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A Family of Conics and Three Special Ruled Surfaces

Hans-Peter Schröcker

Institute for Architecture, University of Applied Arts Vienna
Oskar-Kokoschka-Platz 2, A-1010 Wien, Austria
e-mail: hans-peter.schroecker@uni-ak.ac.at

Abstract. In [5] the authors presented a family $\mathcal{F} = \{c_k \mid k \in \mathbb{R}\}$ of conics. The conics c_k are gained by offsetting from a given conic c_0 with proportional distance functions $k\delta(t)$. We investigate certain properties of $\mathcal{F}$ and give the correct version of a result claimed in [5]: The distance function is unique (up to a constant factor) only if c_0 is not a parabola.
Furthermore we deal with the surfaces that are obtained by giving each conic $c_k \in \mathcal{F}$ the z-coordinate λk with a fixed real λ. We find special metric properties of these surfaces and show that they already appeared in other context.

Keywords: conic, ruled surface, surface of conic sections, rational system of conics, refraction

1. Introduction

In this paper we deal with two geometric objects: A family of conics on the one hand and a class of special ruled surfaces of conic sections on the other hand. Both of them are closely related and properties of one object reflect as properties of the other.

Following an idea from [5] we derive a family $\mathcal{F}$ of conics through proportional offsetting from a given conic c_0: $X(t) \ldots \mathbf{x}(t)$. The distance functions are given by $\lambda \varrho(t)^{1/3}$ where $\varrho(t)$ denotes the radius of curvature of c_0 in $X(t)$. This generation leads to a 3D-interpretation of $\mathcal{F}$ – the conics $c_\lambda \in \mathcal{F}$ (we will call them *distance conics*) may be regarded as the top view of the conics on a ruled surface. In Section 2 we use this connection to prove results on the envelope of the distance conics and on the uniqueness of the distance function if c_0 is not a parabola. In addition we find that the conics are projectively linked by the tangents of the evolute of c_0. Despite their connection to many geometric problems (some of them will appear in this paper), families of projectively linked conics have not been studied too extensively up to now.

In Section 3 we will investigate the ruled surfaces Φ_e and Φ_h of the elliptic and hyperbolic case. They are normal surfaces of two quadrics of revolution along a common plane section parallel to the axes. In addition to their attractive shape and ruled character (which makes them interesting for architectural usage, compare [5]), they have geometric applications as well. Only recently they appeared in a paper on the geometry of refraction ([4]).

Section 4 deals with the surface Φ_p of the parabolic case. It is not the normal surface of a quadric along a plane section but has special properties concerning the line of striction. Φ_p is an example of Steiner's Roman surface with a two parameter manifold of parabolas on it. It is a conoid and affinely equivalent to a surface studied in [10].

Returning to families of conics we finally find that the distance function is not unique in the parabolic case. There exists a one parameter family of non-proportional distance functions that can be used for the generation of families of distance conics as desribed above.

2. A family $\mathcal{F}$ of conics and a ruled surface Φ

2.1. Creating conics by proportional offsetting

Let $c_0\colon X(t)\dots\mathbf{x}(t)$ be a parametrized conic in the Euclidean plane $\mathbb{E}^2$. Being given an arbitrary *distance function* $\delta(t)$ of the curve parameter we can construct the *curve at distance* $\delta(t)$

$$c_{\delta(t)}\colon Y(t)\dots\mathbf{y}(t)=\mathbf{x}(t)+\delta(t)\mathbf{n}(t).$$

In this formula $\mathbf{n}(t)$ denotes the unit normal vector of c_0 in $X(t)$. If $\delta(t)\equiv d$ is constant, $c_\delta(t)$ is the ordinary offset curve at distance d.

Of course, any plane curve can be parametrized (at least locally) in this way. One has to make further assumptions on $\delta(t)$ in order to get an object of geometric interest. In [5] the authors asked for a function $\delta(t)$ with the property that for any real k the curve at distance $k\delta(t)$ is a conic. There, they claimed the following result (compare Figure 1):[1]

> *Being given an arbitrary conic c_0 there exists (up to a constant factor) exactly one function $\delta(t)$ such that for any real k the curve c_k at distance $k\delta(t)$ is a conic as well. If $\varrho(t)$ is the radius of curvature of c_0 in $X(t)$ one may use $\delta(t)=\varrho(t)^{1/3}$.*

They provided, however, no detailed proof for the uniqueness of the distance function. Only an outline for the elliptic case was given. Their proof involved the usage of a computer algebra system and was too long to be published. Apparently the authors missed the fact that the offset function is not unique if c_0 is a parabola.

We will give a simple proof for the elliptic and hyperbolic case in Subsection 2.3. In Subsection 4.3 we will investigate the parabolic case and describe a geometric method of constructing all distance functions with the requested property.

[1]We used the software provided in [3] to produce the pictures in this paper.

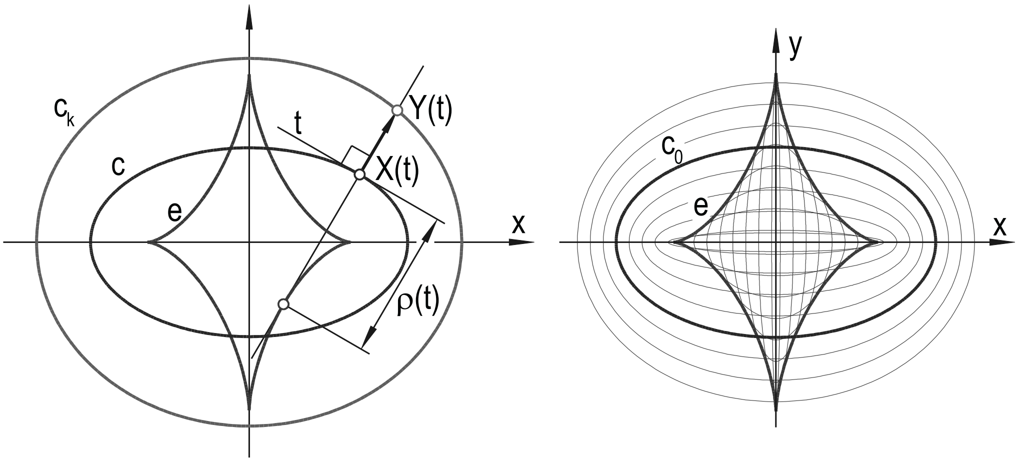

Figure 1: A single distance conic c_k of c_0 (left) and the family $\mathcal{F}$ of distance conics (right).

Before doing so, we investigate the family $\mathcal{F} := \{c_k \mid k \in \mathbb{R}\}$ of *distance conics* to an arbitrary *base conic* c_0. The distance function used to generate c_k shall be $k\varrho(t)^{1/3}$ as proposed in [5]. For the time being only the members of this family $\mathcal{F}$ will be called distance conics. Later on (in Subsection 4.3), we will deal with other families of conics gained through proportional offsetting as well.

A conic's evolute has as many real points at infinity as the conic itself. Each point at infinity corresponds to a pole of the function $\varrho(t)$. Hence, all regular distance conics are of the same type as c_0. In general, they do not belong to a pencil of conics. An exception is the circular case where $\mathcal{F}$ is a set of concentric circles. We will exclude this from our considerations until Subsection 4.3.

If c_0 is not a parabola we can easily express the semiaxes a_k, b_k of c_k in terms of the semiaxes a, b of c_0 ([5]):

$$a_k = a + \frac{kb}{w}, \qquad b_k = b + \varepsilon\frac{ka}{w}, \tag{1}$$

where $w = (ab)^{1/3}$ and $\varepsilon = \pm 1$ in the elliptic and hyperbolic case, respectively. (1) shows that the two distance conics corresponding to

$$k_1 := -\frac{aw}{b} \quad \text{and} \quad k_2 := -\varepsilon\frac{bw}{a}$$

degenerate to line segments. In the elliptic case these segments are finite, in the hyperbolic case they are infinite.

Now to the parabolic case. In an appropriate Cartesian coordinate system c_0 can always be described by the equation $y = ax^2$. Then the conic section c_k is given by $y = p_k x^2 + q_k$, where

$$p_k = \frac{a}{(1+kr^2)^2}, \quad q_k = -\frac{k}{r} \quad \text{and} \quad r = \sqrt[3]{2a}. \tag{2}$$

Here, the single degenerating distance conic corresponds to $k = -r^{-2}$. It is easy to see that in any of the three cases c_0 is the only conic that yields $\mathcal{F}$ as its family of distance conics. I.e., different base conics have different families of distance conics.

On the right hand side of Figure 1 one can see a conic c_0, its evolute e and the corresponding family $\mathcal{F}$ of distance conics. Apparently e is the envelope of all conics c_k and we will soon give a proof for this. But first we need to know a little more about the family $\mathcal{F}$ of distance conics and an interpretation in 3-space. We could use the formulas (1) and (2) to find the envelope of $\mathcal{F}$ by standard methods. Our proof, however, will turn out to be much easier.

2.2. Associated ruled surfaces of conic sections

The possibility of creating a family of conic sections by offsetting with proportional distance functions is the key for a 3D-interpretation of $\mathcal{F}$. We choose a Cartesian coordinate system $\{O; x, y, z\}$ where c_0 is defined by $z = 0$ and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad y = ax^2$$

in the elliptic, hyperbolic or parabolic case, respectively. By assigning the z-coordinate λk ($\lambda \in \mathbb{R} \setminus \{0\}$) to each distance conic we create a surface Φ of conic sections. In [5] this surface has been presented and visualized for all three types of base conics. But still, Φ has many remarkable properties that were not mentioned there and will be contents of this paper. Images of Φ for the elliptic and hyperbolic case may be found in Section 3. The surface of parabolic sections is displayed in Section 4.

Obiously, by the way it has been defined, Φ is not only a surface of conics but a ruled surface as well. The top view of the generating lines just yields the normals of c_0. Thus, according to [1], Φ is algebraic of order $n \le 4$. If c_0 is an ellipse or a hyperbola it has two double lines

$$d_1 \ldots x = 0,\ z = \lambda k_1 \qquad \text{and} \qquad d_2 \ldots y = 0,\ z = \lambda k_2$$

that stem from the two degenerating conics. As a ruled surface of order $n \le 3$ never has two double lines ([6]) we get $n = 4$ in these cases. In the parabolic case we have $n = 3$ as we shall see in Section 4. Now we prove the theorem on the hull curve of $\mathcal{F}$ making essential use of the presented 3D-interpretation (compare Figure 1):

Theorem 1. *The evolute e of c_0 is an envelope of the family $\mathcal{F}$ of distance conics.*

Proof. Projecting Φ orthogonally on the $[x, y]$-plane yields a contour line $l_1 \subset \Phi$. Its projection l_1' on $[x, y]$ consists of the hull curve e of the projected generating lines of Φ plus possible components that result from singular surface points. e is the evolute of c_0 and – at the same time – the envelope of the projected conics of Φ, i.e., the conics of $\mathcal{F}$. □

Theorem 1 is only a special case of a more general theorem.

Theorem 2. *Being given a differentiable plane curve $d\colon Y(t) \ldots \mathbf{y}(t)$ and an arbitrary function $\varepsilon(t)$ we construct a set $\{d_k \mid k \in \mathbb{R}\}$ of curves at distance $k\varepsilon(t)$. Then the evolute of d is envelope of all curves d_k.*

Proof. The proof is completely analogous to the proof of Theorem 1. The relevant property of the curves d_k is the possibility of building a ruled surface in space by assigning the z-coordinate λk to d_k. □

2.3. Uniqueness of the distance function in the elliptic and hyperbolic case

We first present some basic facts on the class curve or ruled surface generated by two projectively linked conic sections (compare [6, 7]). Let c and d be two conics in a common plane that are projectively linked by the map $\pi\colon c \to d$. We regard the set c^* of lines passing through corresponding points $C \in c$ and $\pi(C)$. If c^* happens to be a pencil of lines we require that its vertex S lies in $c \cap d$.[2] Now the following statements are equivalent:

- c^* is a rational curve of class γ ($\gamma = 1, 2, 3, 4$).
- There exist exactly $4 - \gamma$ fixpoints of π.
- There exists a $(5 - \gamma)$-parameter manifold of conics that are projectively linked by the tangents of c^* and, thus, may be used to generate c^* instead of c or d.

If c and d are conics in 3-space they do not generate a class curve but a ruled surface Ψ. We can, however, state three equivalent points that read very similar:

- Ψ is an algebraic surface of order γ ($\gamma = 2, 3, 4$).
- There exist exactly $4 - \gamma$ fixpoints of π.
- On Ψ we find a $(5 - \gamma)$-parameter manifold of conics that are projectively linked by the generators of Ψ and, thus, may be used to generate Ψ instead of c and d.

It is noteworthy that any ruled surface of conic sections can be generated by two projectively linked conics. This was stated in [1] for the first time. For the proof of the following theorem it will be important to observe that any two conics on a ruled surface of conic sections are projectively linked by the generators.

Theorem 3. *Being given an arbitrary ellipse or hyperbola c_0 there exists (up to a constant factor) exactly one function $\delta(t)$ such that for any real k the general offset curve c_k with distance function $\delta_k(t) := k\delta(t)$ is a conic as well. If $\varrho(t)$ is the radius of curvature of c_0 in $X(t)$ one may use $\delta(t) = \varrho(t)^{1/3}$.*

Proof. [5] garantuees that the curve c_k at distance $k\varrho(t)^{1/3}$ is a conic again. In order to prove the uniqueness we take another function $\tilde{\delta}(t)$ that, too, for any $k \in \mathbb{R}$ yields a conic $\tilde{c}_k$ as curve at distance $k\tilde{\delta}(t)$. Then we associate ruled surfaces Φ and $\tilde{\Phi}$ with the families $\mathcal{F} := \{c_k \mid k \in \mathbb{R}\}$ and $\tilde{\mathcal{F}} := \{\tilde{c}_k \mid k \in \mathbb{R}\}$, respectively. The conics on Φ and $\tilde{\Phi}$ are now projectively linked by the generators of these surfaces. Thus, each two conics from $\mathcal{F}$ and/or $\tilde{\mathcal{F}}$ are projectively linked by the normals of c_0. According to Theorem 2 any two of these conics generate the evolute e of the base conic c_0.

A conic $\tilde{c} \in \tilde{\mathcal{F}} \setminus \mathcal{F}$ can now be used to construct a two parameter manifold of conics with this property. Any conic $c^* \in \mathcal{F}$ is projectively linked to $\tilde{c}$ and, thus, determines a ruled surface Φ^*. The top view of the conics on Φ^* yields a one parameter family of generating conics different from $\mathcal{F}$. This is, however, not possible in the elliptic or hyperbolic case as e is of class 4. Therefore we get $\mathcal{F} = \tilde{\mathcal{F}}$ and $\tilde{\delta}(t) = k\varrho(t)^{1/3}$ with some constant $k \in \mathbb{R}$. □

The proof of Theorem 3 is not valid in the parabolic or circular case. The evolutes of these curves are of class 3 or 1, respectively, and the existence of a two parameter set of generating

[2] If $S \notin c \cap d$ we get a case that is of no relevance in this context. It will, however, show up again in Subsection 4.3.

conics is no contradiction. We will solve the problem for these cases in Subsection 4.3. The following corollary follows from the proof of Theorem 3 and is valid in any case:

Corollary 1. *The conics of $\mathcal{F}$ are projectively linked by the normals of c_0.*

For further investigations we need parameter representations of Φ. We will therefore treat the three cases (elliptic, hyperbolic and parabolic) separately. In order to distinguish between them, we will henceforth denote the ruled surface of conic sections by Φ_e (elliptic), Φ_h (hyperbolic) and Φ_p (parabolic).

3. The surfaces Φ_e and Φ_h of elliptic and hyperbolic sections

3.1. The normal surface through the base conic

From (1) it follows that Φ_e can be parametrized according to

$$\Phi_e\colon X(t,u)\ldots\mathbf{X}(t,u) = (1-u)\mathbf{d}_1(t) + u\mathbf{d}_2(t). \tag{3}$$

In this formula $\mathbf{d}_1$ and $\mathbf{d}_2$ are parameter representations of the double lines d_1 and d_2 of Φ_e:

$$d_1\colon D_1(t)\ldots\mathbf{d}_1(t) = \frac{1}{b}\Big(0, (b^2-a^2)\sin t, -\lambda a w\Big)^T,$$
$$d_2\colon D_2(t)\ldots\mathbf{d}_2(t) = \frac{1}{a}\Big((a^2-b^2)\cos t, 0, -\lambda b w\Big)^T.$$

The u-parameter lines of (3) are the generators $g(t)$ of Φ_e, the t-parameter lines are the conics on Φ_e. They are situated in the planes through the line at infinity of the $[x,y]$-plane which is a double line of the surface. The base conic c_0 lies in $[x,y]$ and belongs to $u = a^2(a^2-b^2)^{-1}$. Analogous formulas and statements are true for Φ_h. We will, however, not display them here. They can be obtained by replacing the circular functions $\sin(t)$ and $\cos(t)$ by their hyperbolic relatives $\sinh(t)$ and $\cosh(t)$, respectively, and changing the sign of certain terms. In Figure 2 both surfaces are to be seen.

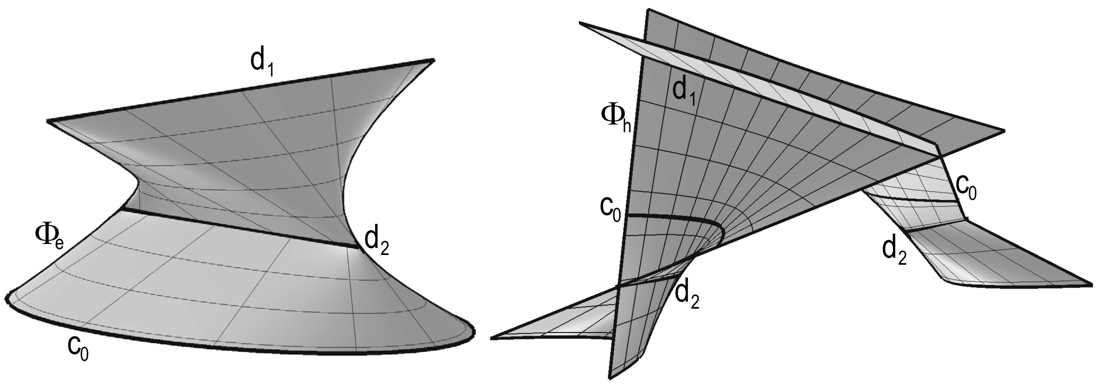

Figure 2: The ruled surfaces Φ_e and Φ_h.

c_0 is a special conic in the family $\mathcal{F}$ of distance conics in the way that it is characteristic of $\mathcal{F}$. No other conic produces the same family of distance conics. So we might expect certain special properties of c_0 as conic on Φ_e as well. The investigations to follow will show that Φ_e has already been studied in earlier works and in other context. Besides, it will be the basis for a very important application of Φ_e in geometrical optics.

By the scaling transformation $x \mapsto ab^{-1}x$ or $y \mapsto ba^{-1}y$ we can map $D_1(t)$ and $D_2(t)$ to points of constant distance

$$d = \frac{|\, a^2 - b^2 \,|}{ab}\sqrt{\lambda^2 w^2 + a^2} \qquad \text{or} \qquad d = \frac{|\, a^2 - b^2 \,|}{ab}\sqrt{\lambda^2 w^2 + b^2},$$

respectively. Thus, Φ_e may be transformed to a ruled surface Φ_e^* that consists of all straight lines that intersect two perpendicular straight lines d_1 and d_2 in points of constant distance. This surface was presented in [2] where it was called *Reiterfläche* (*rider surface*). Note that there exists no family of distance conics that can be associated to the rider surface. In a top view the conics of Φ_e^* envelope an astroid that can never be the evolute of a conic.

Now we want to solve the following task: Find a point $S \in z$ and a conic c: $X(t) \ldots \mathbf{X}(t, u_0)$ of Φ_e such that each generating line $g(t)$ of Φ_e is

1. perpendicular to the straight line $[S, X(t)]$.
2. perpendicular to the tangent of c in $X(t)$.

In other words: *Find a cone Γ of second order with apex on z such that Φ_e is the normal surface of Γ along the plane section c.* It is easy to answer this question algebraically. There exists a unique point S and a unique conic c on Φ_p satisfying the first condition:

$$S\Big(0, 0, w^2\lambda^{-1}\Big)^T, \quad u_0 = \frac{a^2}{a^2 - b^2}.$$

But, as c is just the base conic c_0, it meets the second condition as well. Thus we have

Theorem 4. *Φ_e is the normal surface along c_0 of the cone Γ with center S.*

Surfaces of that kind have been studied in a more general context in [6] and in some earlier publications as well. However, because of its symmetry (the plane section c_0 shares two planes of symmetry with Γ), our case is rather special. E. g., the following remarkable theorem (it follows from some considerations in [6]) is true:

The line of striction on Φ_e is the contour with respect to the projection center S.

3.2. Two quadrics of revolution through the base conic

Of course, Φ_e is not only normal surface of Γ but of any quadric being tangent to Γ along c_0. These quadrics belong to a pencil $\mathcal{Q}$. The implicit equation of any member of $\mathcal{Q}$ can be written as

$$\alpha G(x, y, z) + \beta H(z) = 0,$$

where

$$G(x, y, z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{(w^2 - \lambda z)^2}{w^4} \quad \text{and} \quad H(z) := z^2$$

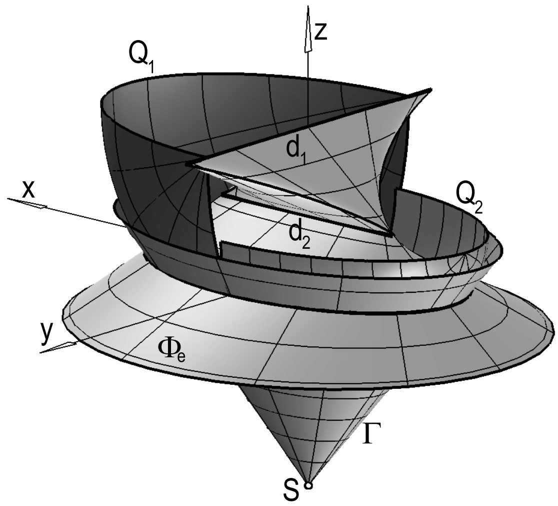

Figure 3: Φ_e, its normal cone Γ and the ellipsoids of revolution Q_1, Q_2.

are the implicit equations of Γ and $[x, y]$ as double plane, respectively. $\alpha : \beta$ is a homogeneous parameter of the pencil $\mathcal{Q}$. Two quadrics Q_1 and Q_2 are of special interest (compare Subsection 3.3). They belong to

$$\alpha : \beta = a^2 w^4 : w^4 + a^2\lambda^2 \quad \text{and} \quad \alpha : \beta = b^2 w^4 : w^4 + b^2\lambda^2$$

and are ellipsoids of revolution with axes d_1 and d_2, respectively.

Corollary 2. *Φ_e is the normal surface of two ellipsoids of revolution with axes d_1 and d_2, respectively, along c_0.*

Figure 3 shows Φ_e together with its normal cone Γ and two quarters of the ellipsoids of revolution Q_1 and Q_2.

3.3. Φ_e, Φ_h and refraction on a plane

Besides the architectural suggestions made in [5] both surfaces, Φ_e and Φ_h, have an important application in the geometric theory of refraction that shall be explained here (compare [4]).[3] We will shortly outline some basics on refraction in the plane case.

Let σ be a plane (the *refracting plane*) and r a positive real (the *index of refraction*). A straight line b_1 intersecting σ in a point B is now refracted to a straight line $b_2 = R(b_1)$ according to *Snell's law*

$$\sin\alpha_1 = r \sin\alpha_2 \tag{4}$$

where α_i denotes the angle between b_i and the normal n of σ (Figure 4). Now we choose a point E (the *eye point*) and reflect all rays through E on σ. It is not hard to prove that the congruence of all reflected rays is the normal congruence of a quadric of revolution Q. To be

[3]The author would like to express his gratitude to M. Husty for pointing out this connection.

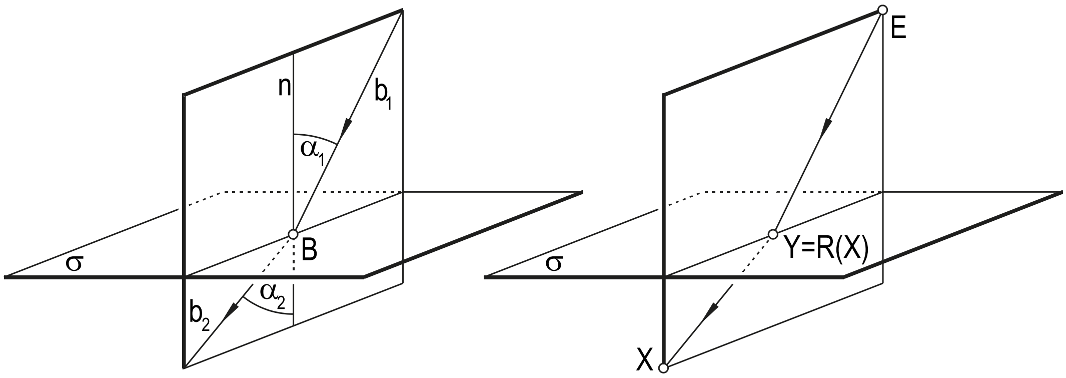

Figure 4: Snell's law and the "refrax" $R(X)$ of a space point X.

more precise, Q is an ellipsoid if $r < 1$ and a hyperboloid if $r > 1$.[4] In both cases the axis of revolution is the normal of σ through E and this point a focal point of Q. The half length of the axes of Q can easily be expressed in terms of e (the distance of E and σ) and r. Now we can define a map

$$R : \mathbb{E}^3 \to \sigma, \qquad X \mapsto R(X) = Y$$

that assigns to each point $X \in \mathbb{E}^3$ the point $Y \in \sigma$ with $R([E, Y]) = [X, Y]$, i.e., the point we practically try to look at when we want to see X through the refracting plane σ (Figure 4).[5] This map is important in computer graphics for the direct and efficient computation of refraction images. Furthermore, it can be applied to create curved perspectives as well.

The counter image $\Psi := R^{-1}(s)$ of a straight line $s \subset \sigma$ is a ruled surface that can be used to solve certain problems of theoretical and practical relevance. E. g., it helps to answer questions on the R-image of an algebraic curve a of order n (in general $R(a)$ is an algebraic curve of order $4n$) and to handle problems with standard filling algorithms for polygons in computer graphics (the R-image of a polygon P may have up to two overlappings; the criterion for this is the number of generators of Ψ that intersect P).

Probably you already guessed that $\Psi = \Phi_e$ or $\Psi = \Phi_h$ and, of course, you are right. For a change we will deal with the hyperbolic case in this subsection. We will use a Euclidean coordinate system where σ is the $[x, y]$-plane, E lies on the z-axis and the straight line s is described by $z = 0$, $y = s_y$. Then all reflected rays are perpendicular to a hyperboloid H of revolution

$$\mathrm{H}\colon H(u, v) \ldots \mathbf{H}(u, v) = \begin{pmatrix} a \cosh u \sin v \\ a \cosh u \cos v \\ b \sinh u \end{pmatrix}. \tag{5}$$

Now we compute the normals of H that intersect s. The characteristic condition for this is

$$\cosh u \cos v = \frac{a s_y}{a^2 - b^2}.$$

[4]If $r = 1$ the refraction is actually just a reflection and Q degenerates.

[5]Actually R is not well defined by this condition as there may be up to four real solutions. There exists, however, exactly one point that is relevant for practical purposes and it can easily be characterized ([4]).

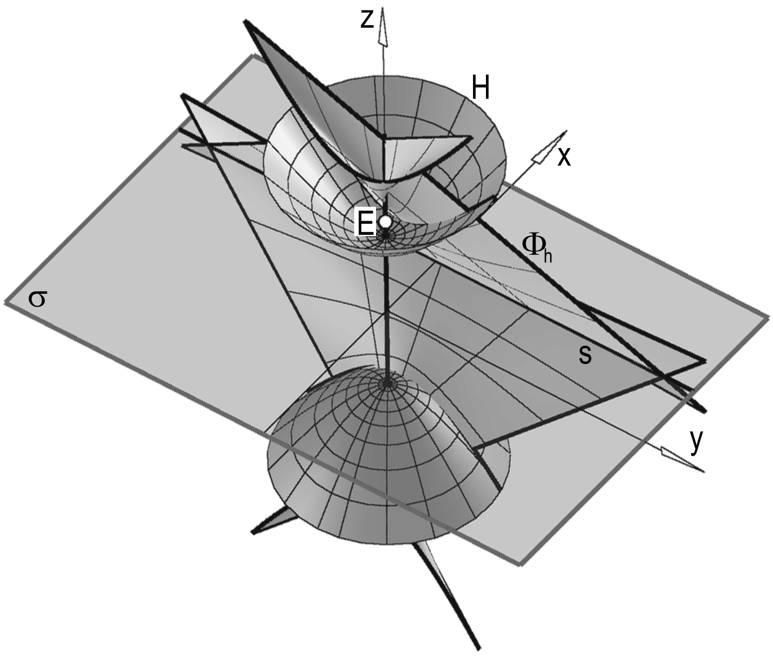

Figure 5: The counter image $\Phi_h = R^{-1}(s)$ of a straight line s.

Together with (5) this yields

> *The counter image Ψ of a straight line s is the normal surface of a hyperboloid* H *of revolution along a plane section parallel to the axis of* H. *Hence, it is an example of the ruled surface Φ_h.*

Figure 5 shows the situation. The double lines of Φ_h are the z-axis and the straight line s, the eye point E is focal point of H. For practical applications only one sheet of Φ_h is relevant. But taking into account that, from the mathematical point of view, equation (4) has two solutions in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we get the whole surface Φ_h.

4. The surface Φ_p of parabolic sections

4.1. Basic properties

The parameter representation of the surface Φ_p of parabolic sections given by [5] can easily be derived from (2). With the abbreviation $r = (2a)^{1/3}$ it reads

$$\Phi_p\colon X(t,u)\ldots \mathbf{X}(t,u) = \begin{pmatrix} t + tur^2 \\ at^2 - ur^{-1} \\ \lambda u \end{pmatrix}. \tag{6}$$

The t-lines are parabolas, the u-lines are generators $g(t)$ of the surface. The base parabola c_0 belongs to $u = 0$. Φ_p has the double line

$$d \ldots x = 0,\ z = -\lambda r^{-2}$$

and is symmetric with respect to $[y,z]$. The $[y,z]$-plane intersects Φ_p in the straight line

$$s \ldots x = 0,\ ry + z = 0$$

and the double line d. It is easy to see that Φ_p is a conoidal surface. All its generating lines are parallel to the plane $ry + z = 0$. The parametrization (6) has the characteristic shape of the Weierstrass-representation of *Steiner's Roman Surface* ([8, 11]): The homogeneous coordinates $x_0 : x_1 : x_2 : x_3$ of the surface points are quadratic polynomials of the two parameters t and u. In our case, already the Euclidean coordinates fulfil this condition. The Roman Surface and any of its tangent planes have two conics in common. Thus, there exists a two parameter manifold of conics on Φ_p. Its algebraic order must therefore be three.[6]

A surface very similar to Φ_p has already been investigated more than 30 years ago in [10]. Using an affine coordinate system with origin $S := s \cap d$ and coordinate axes parallel to x, y and s we can parametrize Φ_p as

$$\Phi_p^*\colon X^*(t,u) \ldots \mathbf{X}^*(t,u) = \begin{pmatrix} tu \\ t^2 \\ u \end{pmatrix}. \tag{7}$$

This is – up to a permutation of the coordinates – identical to a parameter representation that was used in [10] for the investigation of a certain ruled surface of order three. There, the author studied affine properties of Ψ (that, of course, hold for Φ_p as well). So he could afford choosing the normalized parameter representation (7). Φ_p has, however, an interesting metric property that is lost by transforming it to Ψ.

4.2. The line of striction

Some computation following the example of Section 3 shows that Φ_p is not the normal surface of any cone of second order along a plane section. To be more detailed: The conics (parabolas) on Φ_p are defined by relations of the kind $u = \alpha t + \beta$. Evaluating the condition that $SX(t)$ and $g(t)$ are perpendicular immediately yields $\alpha = 0$ and $\beta = -r(4a\lambda)^{-1}$ which defines a unique parabola $c_p \subset \Phi_p$. The normals of Φ_p along c_p are, however, not the generators of a cone or cylinder.

Instead we will have a closer look at the line of striction $l\colon L(t) \ldots \mathbf{l}(t)$ of Φ_p. Φ_p is a conoid, so l is the contour for the normal projection on $[x,s]$. As a consequence, all tangent planes of Φ_p along l are parallel to $\mathbf{v} := (0, r, 1)^T$. Their hull surface is a cylinder that will be denoted by ζ_s in the following. The line of striction can be computed from (6). It has the polynomial representation

$$l\colon L(t) \ldots \mathbf{l}(t) = \frac{1}{2a(1+r^2)} \begin{pmatrix} -4a^3t^3 \\ 2a^2t^2(r^2+3) + r^2 + 1 \\ -4a^2rt^2 - 2a - r \end{pmatrix}.$$

[6] A ruled surface of conics is of order three iff there is a two parameter family of conics on it ([6]). Of course, all conics on Φ_p are parabolas.

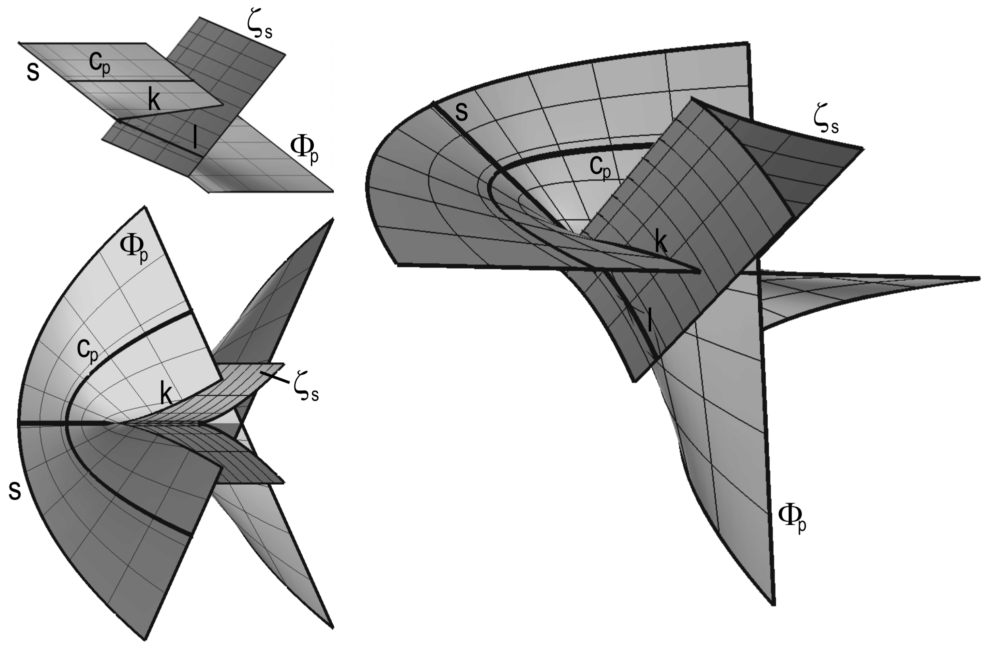

Figure 6: The surface Φ_p and the cylinder ζ_s.

Each generating line of ζ_s intersects Φ_p in three points. Two of them coincide in a point on l, the third one lies on a curve k with parameter representation

$$k\colon K(t)\dots\mathbf{k}(t) = \frac{1}{4ar(r+2a)}\begin{pmatrix} 2a^3t^3r^2 \\ r^2[2r^2(2a^2t^2+1)+3a^2t^2+2] \\ -2a(2r^2-a^2t^2+2) \end{pmatrix}$$

It is easy to verify that l and k are plane curves. As we have $\dot{\mathbf{l}}(0) = \dot{\mathbf{k}}(0) = \mathbf{o}$, the points $L(0)$ and $K(0)$ are cusps of l and k. Thus, these curves are *cubic parabolas* (evolutes of parabolas). As $\mathbf{v}$ is orthogonal to s we can state (compare Figure 6)

Theorem 5. *The line of striction l of Φ_p is a plane cubic parabola. It is the contour for the normal projection on the plane $[x, s]$. The projection rays through l intersect Φ_p in another plane cubic parabola k.*

4.3. Distance conics to parabola and circle

We already mentioned the result stated in [5]: *The general offset curve of a base conic c_0 is a conic if the distance function $\delta(t)$ is of the shape $\delta(t) = k\varrho^{1/3}(t)$ where $\varrho(t)$ is the radius of curvature at $X(t)$ and k an arbitrary real.* In Theorem 3 we gave a proof for the uniqueness (up to a constant factor) of the distance function if c_0 is not a parabola. This proof would not work in the parabolic case, however. In fact, the distance function is not unique as we will see. We already know some basic facts (compare Theorem 2 and the proof of Theorem 3):

- There exists a two parameter manifold $\mathcal{P}$ of parabolas that are projectively linked by the tangents of the evolute e of c_0.[7]
- Only these parabolas can be members of families of distance conics with proportional distance functions.
- All parabolas of $\mathcal{P}$ can be parametrized by substituting $u = \alpha t + \beta$ in (6) and cancelling the z-coordinate. In other words: They are just the top view of the parabolas on Φ_p.

Because of the last point, any parabola $p = p(\alpha, \beta)$ from $\mathcal{P}$ corresponds to a straight line $p^* \ldots u = \alpha t + \beta$ in the $[t, u]$-parameter plane. The straight lines $t =$ const. belong to the generators of Φ_p. Being given an arbitrary parabola $p(\alpha, \beta)$ and a real $k \in \mathbb{R}$ we build the linear combination

$$c_k \colon X_k(t) \ldots (1-k)\mathbf{X}(t, 0) + k\mathbf{X}(t, \alpha t + \beta) \tag{8}$$

(8) is the parameter representation of a parabola c_k of $\mathcal{P}$. Furthermore, c_k is the curve at distance $\delta(t)$ from c_0 where

$$\delta(t) = \Big|\mathbf{X}(t, \alpha t + \beta) - \mathbf{X}(t, 0)\Big|. \tag{9}$$

Thus, we found a one parameter set $\mathbb{F}$ of families of conics with the requested property (see Figure 7). Because of Theorem 2 the hull curve of any of these families is the evolute e of c_0 again. There exists exactly one parameter value t_0 with $\mathbf{X}(t_0, 0) = \mathbf{X}(t_0, \alpha t_0 + \beta)$, i.e., t_0 yields the same point P on c_0 and $p(\alpha, \beta)$. All parabolas of the family defined by c_0 and $p(\alpha, \beta)$ pass through P. Two families $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}$ have only the base parabola c_0 in common. Thus, $\mathbb{F}$ is maximal and we get

Theorem 6. *Being given a parabola c_0 there exists a one parameter family of distance functions* (9) *that can be used to create different families of distance conics* (8)*.*

The parabolas of a family $\mathcal{F}_1 \in \mathbb{F}$ correspond to a family $\mathcal{F}_1^*$ of parabolas on the surface Φ_p. There, the parabolas are located in planes passing through a fixed point $P \in c_0$ and being tangent to Φ_p. We denote this set of tangent planes by $\tau(P)$. They envelope a quadratic cone Γ and all generators of Φ_p are tangents of Γ. As Φ_p is a conoid, the planes of $\tau(P)$ intersect any two generators of Φ_p in points corresponding in an affine transformation.

In the $[t, u]$-parameter plane the families of $\mathbb{F}$ correspond to the pencils of lines with vertices on the straight line $s_0 \ldots t = 0$ that itself corresponds to the base parabola c_0. The distance function $\delta(t) = \varrho(t)^{1/3}$ yields the pencil of lines parallel to s_0. This distance function – the choice of [5] – is characterized by the existence of a singular distance conic – not only in the parabolic but also in the elliptic and hyperbolic case.

Theorem 6 is only possible because a parabola's evolute is of class 3 only. But there is another conic with an evolute of class $\gamma \leq 4$: the circle. Its evolute e degenerates to a single point and is of class 1. The proof of Theorem 3 is, thus, not valid for a circle.[8] To complete

[7] Certain parabolas may degenerate to straight lines. This has, however, no effects on the reasoning to follow.

[8] Remember that we excluded this case at the very beginning of this paper.

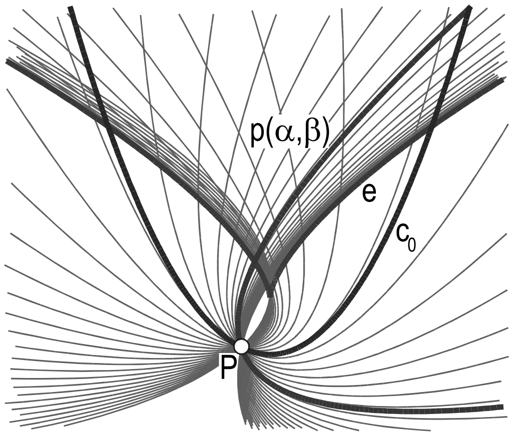

Figure 7: A parabola c_0, its evolute e and a set of proportional offset parabolas.

the picture we will investigate this case right here. This can be done by straightforward computation. We parametrize c_0 according to

$$c_0\colon X(t)\ldots\mathbf{x}(t)=\begin{pmatrix} r\cos t\\ r\sin t\end{pmatrix}. \tag{10}$$

As the apex O of the degenerated evolute e does not lie on c_0 there exists a three parameter manifold $\mathcal{C}$ of conics that are projectively linked by the tangents of e.[9] An arbitrary conic $c\in\mathcal{C}$ can be parametrized according to

$$c\colon Y(t)\ldots\mathbf{y}(t)=\frac{1}{r(a\cos t+b\sin t)+c}\begin{pmatrix} r\cos t\\ r\sin t\end{pmatrix}$$

with three reals a, b and c. Now we solve the vector equation $\mathbf{y}(t)=\mathbf{x}(t)+kr^{-1}f(t)\mathbf{x}(t)$ for the function $f(t)$ and find

$$f(t)=-\frac{r^2(a\cos t+b\sin t)+r(c-1)}{kr(a\cos t+b\sin t)+kc}.$$

Varying k yields a proportional function $f(t)$ only if $a=b=0$ and we get the trivial family of concentric circles as the only solution to the problem. *Theorem 3 is therefore valid for the circular case as well.*

5. Conclusion and future work

In this paper we solved the question of finding all families of proportional distance conics to a given conic. In contrast to claims in [5] there exists more than one family of that kind in the parabolic case. These families are closely linked to certain ruled surfaces of conic sections that have remarkable geometric properties and applications. Perhaps there will be found

[9]See [7]; all conics may be found as the top view of the plane sections of an arbitrary cone of revolution with axis z.

more then we gave in the above text. Of special interest seems the relation of Φ_e and Φ_h to the surfaces presented in [2].

Throughout the text we dealt with families of conic sections that are projectively linked by straight lines (the tangents of a rational curve of class $\gamma \leq 4$ or the generators of a ruled surface of conic sections). Many results (e. g., those of Subsections 2.3 and 4.3) on these families can be obtained by investigating linear combinations of two rational parametrizations that realize the projective mapping by identical parameter values.

It is possible to combine more than two parameter representations and create moredimensional *rational systems of conics* (see [7]). They have the structure of a projectiv space and can be applied to many geometric problems. In contrast to the theory of pencils of conics there exists no theory of rational systems of conics yet. Hopefully, this paper has shown their usefullness and will promote their further investigation.

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