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On Subdirectly Irreducible Steiner Loops of Cardinality $2n$

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Abstract. Let $\mathbf{L}_1$ be a finite simple sloop of cardinality n or the 8-element sloop. In this paper, we construct a subdirectly irreducible (monolithic) sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ of cardinality $2n$, for each $n \geq 8$, with $n \equiv 2$ or $4 \pmod 6$, in which each proper homomorphic image is a Boolean sloop. Quackenbush [12] has proved that the variety $V(\mathbf{L}_1)$ generated by a finite simple planar sloop $\mathbf{L}_1$ covers the smallest non-trivial subvariety (the class of all Boolean sloops). For any finite planar sloop $\mathbf{L}_1$, the variety $V(\mathbf{L})$ generated by the constructed sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ covers the variety $V(\mathbf{L}_1)$.

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1. Introduction

A Steiner loop (or sloop) is a groupoid $\mathbf{S} = (S, \cdot, 1)$ with neutral element 1 satisfying the identities:

$$x \cdot x = 1, \quad x \cdot y = y \cdot x, \quad x \cdot (x \cdot y) = y.$$

We use the abbreviation SL(n) for a sloop of cardinality n. If a sloop satisfies the associative law $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, then it will be a Boolean group that is also called a Boolean sloop. An extensive study of sloops can be found in [4], [8] and [12].

A Steiner triple system is a pair $(P; B)$, where P is a set of points and B is a set of 3-element subsets of P called blocks such that for distinct points $p_1, p_2 \in P$, there is a unique block $b \in B$ with $\{p_1, p_2\} \subseteq b$.

There is a one to one correspondence between the sloops and the Steiner triple systems [8], [12]. If the cardinality of the set of points P is equal to n, the Steiner triple system $(P; B)$ will be denoted by STS(n). It is well known that a necessary and sufficient condition for the existence of an STS(n) is $n \equiv 1$ or 3 (mod 6).

Quackenbush [12] proved that the congruences of sloops are permutable, regular, and Lagrangian. A subsloop S of a sloop L is called normal iff $(x \cdot y) \cdot S = x \cdot (y \cdot S)$ for all $x, y \in L$.

We have that the lattice of normal subsloops of a sloop $\mathbf{L}$ is isomorphic to the lattice of the congruence relation of $\mathbf{L}$. Quackenbush [12] has also proved that if $\mathbf{S}$ is a subsloop of $\mathbf{L}$ and $|L| = 2|S|$, then $\mathbf{S}$ is normal.

There is a well known method for turning a Steiner triple system into another algebra called a Steiner quasigroup (or squag) [12].

In the comments and problems section of [12], Quackenbush has stated that there should be non-simple subdirectly irreducible sloops in which any proper homomorphic image must be a Boolean sloop. He stated that there should be non-simple subdirectly irreducible squags in which any proper homomorphic image must be a medial squag.

The author in [3] has given a construction of finite subdirectly irreducible squags in which all proper homomorphic images are medial squags.

In [1] and [2] the author has also given a construction of a subdirectly irreducible (monolithic) sloop of cardinality 2^n, in which the cardinality of the congruence class of the unique atom of its congruence lattice is equal to 2 (the minimal possible size of a proper normal subsloop).

In this paper, we construct a subdirectly irreducible sloop of cardinality $2n$, for each $n \geq 8$, with $n \equiv 2$ or 4 (mod 6), in which its congruence lattice is a chain of length 2 and its proper homomorphic image is the 2-element Boolean group. Moreover, the cardinality of the congruence class of its unique atom is equal to n (the maximal possible size of a proper normal subsloop).

We will use in this article some basic concepts of universal algebra [9] and other concepts of graph theory [10].

2. Construction of $\mathbf{2} \otimes_\alpha \mathbf{L_1}$

Let $(P_1^*; B_1)$ be an STS$(n-1)$ and its corresponding sloop $\mathbf{L}_1 = (P_1; \cdot, 1)$, where $P_1^* = \{a_0, a_1, \ldots, a_{n-2}\}$ and $P_1 = P_1^* \cup \{1\}$. Consider the set of 1-factors on P_1 defined by $F_i = \{a_i a_k \colon\ a_i \cdot a_k = a_i\}$, then the class $\mathbf{F} = \{F_0, F_1, \ldots, F_{n-2}\}$ forms a 1-factorization of the complete graph K_n on the set of vertices P_1.

By taking the set $P_2 = \{b, b_0, b_1, \ldots, b_{n-2}\}$ with $P_1 \cap P_2 = \emptyset$ and $G_i = \{bb_i\} \cup \{b_l b_k \colon$ $a_l \cdot a_k = a_i$ for $i \notin \{l, k\}\}$, then the class of 1-factors $\mathbf{G} = \{G_0, G_1, \ldots, G_{n-2}\}$ forms a 1-factorization of the complete graph K_n on the set of vertices P_2. There is a well known construction of an STS$(2n-1) = (P^*; B)$ [11], where $P^* = P_1^* \cup P_2$ and the set of triples $B = B_1 \cup \{\{b_l, b_k, a_i\} \colon\ b_l b_k \in G_{\alpha(i)}\}$ for any permutation α on the set $\{0, 1, \ldots, n-2\}$.

The corresponding sloop SL$(2n)$ of the STS$(2n-1) = (P^*; B)$ will be denoted by $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1 = (P; \cdot, 1)$ where $P = P_1 \cup P_2$ and $P^* = P - \{1\}$.

If we choose the permutation α equal to the identity map on the set $\{0, 1, \ldots, n-2\}$, then the constructed sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ is isomorphic to the direct product of $\mathrm{SL}(n) = \mathbf{L}_1$ and the 2-element sloop SL(2). We observe that $\mathbf{L}_1$ is a subsloop of $2 \otimes_\alpha \mathbf{L}_1$ for any permutation α.

In the following section, we choose a simple sloop $\mathbf{L}_1$ of cardinality n and a suitable permutation α to construct a subdirectly irreducible sloop of cardinality $2n$.

2.1. Subdirectly irreducible sloops SL($2n$)

An STS is planar if it is generated by every triangle and contains a triangle. A planar STS(n) exits for each $n \geq 7$ and $n \equiv 1$ or 3 (mod 6) [6]. Quackenbush [12] proved in the next theorem that almost all planar SL(n)'s are simple.

Theorem 1. [12] *Let* $(P^*; B)$ *be a planar* STS$(n-1)$ *and* $(P; \cdot, 1)$ *be its corresponding sloop, then either* $(P; \cdot, 1)$ *is simple or* $n = 8$.

Accordingly, we may say that for any $n > 8$ with $n \equiv 2$ or 4 (mod 6) there is a simple SL(n).

Lemma 2. *Let* $\mathbf{F}$ *be a 1-factorization of the complete graph* K_n*. For any two distinct 1-factors* F_1 *and* F_2 *of* $\mathbf{F}$*, there is always a 1-factor* F_3 *of* $\mathbf{F}$ *satisfying that the three factors* F_1, F_2*, and* F_3 *do not contain any sub 1-factorization of the complete graph* K_4*.*

Proof. The number of edges of a 1-factor F_i of $\mathbf{F}$ is $n/2$. Then the maximum number of sub 1-factorizations on K_4 of $\mathbf{F}$ with sub 1-factors $f_1 \subseteq F_1$ and $f_2 \subseteq F_2$ on a 4-element subset of vertices is $[n/4]$ (the greatest integer in $n/4$).

For any possible 4-element subset of vertices $\{x, y, z, w\}$, if $f_1 = \{xy, zw\}$ and $f_2 = \{xz, yw\}$, then there is at most only one 1-factor F_i of $\mathbf{F}$ containing the third sub 1-factor $f_i = \{xw, yz\}$ of the sub 1-factorization of K_4.

If the 1-factorization $\mathbf{F}$ contains the maximum number of sub 1-factorizations of K_4 with sub 1-factors $f_1 \subseteq F_1$ and $f_2 \subseteq F_2$ on a 4-element subset of vertices, then there are at most $[n/4]$ distinct 1-factors of $\mathbf{F} - \{F_1, F_2\}$, each containing the third sub 1-factor f_i of a sub 1-factorization of K_4.

Since $|\mathbf{F}| = n - 1$ and $n - 3 > [\frac{1}{4}n]$ for $n > 4$, then we may say that for $n > 4$ there is always at least one 1-factor F_j not containing the third sub 1-factor $\{xw, yz\}$ for all possible 4-element subset of vertices $\{x, y, z, w\}$. This completes the proof of the lemma. □

Now, we are ready to construct a sloop $\mathbf{L}$ of cardinality $2n$ having only one proper congruence relation ϕ, in which its homomorphic image $\mathbf{L}/\phi$ is the 2-element Boolean group.

Theorem 3. *Let* $\mathbf{L}_1$ *be a simple sloop of cardinality* $n > 8$*, then there is a permutation* α *on the set* $\{0, 1, \ldots, n-2\}$ *such that the construction* $2 \otimes_\alpha \mathbf{L}_1$ *will be a subdirectly irreducible sloop of cardinality* $2n$*, in which each proper homomorphic image is Boolean.*

Proof. Without loss of generality, we may assume that $a_0 \cdot a_1 = a_2$ in $\mathbf{L}_1$; then

$$1a_0, a_1a_2 \in F_0, \quad 1a_1, a_0a_2 \in F_1 \quad \text{and} \quad 1a_2, a_0a_1 \in F_2.$$

And according to the definition of the 1-factorization $\mathbf{G}$ in the construction $2 \otimes_\alpha \mathbf{L}_1$, there is a sub 1-factorization of $\mathbf{G}$ on K_4 namely:

$$bb_0, b_1b_2 \in G_0, \quad bb_1, b_0b_2 \in G_1 \quad \text{and} \quad bb_2, b_0b_1 \in G_2.$$

By Lemma 2, we may also say that there is a 1-factor G_i; $i \neq 0, 1, 2$ such that the three 1-factors G_0, G_1, G_i do not contain any sub 1-factorization of the complete graph K_4.

By choosing the permutation $\alpha = (2\ i)$ on the set $\{0, 1, \dots, n-2\}$, we will prove that the constructed sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ is a subdirectly irreducible sloop, in which each proper homomorphic image is Boolean.

Since $\mathbf{L}_1$ is simple and $|\mathbf{L}| = 2|\mathbf{L}_1|$, then for any permutation α the constructed sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ contains the subsloop $\mathbf{L}_1$ as the unique subsloop of cardinality n; $\mathbf{L}_1$ is necessary normal.

If there is an isomorphism f between $\mathbf{L}$ and the direct product of $\mathbf{L}_1$ and $\mathrm{SL}(2) := (\{0,1\}; +, 0)$, then it must be $f(P_1) = P_1 \times \{0\}$ and $f(P_2) = P_2 \times \{1\}$. But any 4-element subsloop $\{1, a_i, a_j, a_k\} \times \{0\}$ of $P_1 \times \{0\}$ lies in an 8-element subsloop $Y = \{1, a_i, a_j, a_k\} \times \{0, 1\}$ of the direct product $L_1 \times \{0, 1\}$. Since the image $f^{-1}(Y)$ is a subsloop of $\mathbf{L}$, hence we may say that if $\mathbf{L} \cong L_1 \times \{0,1\}$, then any 4-element subsloop $U = \{1, a_i, a_j, a_k\}$ of P_1 lies in an 8-element subsloop $X = \{1, a_i, a_j, a_k, b', b_{i'}, b_{j'}, b_{k'}\}$ of $\mathbf{L}$ with $X - U \subseteq P_2$.

Accordingly, to prove that $\mathbf{L}$ is not isomorphic to the direct product of L_1 and the 2-element sloop, it is enough to show that there is no subsloop $\mathbf{X}$ of $\mathbf{L}$ of cardinality 8 containing the subsloop $U = \{1, a_0, a_1, a_2\}$ with $X - U \subseteq P_2$.

Assume there is a subsloop $\mathbf{X}$ of cardinality 8 containing $U = \{1, a_0, a_1, a_2\}$ with $X - U = \{b_i, b_j, b_l, b_k\} \subseteq P_2$. We have $\alpha(0) = 0$, so $\{a_0, b_l, b_k\}$ is a block of B iff $b_lb_k \in G_0$. Hence, there is a sub-1-factor $\{b_ib_j, b_lb_k\}$ of G_0 related with a_0. And we have $\alpha(1) = 1$, so there is also a sub-1-factor, say $\{b_ib_l, b_jb_k\}$ of G_1 , related with a_1. If the set $X = \{1, a_0, a_1, a_2, b_i, b_j, b_l, b_k\}$ forms a subsloop, then the operation "$\cdot$" is associative on X. Hence $a_2 = a_0 \cdot a_1 = (b_i \cdot b_j) \cdot (b_i \cdot b_l) = b_j \cdot b_l$ and so $a_2 = a_0 \cdot a_1 = (b_l \cdot b_k) \cdot (b_i \cdot b_l) = b_i \cdot b_k$. Moreover, we have $\alpha(2) = i$; this means that a_2 is related with the 1-factor G_i (i.e. $\{a_2, b_l, b_k\}$ is a block of B iff $b_lb_k \in G_i$). This implies that G_i contains the sub 1-factor $= \{b_jb_l, b_ib_k\}$, contradicting the assumption that G_0, G_1, G_i does not contain any sub 1-factorization of K_4.

Accordingly, the constructed sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ is not isomorphic to the direct product of $\mathbf{L}_1$ and the 2-element sloop $\mathrm{SL}(2)$.

Next, we will show that the constructed sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ has only one proper congruence ϕ satisfying $\mathbf{L}/\phi$ is Boolean.

Since the constructed sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ has $\mathbf{L}_1$ as a subsloop, then $\mathbf{L}_1$ is a normal subsloop of $\mathbf{L}$. This means that there is a congruence relation ϕ of $\mathbf{L}$ defined by $[1]\phi = L_1$.

Let θ be a non-trivial congruence relation of $\mathbf{L}$ and $\theta \neq \phi$; then $[1]\theta \cap L_1 = \{1\}$, otherwise $\theta_{\mathbf{L}_1}$ (θ restricted on $\mathbf{L}_1$) is a non-trivial congruence on $\mathbf{L}_1$, contradicting the assumption that L_1 is simple. Consequently, $|[1]\theta| = 2$; then we may assume that $[1]\theta = \{1, b_i\}$ for $b_i \in P_2$. This means that $\theta \cap \phi = \Delta$ (the diagonal relation) and $\theta \circ \phi = \nabla$ (the largest congruence). This implies that $\mathbf{L}$ is isomorphic to the direct product of $\mathbf{L}_1$ and $\mathrm{SL}(2)$, contradicting the preceding result that $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1 \ncong \mathbf{L}_1 \times \mathrm{SL}(2)$.

This means that the constructed sloop $\mathbf{L}$ has no non-trivial congruence θ with $\theta \neq \phi$. This completes the proof of the theorem. $\square$

In fact, a sloop SL(m) is simple if m can not be factored into $(6n_1+i)(6n_2+j)$ for some n_1, n_2 and some $i, j \in \{2, 4\}$. In particular, there are only simple SL(m)'s, if m is not divisible by 4.

If $m \equiv 2$ or 4 (mod 6), then $m = 2n$. Moreover, if $n \not\equiv 2$ or 4 (mod 6), then there are only simple sloops of cardinality m.

Let $m \equiv 2$ or 4 (mod 6) and $m = 2n$ with $n > 8$. For $n \equiv 2$ or 4 (mod 6), so by Theorem 1, there is a simple sloop $\mathbf{L}_1$ of cardinality n. And by Theorem 3, there is a subdirectly irreducible sloop $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ of cardinality m with only one proper congruence ϕ satisfying that $\mathbf{L}/\phi$ is isomorphic to the 2-element Boolean sloop.

For $m \leqslant 16$, there are only Boolean sloops SL(m) for $m = 2, 4, 8$ and only simple sloops SL(m) for $m = 10, 14$. To complete the result of Theorem 3 we will show that there is a subdirectly irreducible $\mathbf{L}$ =SL(16) having only one proper congruence ϕ with $|\mathbf{L}/\phi| = 2$ as follows:

In the catalogue of all 80 STS(15)'s, see [5], choose one which has exactly one 7-element subsystem. The corresponding sloop has exactly one 8-element subsloop. It has no other proper non-trivial normal subsloops, since otherwise one could construct a second 7-element subsystem.

Finally, we may say that:

For each $n > 4$ *with* $n \equiv 2$ *or* 4 (mod 6)*, there is a subdirectly irreducible (monolithic) sloop* $\mathbf{L}$ *of cardinality* $2n$ *having only one proper non-trivial congruence relation* ϕ*; furthermore* $\mathbf{L}/\phi$ *is the 2-element Boolean sloop.*

Quackenbush [12] has proved that the variety $V(\mathbf{L}_1)$ generated by a simple planar sloop $\mathbf{L}_1$ has only two subdirectly irreducible sloops $\mathbf{L}_1$ and the 2-element sloop SL(2) and then $V(\mathbf{L}_1)$ covers the smallest nontrivial subvariety (the class of all Boolean sloops).

Quackenbush [12] has also showed that the variety $V(\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m)$ generated by pairwise non-isomorphic finite simple planar sloops $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$ equal to $P_s(\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m, \mathrm{SL}(2))$. This implies that the variety $V(\mathbf{L})$ generated by the constructed sloop SL($2n$) $= \mathbf{L}$, is not subvariety of the variety $V(\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m)$ generated by any set of pairwise non-isomorphic finite simple planar sloops $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$.

On the other hand, let $\mathbf{L}_1$ be a planar sloop of cardinality n and $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ be the constructed sloop SL($2n$). For any subsloop S other than $\mathbf{L}_1$ of $\mathbf{L}$ with $|S| > 4$, one can easily prove that $|L_1 \cap S| = \frac{1}{2}|S|$. Accordingly, if $\mathbf{L}_1$ is planar, then the class of all proper subsloops of $\mathbf{L} = 2 \otimes_\alpha \mathbf{L_1}$ are exactly $\mathbf{L}_1$ and SL(n) for $n = 8, 4, 2$. This means that the variety $V(\mathbf{L})$ generated by $\mathbf{L}$ properly contains the subvariety $V(\mathbf{L}_1)$.

Finally, we want to sharpen the previous result by proving that the variety $V(\mathbf{L})$ covers the variety $V(\mathbf{L}_1)$. For it we need some concepts and theorems on the congruence modular varieties due to R. Freese and R. McKenzie [7] and H. Werner [13].

In [12] Quackenbush has proved that any finite simple sloop $\mathbf{L}_1$ with $|L_1| > 2$ is functionally complete; i.e. $\mathbf{L}_1^n$ has no skew congruence for any $n \geq 2$. He also proved that $\mathbf{L}_1^n \times \mathbf{C}_2^m$ has no skew congruence for any positive integers n and m, where $\mathbf{C}_2$ denotes to the 2-element sloop SL(2).

Theorem 4. [13] *Let* $\mathbf{K}$ *be a permutable variety with* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n \in \mathbf{K}$*. Then* $\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n$ *has a skew congruence iff for some* $i \neq j$*;* $\mathbf{A}_i \times \mathbf{A}_j$ *has a skew congruence.*

Accordingly, we may say that $\mathbf{L}^r \times \mathbf{L}_1^n \times \mathbf{C}_2^m$ has a skew congruence iff $\mathbf{L} \times \mathbf{L}_1, \mathbf{L}^2, \mathbf{L} \times \mathbf{C}_2$ or $\mathbf{C}_2^2$ has a skew congruence.

We want to prove that if $\mathbf{L}^2, \mathbf{L}\times\mathbf{C}_2$ or $\mathbf{C}_2^2$ has a skew congruence ϕ, then the homomorphic image of any of them by ϕ is isomorphic to $\mathbf{C}_2$.

Theorem 5. [13] *Let* $\mathbf{K}$ *be a permutable variety with* $\mathbf{A}, \mathbf{B} \in \mathbf{K}$. *Then* $\mathbf{A} \times \mathbf{B}$ *has a skew congruence iff there are homomorphic images* $\mathbf{A}'$ *of* $\mathbf{A}$ *and* $\mathbf{B}'$ *of* $\mathbf{B}$ *and a 1-1 map* μ *with* $dom(\mu) \subseteq \mathbf{A}'$, $|dom(\mu)| > 1$, $range(\mu) \subseteq \mathbf{B}'$ *such that* $\{(a, \mu(a))\colon\ a \in dom(\mu)\}$ *is a congruence class on* $\mathbf{A}' \times \mathbf{B}'$.

Theorem 6. [13] *Let* $\mathbf{K}$ *be a permutable variety with* $\mathbf{A}, \mathbf{B} \in K$. *If* $\{(a_i, b_i)\colon\ i \in I\}$ *is a congruence class of* $\mathbf{A} \times \mathbf{B}$, *then* $\{a_i\colon\ i \in I\}$ *is a congruence class of* $\mathbf{A}$ *(i.e. the projection of a congruence class is a congruence class).*

Theorem 7. *Let* $\mathbf{L}_1$ *be a finite simple sloop and* $\mathbf{L} = 2 \otimes_\alpha \mathbf{L}_1$ *be the constructed sloop. Then* $\mathbf{L} \times \mathbf{L}_1$ *has no skew congruence and each of* $\mathbf{L}^2, \mathbf{L} \times \mathbf{C}_2$ *and* $\mathbf{C}_2^2$ *has only one skew congruence* ψ *in which the homomorphic image of any of them by* ψ *is isomorphic to* $\mathbf{C}_2$.

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be two finite non-trivial sloops. By applying the previous two theorems then we may say that:
$\mathbf{A}\times\mathbf{B}$ has a skew congruence iff there are $\mathbf{A}' \in H(\mathbf{A})$, $\mathbf{B}' \in H(\mathbf{B})$ and a 1-1 map $\mu\colon\ \mathbf{A}'_1 \to \mathbf{B}'$ for a subsloop $\mathbf{A}'_1$ of $\mathbf{A}'$ with $|\ \mathbf{A}'_1\ | > 1$ such that $(\mathbf{A}'_1,\ \mu(\mathbf{A}'_1))\colon= \{(a, \mu(a))\colon\ a \in \mathbf{A}'_1 = dom(\mu)\}$ is a congruence class of $\mathbf{A}'\times\mathbf{B}'$. We observe that $\mathbf{A}'_1$ and $\mu(\mathbf{A}'_1)$ are also congruence classes of $\mathbf{A}'$ and $\mathbf{B}'$ respectively. Moreover, $(\mathbf{A}'_1, \mu(\mathbf{A}'_1))$ is a non-trivial congruence class of $\mathbf{A}'_1 \times \mathbf{B}'$.

This means that if $\mathbf{A} = \mathbf{L}$ and $\mathbf{B} = \mathbf{L}_1$, since $\mathbf{L}_1$ is simple, then $\mu(\mathbf{A}'_1)$ must be equal to $\mathbf{L}_1$. Thus $\mathbf{A}' = \mathbf{L}$ and $\mathbf{A}'_1$ must be equal to $\mathbf{L}_1$. Hence, if there is a skew congruence on $\mathbf{L} \times \mathbf{L}_1$, then $\mathbf{A}'_1 \times \mathbf{B}' = \mathbf{L}_1 \times \mathbf{L}_1$ has also a skew congruence, contradicting the fact that $\mathbf{L}_1$ is functional complete. Therefore, $\mathbf{L} \times \mathbf{L}_1$ has no a skew congruence.

Let $\mathbf{A} = \mathbf{L}$ and $\mathbf{B} = \mathbf{L}$, so if $\mathbf{A}' = \mathbf{L}$ and $\mathbf{A}'_1 = \mathbf{L}_1$, then $\mu(\mathbf{A}'_1) = \mathbf{L}_1$. In this case, if $\mathbf{L} \times \mathbf{L}$ has a skew congruence, then $(\mathbf{A}'_1, \mu(\mathbf{A}'_1)) = (\mathbf{L}_1, \mathbf{L}_1)$ is a congruence class of a skew congruence of $\mathbf{A}'_1 \times \mathbf{B}' = \mathbf{L}_1 \times \mathbf{L}_1$. Which is impossible, for the same reason given above. This means that if $\mathbf{A} = \mathbf{L}$ and $\mathbf{B} = \mathbf{L}$, then $\mathbf{A}'_1$ must be a congruence class of $\mathbf{A}' = \mathbf{L}/\theta \cong \mathbf{C}_2$, hence $\mathbf{A}'_1 = \mathbf{A}' \cong \mathbf{C}_2$ and $\mu(\mathbf{A}'_1) \cong \mathbf{C}_2$. Since $\mu(\mathbf{A}'_1)$ is a congruence class of the homomorphic image $\mathbf{B}'$, then $\mathbf{B}' \cong \mathbf{C}_2$. Which implies that $\mathbf{L} \times \mathbf{L}$ has only one skew congruence ψ, if $\psi/\theta \times \theta$ is a skew congruence of $\mathbf{L}/\theta \times \mathbf{L}/\theta = \mathbf{A}' \times \mathbf{B}'$. Therefore, the only skew congruence ψ of $\mathbf{L} \times \mathbf{L}$ satisfies that $(\mathbf{L} \times \mathbf{L})/\psi \cong \mathbf{C}_2$.

Let $\mathbf{A} = \mathbf{L}$ and $\mathbf{B} = \mathbf{C}_2$, then $\mathbf{A}'_1$ must be equal to $\mathbf{C}_2$. Hence $\mathbf{L} \times \mathbf{C}_2$ has a skew congruence iff $(\mathbf{A}'_1, \mu(\mathbf{A}'_1)) \cong (\mathbf{C}_2, \mathbf{C}_2)$ as a congruence class of a skew congruence of $\mathbf{A}' \times \mathbf{B}' = \mathbf{C}_2 \times \mathbf{C}_2$. Which implies that $\mathbf{L} \times \mathbf{C}_2$ has only one skew congruence ψ satisfying that $(\mathbf{L} \times \mathbf{C}_2)/\psi \cong \mathbf{C}_2$. This completes the proof of the theorem. □

Quackenbush [12] has proved that $HSP_f(\mathbf{L}_1) = P_f\{\mathbf{L}_1, \mathbf{C}_2\}$. Similarly and according to the previous discussion, one may show that $HSP_f(\mathbf{L}) = P_f\{\mathbf{L}, \mathbf{L}_1, \mathbf{C}_2\}$. Consequently, we may say that $\mathbf{L}, \mathbf{L}_1, \mathbf{C}_2$ are the only subdirectly irreducible (monolithic) sloops in the variety $V(\mathbf{L})$, then $V(\mathbf{L}) = P_s\{\mathbf{L}, \mathbf{L}_1, \mathbf{C}_2\}$. Which implies that the variety $V(\mathbf{L})$ covers the variety $V(\mathbf{L}_1)$.

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