Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 2, 575-582.

On Propellers from Triangles

Zvonko Čerin

Kopernikova 7, 10010 Zagreb, Croatia
e-mail: CERIN@MATH.HR

Abstract. We prove several improvements and analogs of Leon Bankoff's theorem on asymmetric propellers from directly similar triangles.

1. Propellers from directly similar triangles

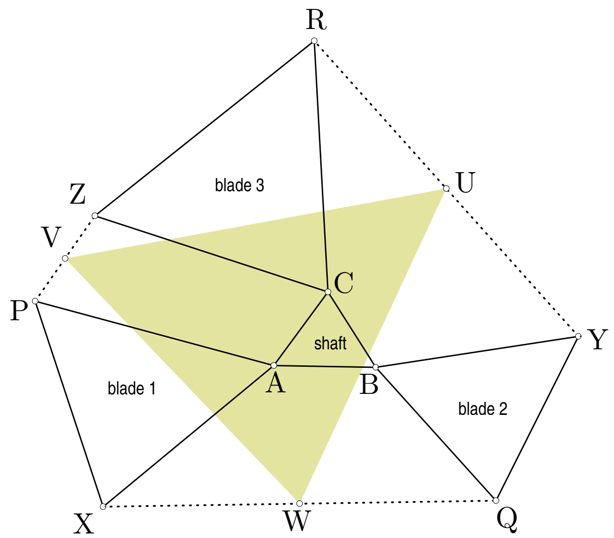

Figure 1. Asymmetric propeller from directly similar triangles determines in every position of blades directly similar triangle UVW

The propeller from triangles has a triangle ABC as a shaft and three triangles APX, YBQ, and RZC as blades. There are some interesting results on propellers from triangles which

0138-4821/93 $ 2.50

show that regardless of the way in which the blades are glued to the shaft the triangle UVW on the midpoints of segments YR, ZP, and XQ shares certain properties with the shaft and the blades. For example, if the shaft and the blades are equilateral triangles, then UVW is also an equilateral triangle. This equilateral propeller theorem and its special cases and extensions have long history and numerous contributions. Many of them are collected in Section 2 of the excellent survey [8]. From it we recall papers by A. Hofmann [6] referring to the historical origin of the propeller theorem, by W. Götz [4] showing a subcase of Bankoff's theorem which corresponds with a joint figure from kinematics already known in the seventies of the 19th century and connected with names like W. Clifford and A. Cayley, and (regarding the interesting variety of proof methods) the papers [12], [7], [9], and [11]. (The latter paper shows the connection between the famous three-circles theorem and propellers, see also Math. Reviews **81c**:51010.)

Another example is the following improvement of the equilateral propeller theorem by Leon Bankoff [3] where the shaft and the blades are directly similar (see the above Figure 1). Its proof in [3] is geometric while our proof below uses complex numbers and is algebraic.

Theorem 1. *Let ABC, PQR, and XYZ be triangles such that ABC, APX, YBQ, and RZC are directly similar. Then the triangle UVW on midpoints of segments YR, ZP, and XQ is also directly similar to ABC.*

Proof. Let us use the convention that a point with a capital Latin letter as a label is represented with the complex number (its *affix*) which is denoted with the same small Latin letter. The complex conjugate of p is $\bar{p}$ and $\bar{P}$ (the reflection of P in the real axis) is a point whose affix is $\bar{p}$.

It is well-known (see [5, p.57]) that triangles ABC and PQR are directly similar if and only if the determinant $D(ABC,\, PQR)$ of the matrix with the first row a, b, c, the second row p, q, r, and the third row 1, 1, 1 is zero.

Hence, if ABC and APX are directly similar, then $D(ABC,\, APX) = 0$ so that we can solve for x to get $x = -(d_b\, p + a\, d_a)/d_c$, where $d_a = b - c$, $d_b = c - a$, and $d_c = a - b$. Similarly, $y = -(d_c\, q + b\, d_b)/d_a$ and $z = -(d_a\, r + c\, d_c)/d_b$.

The determinant $D(ABC,\, UVW)$ is $(D_1 - D_2 - D_3)/2$, where

$$D_1 = \begin{vmatrix} a & b & c \\ r & p & q \\ 1 & 1 & 1 \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a & b & c \\ \frac{q\, d_c}{d_a} & \frac{r\, d_a}{d_b} & \frac{p\, d_b}{d_c} \\ 1 & 1 & 1 \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a & b & c \\ \frac{b\, d_b}{d_a} & \frac{c\, d_c}{d_b} & \frac{a\, d_a}{d_c} \\ 1 & 1 & 1 \end{vmatrix}.$$

One can easily check that $D_1 = D_2$ and $D_3 = 0$. Hence, $D(ABC,\, UVW) = 0$ so that triangles ABC and UVW are directly similar. □

From the above theorem we can get many related results if we observe that for directly similar triangles ABC and PQR and for any real number λ the triangle $\lambda_A^P \lambda_B^Q \lambda_C^R$ is also directly similar to ABC, where λ_A^P is A for $A = P$ and for $A \neq P$ and $\lambda = -1$ it is P while for $A \neq P$ and $\lambda \neq -1$ it is the unique point X on the line AP such that the quotient $|AX|/|XP|$ of oriented distances is λ. Since the affix of λ_A^P is $(a + \lambda\, p)/(\lambda + 1)$ (for $\lambda \neq -1$), the above claim follows from the fact that the determinant is a linear functional (of rows). In particular, from this we immediately get the following result.

Corollary 1. *Let ABC, PQR, and XYZ be triangles such that ABC, APX, YBQ, and RZC are directly similar. Then the triangle DEF on centroids of triangles ARY, BPZ, and CQX is directly similar to ABC.*

Proof. Since the centroids D, E, F divide segments AU, BV, CW in ratio 2, we can use Theorem 1 and the above observation for $\lambda = 2$ and triangles ABC and UVW. □

Another supply of directly similar to ABC triangles in Bankoff's propellers is offered by the following lemma.

Lemma 1. *Let ABC, PQR, and XYZ be triangles such that PX, QY, RZ have equal oriented length and are parallel to segments joining A, B, and C with the circumcentre O of ABC. Then the triangles ABC and XYZ are directly similar if and only if the triangles ABC and PQR are directly similar.*

Proof. Let the common oriented length of segments PX, QY, and RZ be a real number h. We can assume that the unit circle of all unimodular complex numbers is the circumcircle of ABC. Then $x = p + h\,a$, $y = q + h\,b$, and $z = r + h\,c$. The linearity of the determinant with respect to rows implies that $D(ABC, PQR) = 0$ if and only if $D(ABC, XYZ) = 0$. □

Corollary 2. *With the notation and the assumptions of Theorem 1, the triangle DEF on endpoints of segments at midpoints U, V, and W of equal oriented length and parallel to segments joining A, B, and C with the circumcentre O of ABC is directly similar to ABC.*

2. Rotating blades in Bankoff's propellers

In this section we explore what happens with the triangle UVW while points P, Q, and R rotate with constant angular speed on circles of equal radius with centres at A, B, and C, respectively.

Theorem 2. *Let points P, Q, and R rotate with the constant angular speed φ around the vertices of ABC on circles of radius k. Let X, Y, and Z be points such that triangles APX, YBQ, and RZC are directly similar to ABC. The midpoints U, V, and W of segments YR, ZP, and XQ rotate with the angular speed φ on circles with centres at midpoints A_m, B_m, and C_m of BC, CA, and AB and with radii k times $\frac{|BB_m|}{|BC|}$, $\frac{|CC_m|}{|CA|}$, and $\frac{|AA_m|}{|AB|}$, respectively.*

Proof. We again assume that the unit circle of all unimodular complex numbers is the circumcircle of ABC. Then $p = a + k\,e^{i\varphi}$, $q = b + k\,e^{i\varphi}$, and $r = c + k\,e^{i\varphi}$. It follows that $x = a - k\,e^{i\varphi}\,d_b/d_c$, $y = b - k\,e^{i\varphi}\,d_c/d_a$, and $z = c - k\,e^{i\varphi}\,d_a/d_b$, so that $u = (b+c)/2 + k\,e^{i\varphi}\,(d_a - d_c)/(2\,d_a)$. From this the claim about the rotation of the point U follows easily. The points V and W are treated similarly. □

We shall use S to denote the area of ABC and ℓ_a, ℓ_b, and ℓ_c for lengths of its sides. The symmetric functions of ℓ_a, ℓ_b, and ℓ_c we denote as follows.

$$s = \ell_a + \ell_b + \ell_c, \quad m = \ell_a\,\ell_b\,\ell_c, \quad t = \ell_b\,\ell_c + \ell_c\,\ell_a + \ell_a\,\ell_b, \quad s_a = -\ell_a + \ell_b + \ell_c,$$

$$s_b = \ell_a - \ell_b + \ell_c, \quad s_c = \ell_a + \ell_b - \ell_c, \quad m_a = \ell_b\,\ell_c, \quad m_b = \ell_c\,\ell_a, \quad m_c = \ell_a\,\ell_b,$$

$$r_a = \ell_b - \ell_c, \quad r_b = \ell_c - \ell_a, \quad r_c = \ell_a - \ell_b, \quad z_a = \ell_b + \ell_c, \quad z_b = \ell_c + \ell_a, \quad z_c = \ell_a + \ell_b.$$

For each $k \geq 2$, s_k, s_{ka}, s_{kb}, and s_{kc} are derived from s, s_a, s_b, and s_c with the substitution $\ell_a = \ell_a^k$, $\ell_b = \ell_b^k$, $\ell_c = \ell_c^k$. In a similar fashion we can define analogous expressions using letters m, r, t, and z.

Theorem 3. *Under the assumptions of Theorem 2,*

- *The circumcentre of UVW rotates with the angular speed φ on the circle with the centre at the centre of the nine-point circle of ABC and with the radius k times $\frac{\sqrt{2\,s_4 - 5\,t_2}}{8\,S}$.*
- *The centroid of UVW rotates with the angular speed φ on the circle with the centre at the centroid of ABC and with the radius k times $\frac{\sqrt{21\,m_2 + 5\,g_1 - s_6 - 3\,g_2}}{6\,m}$, where $g_1 = g_{4,2}^{a,c} = \ell_a^4\,\ell_c^2 + \ell_c^4\,\ell_b^2 + \ell_b^4\,\ell_a^2$ and $g_2 = g_{4,2}^{a,b}$.*
- *The orthocentre of UVW rotates with the angular speed φ on the circle with the centre at the circumcentre of ABC and with the radius k times $\frac{\sqrt{g}}{8\,S\,m}$, where $g = g_{10}^{a} - g_{8,2}^{a,b} - 4\,g_{8,2}^{a,c} - 3\,g_{6,4}^{a,b} + 6\,g_{6,4}^{a,c} + g_{6,2,2}^{a,b,c} + 3\,g_{4,4,2}^{a,b,c}$.*

Proof. While it is possible to prove this theorem using complex numbers, with the help of computers, it is simpler to apply Cartesian coordinates because in this approach it is easier to identify the radii of revolving circles.

Here is an outline of the key steps in the proof for the circumcentre of UVW. Let $A(0,0)$, $B(0,\, h\,(f+g))$, and $C(\frac{(f^2-1)\,g\,h}{f\,g-1},\, \frac{2\,f\,g\,h}{f\,g-1})$, where h is the inradius of ABC and f and g are cotangents of angles $A/2$ and $B/2$. The coordinates of P are $\frac{k\,(1-\psi^2)}{1+\psi^2}$ and $\frac{2\,k\,\psi}{1+\psi^2}$, where $\psi = \tan\frac{\varphi}{2}$. In coordinates of Q and R we must add coordinates of B and C to these values.

In the next step we determine the coordinates of points X, Y, and Z and then U, V, and W. Let T be the circumcentre of UVW with coordinates T_x and T_y. When we eliminate the parameter ψ from equalities $x = T_x$ and $y = T_y$ the equation of the required locus will emerge. This equation has the form $(x - F_x)^2 + (y - F_y)^2 - r_O^2 = 0$, where F_x and F_y are coordinates of the centre F of the nine-point circle of ABC and r_O is an expression in terms of f, g, h, and k which has the above form when we apply equalities

$$f = \frac{z_a^2 - \ell_a^2}{4\,S}, \quad g = \frac{z_b^2 - \ell_b^2}{4\,S}, \quad h = \frac{2\,S}{s}. \qquad \square$$

3. Propellers from reversely similar and orthologic triangles

Recall that triangles ABC and PQR are *reversely similar* provided triangles ABC and $\bar{P}\bar{Q}\bar{R}$ are directly similar. In other words, $D(ABC,\, \bar{P}\bar{Q}\bar{R}) = 0$ is a necessary and sufficient condition for ABC and PQR to be reversely similar. In the next theorem we shall show that reverse similarity is equivalent with two interesting geometric properties of triangles known as paralogy and orthology whose definitions are as follows.

Triangles ABC and XYZ are *orthologic* provided the perpendiculars from the vertices of ABC on the sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC,\, XYZ]$. It is well-known (see [2] or [10]) that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars from the vertices of XYZ on the sides BC, CA, and AB of ABC are concurrent at the point $[XYZ,\, ABC]$.

Lemma 2. *Triangles XYZ and PQR are orthologic if and only if*

$$O(XYZ,\, PQR) = x(\bar{q}-\bar{r}) + \bar{x}(q-r) + y(\bar{r}-\bar{p}) + \bar{y}(r-p) + z(\bar{p}-\bar{q}) + \bar{z}(p-q) = 0.$$

Proof. The line QR is $[\bar{q}-\bar{r},\, r-q,\, q\,\bar{r}-\bar{q}\,r]$ so that the perpendicular per_{QR}^{X} through X onto QR is the line $[\bar{q}-\bar{r},\, q-r,\, x\,(\bar{r}-\bar{q})+\bar{x}\,(r-q)]$. The perpendiculars per_{RP}^{Y} and per_{PQ}^{Z} through Y and Z onto RP and PQ are relatives of per_{QR}^{X}. These three perpendiculars are concurrent if and only if $\Theta = 0$, where Θ denotes the determinant

$$\begin{vmatrix} \bar{q}-\bar{r} & q-r & x\,(\bar{r}-\bar{q})+\bar{x}\,(r-q) \\ \bar{r}-\bar{p} & r-p & y\,(\bar{p}-\bar{r})+\bar{y}\,(p-r) \\ \bar{p}-\bar{q} & p-q & z\,(\bar{q}-\bar{p})+\bar{z}\,(q-p) \end{vmatrix}.$$

But, $\Theta = O(XYZ,\, PQR)\,\Omega$, where $\Omega = p\,(\bar{q}-\bar{r}) + q\,(\bar{r}-\bar{p}) + r\,(\bar{p}-\bar{q})$. Since $\Omega = 0$ if and only if points P, Q, and R are collinear (and our assumptions exclude this possibility), we conclude that the triangles XYZ and PQR are orthologic if and only if $O(XYZ,\, PQR) = 0$. □

Triangles ABC and XYZ are *paralogic* provided the parallels through the vertices of ABC with the sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these parallels is denoted by $\|ABC,\, XYZ\|$. The relation of paralogy for triangles is symmetric so that the parallels through the vertices of XYZ with the sides BC, CA, and AB of ABC are concurrent at the point $\|XYZ,\, ABC\|$.

Lemma 3. *Triangles XYZ and PQR are paralogic if and only if*

$$P(XYZ,\, PQR) = x(\bar{q}-\bar{r}) - \bar{x}(q-r) + y(\bar{r}-\bar{p}) - \bar{y}(r-p) + z(\bar{p}-\bar{q}) - \bar{z}(p-q) = 0.$$

Proof. The line QR is $[\bar{q}-\bar{r},\, r-q,\, q\,\bar{r}-\bar{q}\,r]$ so that the parallel par_{QR}^{X} through X with QR is the line $[\bar{q}-\bar{r},\, r-q,\, x\,(\bar{r}-\bar{q})-\bar{x}\,(r-q)]$. The parallels par_{RP}^{Y} and par_{PQ}^{Z} through Y and Z with RP and PQ are relatives of par_{QR}^{X}. The rest of the proof is the same as the proof of Lemma 2. □

Theorem 4. *Triangles ABC and XYZ are reversely similar if and only if they are both paralogic and orthologic.*

Proof. Let $\Delta = D(XYZ,\, \bar{P}\bar{Q}\bar{R})$, $\Omega = O(XYZ,\, PQR)$, and $\Pi = P(XYZ,\, PQR)$. Since $\Pi + \Omega = 2\,\Delta$ and $\Pi - \Omega = -2\,\bar{\Delta}$, we see that $\Delta = 0$ if and only if $\Pi = 0$ and $\Omega = 0$. □

Theorem 5. *Let ABC, PQR, and XYZ be triangles such that APX, YBQ, and RZC are reversely similar to ABC. Then the triangle UVW on midpoints of segments YR, ZP, and XQ is orthologic to ABC and never paralogic to it.*

Proof. Since ABC and APX are reversely similar, $D(ABC,\, \bar{A}\bar{P}\bar{X}) = 0$ so that we can solve for x to get $x = -(\bar{d}_b\, p + a\, \bar{d}_a)/\bar{d}_c$. Similarly, $y = -(\bar{d}_c\, q + b\, \bar{d}_b)/\bar{d}_a$ and $z = -(\bar{d}_a\, r + c\, \bar{d}_c)/\bar{d}_b$. We can now easily compute affixes of midpoints U, V, and W and verify that $O(ABC, UVW) = 0$. On the other hand, $P(ABC,\, UVW)$ is equal (up to a sign) to $\Lambda = \bar{a}\, d_a + \bar{b}\, d_b + \bar{c}\, d_c$, where $\Lambda = 0$ is the condition for points A, B, and C to be collinear. □

With a similar argument, the above theorem could be slightly improved to the following statement: If ABC, PQR, and XYZ are triangles such that APX, YBQ, and RZC are orthologic to ABC, then the triangle UVW on midpoints of segments YR, ZP, and XQ is also orthologic to ABC. On the other hand, we have: If ABC, PQR, and XYZ are triangles such that APX, YBQ, and RZC are paralogic to ABC, then the triangle UVW on midpoints of segments YR, ZP, and XQ is never paralogic to ABC.

The method employed in the proof of Corollary 1 implies the following.

Corollary 3. *Let ABC, DEF, PQR, and XYZ be triangles such that APX, YBQ, and RZC are orthologic to ABC. Then the triangle UVW on centroids of triangles DYR, PEZ, and XQF is orthologic to ABC if and only if DEF is orthologic to ABC.*

Our last theorem on propellers with the blades orthologic to the shaft is similar to Corollary 2 and gives a large supply of triangles orthologic to the shaft ABC. For an efficient formulation of this result we need the following notation and definitions.

For an expression ε in terms of ℓ_a, ℓ_b, and ℓ_c and a real number h, let $\varepsilon[h]$ denote the triple $(h\,\varepsilon,\, h\,\varphi(\varepsilon),\, h\,\psi(\varepsilon))$. More precisely, the coordinates $\varepsilon[h]_1$, $\varepsilon[h]_2$, $\varepsilon[h]_3$ of $\varepsilon[h]$ are products with h of ε, the first cyclic permutation $\varphi(\varepsilon)$ of ε, and the second cyclic permutation $\psi(\varepsilon)$ of ε, respectively. For example, $a[h] = (h\,a,\, h\,b,\, h\,c)$ and $s_a[h] = (h\,s_a,\, h\,s_b,\, h\,s_c)$.

Let T_i, for $i = 1, \ldots, 13$, denote the following expressions: 1, ℓ_a^2, s_{2a}, ℓ_a/z_a, $1/z_a$, $3\,s_{2a} + 4\sqrt{3}\,S$, $3\,s_{2a} - 4\sqrt{3}\,S$, $s_{2a} + 4\sqrt{3}\,S$, $s_{2a} - 4\sqrt{3}\,S$, z_{2a}, $\ell_a^2\,(z_{2a}^2 - m_a^2)$, $(r_{2a}^2 - \ell_a^2\, z_{2a})(s_2^2 - 2\,m_a^2)$, $z_{4a} - \ell_a^2\, z_{2a}$.

For a triple $h = (s_1,\, s_2,\, s_3)$ of real numbers and for triangles ABC and XYZ, let $[ABC,\, XYZ,\, h]$ denote the triangle UVW such that the segments XU, YV, ZW are parallel to lines AO, BO, CO joining the circumcentre O of ABC with its vertices and the directed distances $|XU|$, $|YV|$, $|ZW|$ are equal to s_1, s_2, s_3, respectively. When $s_1 = 0$, we put $U = X$, and we do similar assignments when s_2 and s_3 are zero. For $s_1 > 0$ the vector $\vec{XU}$ points towards outside of ABC while for $s_1 < 0$ it points towards inside.

Theorem 6. *Let ABC, PQR, and XYZ be triangles such that triangles APX, YBQ, and RZC are orthologic to ABC. Let U, V, and W be midpoints of segments YR, ZP, and XQ. Then for every real number h and every integer i from 1 to 13 the triangle $[ABC,\, UVW,\, T_i[h]]$ is also orthologic to ABC.*

Proof. We shall give proof only for $i = 2$ because the procedure for other values of i is similar.

Let us assume again that the circumcircle of ABC is the set of all unimodular complex numbers. Since ABC and APX are orthologic, $O(ABC, APX) = 0$ so that we can solve for x to get that x is any complex number whose complex conjugate is

$$\frac{x}{a\,b} + \frac{(c-a)(p - c\,a\,\bar{p})}{c\,a\,(a-b)} + \frac{(b-c)(a^2 - b\,c)}{a\,b\,c\,(a-b)}.$$

Similarly, y and z are any complex numbers whose complex conjugates are related to themselves by relations that are cyclic relatives of the above relation between $\bar{x}$ and x.

We can now easily find that the affixes of the vertices of $[ABC,\, UVW,\, T_2[h]]$ are

$$\frac{y+r}{2}+\frac{h\,a\,(b-c)^2}{b\,c},\qquad \frac{z+p}{2}+\frac{h\,b\,(c-a)^2}{c\,a},\qquad \frac{x+q}{2}+\frac{h\,c\,(a-b)^2}{a\,b},$$

and verify that $O(ABC,\, [ABC,\, UVW,\, T_2[h]]) = 0$. □

4. Propellers from isocentric triangles

In the rest of this paper we consider propellers in which the blades share centroids with the shaft. Once again this property is retained by the triangle UVW.

Recall that triangles ABC and PQR are *isocentric* provided they have the same centroid. For example, ABC and its Brocard triangle $A_bB_bC_b$ are isocentric.

Theorem 7. *Let ABC, PQR, and XYZ be triangles such that triangles ABC, APX, YBQ, and RZC are isocentric. Then the triangle UVW on midpoints of segments YR, ZP, and XQ is isocentric with ABC.*

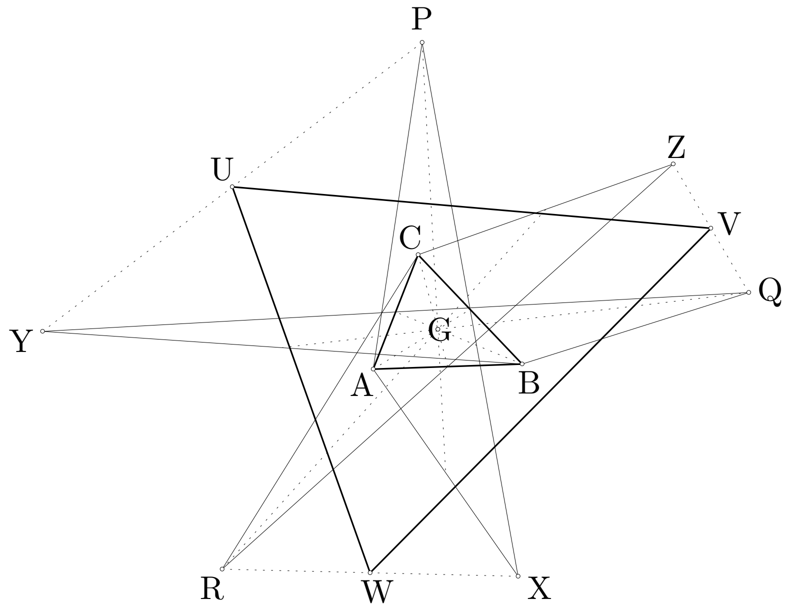

Figure 2. For a propeller from isocentric triangles the triangle UVW is also isocentric with ABC.

Proof. The affixes of the centroids of ABC and APX are $(a+b+c)/3$ and $(a+p+x)/3$. They will coincide if and only if $x = b+c-p$. In the same way we get that the affixes of Y and Z are $y = c+a-q$ and $z = a+b-r$. Then $u = (c+a-q+r)/2$, $v = (a+b-r+p)/2$, and $w = (b+c-p+q)/2$. Hence, ABC and UVW are isocentric. □

The following theorem is analogous to Corollaries 1 and 3.

Theorem 8. *Let ABC, DEF, PQR, and XYZ be triangles such that ABC, APX, YBQ, and RZC are isocentric. Then the triangle UVW on centroids of triangles DYR, PEZ, and XQF is isocentric with ABC if and only if DEF is isocentric with ABC.*

Proof. Let U, V, and W be the centroids of triangles DYR, PEZ, and XQF. Since $x = b + c - p$, $y = c + a - q$, and $z = a + b - r$ it follows that

$$u = \frac{d + c + a - q + r}{3}, \quad v = \frac{e + a + b - r + p}{3}, \quad w = \frac{f + b + c - p + q}{3}.$$

The centroid G of UVW has affix $(d + e + f + 2(a + b + c))/9$. This complex number will be $(a + b + c)/3$ (the affix of the centroid of ABC) if and only if $d + e + f = a + b + c$ which holds if and only if ABC and DEF are isocentric. □

References

[1] Bankoff, L.; Erdös, P.; Klamkin, M.: *The asymmetric propeller.* Math. Mag. **46** (1973), 270–272.

[2] Gallatly, W.: *The Modern Geometry of the Triangle.* Francis Hodgson, London 1910.

[3] Gardner, M.: *The Asymmetric Propeller.* College Math. J. **30** (1999), 18–22.

[4] Götz, W.: *Einige Sätze mit rechnungsarmen Vectorbeweisen.* Math.-Naturwiss. Unterr. **12** (1959/60), 368–370.

[5] Hahn, L.: *Complex numbers and geometry.* Math. Assoc. America, Washington 1994.

[6] Hofmann, A.: *Vektorieller Beweis einiger Sätze der ebenen Geometrie.* Math.-Naturwiss. Unterr. **2** (1949), 53–54.

[7] Jeger, M.: *Komplexe Zahlen in der Elementargeometrie.* Elem. Math. **37** (1982), 136–147.

[8] Martini, H.: *Zur Geometrie der Hexagone.* Nieuw Arch. Wisk. (4) **14** (1996), 329–342.

[9] Mauldon, J. G.: *Similar triangles.* Math. Mag. **39** (1966), 165–174.

[10] Neuberg, J.: *Cours de Géométrie Analytique.* A. Hermann, Paris 1908.

[11] Popescu, M.: *Improvements of Titeica's theorem.* Gaz. Mat. (Bukarest) **81** (1976), 161–163. (Rumanian)

[12] Rouff, D.: *Eine kleine n-Ecks-Lehre.* Elem. Math. **43** (1988), 129–160.

[13] Dobrilo Tošić, Dragomir Đoković: *Denk-ov problem i neke njegove generalizacije.* Uvođenje mladih u naučni rad, Matematička biblioteka **19**, Beograd 1961, 109–113.

Received December 1, 1999