \mathbb{F})$ and $\phi \in \mathrm{Aut}^+(\mathbb{F})$ (so that $\sigma \in \mathrm{P\Gamma L}^-(2, \mathbb{F})$. Then $\sigma^2 = \pi\phi\pi\phi = \pi\tilde{\pi}\phi^2 \in \mathrm{PGL}^+(2, \mathbb{F})$ for some $\tilde{\pi} \in \mathrm{PGL}^-(2, \mathbb{F})$. Hence $\phi^2 = id$. But now $x - \phi(x) = \phi^2(x) - \phi(x) = \phi(\phi(x) - x) = -\phi(x - \phi(x))$ for all $x \in \mathbb{F}$. Since $\phi$ is order-preserving and because $-1 < 0$ this implies that $x - \phi(x) = 0$, that is, $\phi = id$. This shows that $\sigma \in \mathrm{PGL}^-(2, \mathbb{F})$. □

**Remark 4.6.** A symmetric Minkowski plane (or a Minkowski plane satisfying the symmetry axiom) is a Minkowski plane that admits a symmetry at each circle and such that the composition of circle-symmetries at three circles passing through two nonparallel points is again a circle-symmetry, see [4] for a different definition of symmetric Minkowski plane. It is well-known that a symmetric Minkowski plane is Miquelian, cf. [1], [2], [3, Satz 2] or [8]. Minkowski planes admitting a symmetry at each circle were studied in [3]. The finite planes are precisely those that can be represented over a Tits near-field, cf. [9, Theorem B]. These planes can also be characterized as those Minkowski planes for which the collection of circle describing permutations is a group. In our family of Minkowski planes the finite examples of these planes are, up to isomorphism, precisely the planes $\mathcal{M}(GF(p^{2n}), x \mapsto x^{p^n}, id)$.

## References


[1] Artzy, R.: *A Pascal theorem applied to Minkowski geometry.* J. Geom. **3** (1973), 93–105.

[2] Dienst, K. J.: *Minkowski-Ebenen mit Spiegelungen.* Mh. Math. **84** (1977), 197–208,

[3] Hartmann, E.: *Zykel- und Erzeugendenspiegelungen in Minkowski-Ebenen.* Geom. Dedicata **11** (1981), 489–503.

[4] Heise, W.; Karzel, H.: *Symmetrische Minkowski-Ebenen.* J. Geom. **3** (1973), 5–20.

[5] Jakóbowski, J.: *Conjugate points of a translation plane of order* 9 *and its extensibility to a Minkowski plane.* Zeszyty Nauk. Geom. **17** (1988), 85–96.

[6] Jakóbowski, J.: *A new generalization of Moulton affine planes.* Geom. Dedicata **42** (1992), 243–253.

[7] Jakóbowski, J.: *A full generalization of Moulton's method applied to Minkowski axioms.* Zeszyty Nauk. Geom. **20**, 13–21.

[8] Karzel, H.: *Symmetrische Permutationsmengen.* Aequat. Math. **17** (1978), 83–90.

[9] Percy, N.: *Finite Minkowski planes in which every circle-symmetry is an automorphism.* Geom. Dedicata **10** (1981), 269–282.

[10] Schenkel, A.: *Topologische Minkowskiebenen.* Dissertation, Erlangen-Nürnberg 1980.

[11] Steinke, G. F.: *A remark on Benz planes of order 9.* Ars Comb. **34** (1992), 257–267.

[12] Steinke, G. F.: *A family of 2-dimensional Minkowski planes with small automorphism groups.* Result. Math. **26** (1994), 131–142.

[13] Steinke, G. F.: *Semiclassical projective planes over half-ordered fields.* Geom. Dedicata **58** (1995), 21–44.


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