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On the Volume of Unbounded Polyhedra in the Hyperbolic Space

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1. Introduction

In the Euclidean plane the definition of the area of the polygon harmonizes well with intuition, since by the decomposition theorem of Farkas Bolyai [2] two polygons of the same area can be decomposed into pairwise congruent polygons.

The definition of the measure of an unbounded polyhedron in two- and in three-dimensional Euclidean space [7] is likewise well-founded, since we obtain an inner characterization of this measure by using the notions of endlike decomposition-equality and of endlike completion-equality.

In hyperbolic plane H^2 the definition, given for the measure of polygons (bounded or unbounded) [8], can also be motivated by pointing out that the measure of the whole plane is $-2\pi k^2$ [9], and polygons of equal area are endlike decomposition-equal or endlike completion-equal.

In this paper we investigate the measure of unbounded polyhedra in three-dimensional hyperbolic space H^3. This measure has a property similar to that of the measure of polyhedra in Euclidean space: the measure of the union of a tetrahedron and of a trihedron is equal to the measure of the trihedron. Furthermore, it also has a property in common with the measure of polygons in the hyperbolic plane: the measure of the whole space is 4π. The measure of polyhedra is given by the angles of the boundary polygons of the polyhedra on the boundary sphere (or absolute) of the space H^3. We emphasize that the volume is unbounded.

2. Preliminary notions and theorems

Here we summarize in a dimension free manner some definitions and properties (theorems) [3] which, while well-known, are important to us.

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We shall use the Poincarè's conforme model in the course of our discussion. If H is a subset of the space H^3, then the intersection of the closure (with respect to the natural topology of Euclidean space containing our model) of H with the boundary sphere ω of H^3 will be denoted by $H_\omega(=\bar{H}\cap\omega)$. For example, a circle h of ω is the ideal line of a plane N of the hyperbolic space, hence $h = N_\omega$. In our figures we apply stereographic projection of the boundary sphere ω onto the Euclidean plane.

A *convex polyhedron* is the closure of intersection of finitely many open halfspaces with nonempty interior (it is proper). A point-set P, which can be represented as a union of finitely many convex polyhedra is called a *polyhedron*. By the division (decomposition) into two parts of a polyhedron by a plane we mean taking the closure of intersections of the polyhedron with the open halfspaces defined by the plane.

2.1. A polyhedron can be decomposed in the sense of elementary geometry into finitely many convex polyhedra P_i $(i = 1, 2, \dots, n)$ in symbols

$$P = \cup_{i=1}^{n} P_i \quad (int(P_i \cap P_j) = \emptyset, \quad i \neq j).$$

Two polyhedra A and B are called *decomposition-equal*, denoted by $A \sim B$, if there exist polyhedra A_i, B_i $(i = 1, 2, \dots, m)$ so that the following conditions are satisfied ($\cong$ stands for congruence):

$$A = \cup_{i=1}^{m} A_i, \quad B = \cup_{i=1}^{m} B_i, \quad A_i \cong B_i \quad (i = 1, 2, \dots, m),$$
$$(int(A_i \cap A_j) = \emptyset, \quad int(B_i \cap B_j) = \emptyset \quad i \neq j).$$

2.2. The decomposition-equality of polyhedra is an equivalence relation.

Two polyhedra A and B are called *completion-equal*, denoted by $A \overset{+}{\sim} B$, if there exist polyhedra C, D, such that $int(A \cap C) = \emptyset$, $int(B \cap D) = \emptyset$, $C \sim D$, and $A \cup C \sim B \cup D$.

2.3. The completion-equality of polyhedra is an equivalence relation.

3. Polyhedra in hyperbolic 3-space

We shall represent polyhedra as the unions of special polyhedra called simplexes. We define two kinds of simplexes, which will be called tetrahedron, trihedron base, respectively. The tetrahedron is well-known. It is determined by its four proper or ideal vertices, it can be bounded or else $1-, 2-, 3-, 4-$asymptotic. We denote by $T(LEFG)$ the tetrahedron T with vertices L, E, F, G.
The trihedron base is the proper intersection of the halfspaces H_0, H_1, H_2, H_3 under the condition that $(H_1 \cap H_2 \cap H_3)_\omega$ is a triangle bounded by circle arcs, such that its circumscribed circle h_0 is the ideal line of the boundary plane of H_0 and $H_{0\omega} \supset (H_1 \cap H_2 \cap H_3)_\omega$ (Figure 1). We denote this simplex by $TB(H_0H_1H_2H_3)$. It is clear that the trihedron base TB is uniquely determined by the triangle

$$TB_\omega = (H_1 \cap H_2 \cap H_3)_\omega.$$

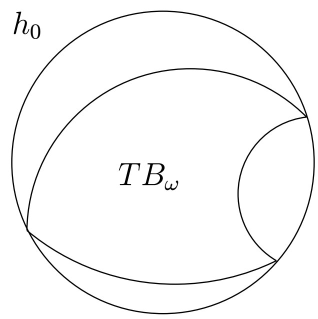

Figure 1

Remark. The terminology trihedron base is motivated by the fact that if the common part of the halfspaces H_1, H_2, H_3 is a trihedron with vertex L, the edges of which intersect ω in the ideal points E, F, G, then, denoting by H_0 the plane through the points E, F, G, the trihedron is the union of the tetrahedron $T(LEFG)$ and of the trihedron base $TB(H_0H_1H_2H_3)$. Let us denote this trihedron by $TI(LEFG)$:

$$TI(LEFG) = T(LEFG) \cup TB(H_0H_1H_2H_3) \quad (int(T \cap TB) = \emptyset).$$

Theorem 1. *A polyhedron is the union, in the sense of elementary geometry, of finitely many simplexes:*

$$P = \cup_{i=1}^{n} S_i \quad (int(S_i \cap S_j) = \emptyset, \quad i \neq j).$$

Proof. By 2.1 it will be sufficient to prove Theorem 1 only for convex polyhedra. The proof proceeds by induction on the number k of halfspaces in the representation of the convex polyhedron.

First we show that Theorem 1 is true for $k = 1$ because a halfspace is a trihedron base, hence a simplex. Let a circle h_0 be the ideal line of the boundary plane of a halfspace H and let R_1, R_2, R_3 be optional points in h_0. Then $H = TB(HHHH)$ and $H_\omega = R_1R_2R_3\triangle$. The sides of the triangle $R_1R_2R_3$ are the arcs of the circle h_0 (Figure 2).

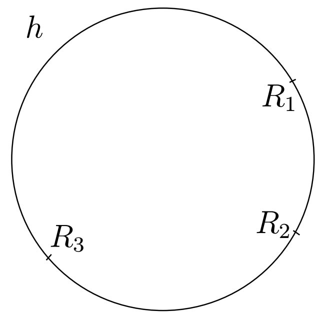

Figure 2

We have to prove two lemmas:

Lemma 1. *In the Euclidean plane a polygon D (simply connected domain bounded by circle arcs) can be decomposed by triangulation (if a vertex V of a triangle T_1 is a point of a triangle T_2 then V is a vertex of the triangle T_2) into triangles bounded by circle arcs (or segments) and triangles are subsets of the circumscribed circle.*

Proof. Let be $D = A_1A_2 \ldots A_n$ (Figure 3). Let $A_i^1, A_i^2, \ldots, A_i^{2r_i+1} (i = 1, 2, \ldots, n)$ be points so that the point A_i^{2j+1} is on the arc A_iA_{i+1} ($A_{n+1} = A_1$), also, the polygonal path

$$L_i = A_i^1, A_i^2, \ldots, A_i^{2r_i+1} \subset D \quad (i = 1, 2, \ldots, n)$$

and either the polygonal path (segment) $A_i^{2r_i+1}A_{i+1}^1 \subset D$ or the polygonal path $A_i^{2r_i+1}A_{i+1}$ $A_{i+1}^1 \subset D$. Also, these polygonal paths are disjoint and they bound the polygon $D' \subset D$, hence

$$D = D' \cup (\cup_{i=1}^{n}(\cup_{j=1}^{r_i} A_i^{2j-1}A_i^{2j}A_i^{2j+1}) \cup (\cup_{i=1}^{n} T_i),$$

where $T_i = A_i^{2r_i+1}A_{i+1}A_{i+1}^1\triangle$ or $T_i = A_i^{2r_i+1}A'_{i+1}A_{i+1}\triangle \cup A_{i+1}A''_{i+1}A_{i+1}^1\triangle$, ($A'_{i+1}$ is a point of the arc $A_i^{2r_i+1}A_{i+1}$, A''_{i+1} is a point of the arc $A_{i+1}A_{i+1}^1$, these triangles are bounded by circle arc or segments.)

The polygon D' can be decomposed by triangulation, so Lemma 1 is true.

Remark. Lemma 1 is true on the sphere too.

Lemma 2. *The common part of the simplex S and of the halfspace H is the union, in the sense of elementary geometry, of finitely many simplexes.*

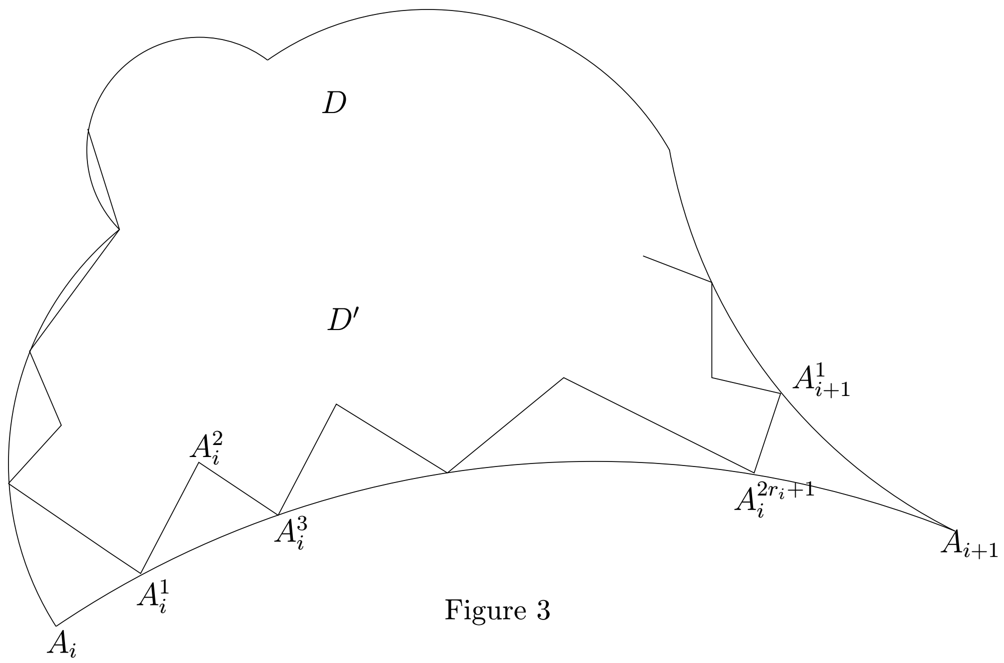

Figure 3

Proof. If the simplex is a tetrahedron then this is clear, the parts will be tetrahedra.

If the simplex S is a trihedron base, then the domain $S_\omega \cap H_\omega$ can be decomposed into polygons (simply connected domains bounded by circle arcs):

$$S_\omega \cap H_\omega = \cup_{i=1}^n D_i \ \ (int(D_i \cap D_j) = \emptyset \ i \neq j).$$

By Remark of Lemma 1 the domain D_i can be decomposed by triangulation:

$$S_\omega \cap H_\omega = \cup_{i=1}^n (\cup_{j=1}^{m_i} G_{ij}) \ \ (int(G_{ij} \neq G_{kl}) \ \ i \neq k \ \ or \ \ j \neq l).$$

We denote by S_{ij} the simplex for which $S_{ij\omega} = G_{ij}$.

Since $(S \cap H) \setminus (\cup_{i=1}^n (\cup_{j=1}^{m_i} S_{ij}))$ is such kind of polyhedron which has ideal points at most on its edges, we can decompose it into tetrahedra:

$$(S \cap H) \setminus (\cup_{i=1}^n (\cup_{j=1}^{m_i} S_{ij})) = \cup_{i=1}^h T_i \ \ (int(T_i \cap T_j), \ \ i \neq j).$$

So $(S \cap H) = (\cup_{i=1}^n (\cup_{j=1}^{m_i} S_{ij})) \cup (\cup_{i=1}^h T_i)$ is the expected decomposition.

Let us now prove the Theorem 1 itself. Let us suppose that Theorem 1 is true for a convex polyhedron $P = \cap_{i=1}^r H_i \ \ (r \leq k, \ \ H_i$ are halfspaces).

If $P = \cap_{i=1}^{k+1} H_i$ then by the condition

$$P_k = \cap_{i=1}^k H_i = \cup_{i=1}^m S_i \ \ (S_i \ \text{ are simplexes, } int(S_i \cap S_j) = \emptyset, i \neq j).$$

By the Lemma 2, $S_i \cap H_{k+1} = \cup_{h=1}^{r_i} S_{ih}$. Hence

$$P = P_k \cap H_{k+1} = (\cup_{i=1}^m S_i) \cap H_{k+1} = \cup_{i=1}^m (S_i \cap H_{k+1}) = \cup_{i=1}^m (\cup_{h=1}^{r_i} S_{ih})$$

$$(int(S_{ih} \cap S_{jl}) = \emptyset, \ \ i \neq j \ \text{ or } \ h \neq l.$$

Thus the Theorem 1 is true for a convex polyhedron $P = \cap_{i=1}^r H_i \ \ (r \leq k+1)$.

4. The volume of polyhedra in hyperbolic 3-space

First we define the volume of simplexes. Let the length measure curvature coefficient k be equal 1.

Volume of the tetrahedron: $M(T) = 0$.

Let S be the trihedron base TB, and let the angles of TB_ω be α, β, γ.

Volume of the trihedron base: $M(TB) = \alpha + \beta + \gamma - \pi$.

Volume of the polyhedron $P = \cup_{i=1}^{n} S_i \ \ (int(S_i \cap S_j) = \emptyset, i \neq j)$:

$$M(P) = \sum_{i=1}^{n} M(S_i).$$

Remark. Thus the volume of $TI(LEFG)$ is the sum of the volume of the tetrahedron $T(LEFG)$ and of the volume of the trihedron base $TB(H_0H_1H_2H_3)$ given by the bounding planes

$$H_0' = EFG, \ H_1' = LFG, \ H_2' = EFL, \ H_3' = EGL$$

of halfspaces H_0, H_1, H_2, H_3, respectively. Since the angles α, β, γ of $TB_\omega = EFG\triangle$ are just the surface angles of the trihedron, $M(TI) = \alpha + \beta + \gamma - \pi$ is the same as the usual area of the spherical triangle.

5. Properties of the volume function

Theorem 2. *The volume of the polyhedra is decomposition-invariant:*

$$P = \cup_{i=1}^{r} S_i' = \cup_{i=1}^{s} S_i'' \ \ (int(S_i' \cap S_j') = \emptyset, \ \ int(S_i'' \cap S_j'') = \emptyset, \ \ i \neq j)$$

implies $\qquad \sum_{i=1}^{r} M(S_i') = \sum_{i=1}^{s} M(S_i'')$.

Proof. Let $A_{ij} = S_i' \cap S_j'' = \cup_{k=1}^{m} A_{ijk}$

$$(i, j, k : intA_{ijk} \neq \emptyset, \ \ int(A_{ijk} \cap A_{ijl}) = \emptyset, \ \ k \neq l).$$

$\cup A_{ijk}$ is a decomposition into simplexes of P.

Lemma. *The volume of the simplex is decomposition-invariant:*

$$S = \cup_{i=1}^{n} S_i \ \ (int(S_i \cap S_j) = \emptyset, \ \ i \neq j) \text{ implies } M(S) = \sum_{i=1}^{n} M(S_i).$$

If the simplex S is a tetrahedron, then S_i can only be tetrahedron as well, hence the lemma is true.

If the simplex S is a trihedron base, then S_i is either a trihedron base or tetrahedron.

Let S_i $(1 \le i \le m)$ be a trihedron base and S_i $(m+1 \le i \le n)$ a tetrahedron. Let the angles of S_ω be α, β, γ and those of $S_{i\omega}$ $(1 \le i \le m)$ be $\alpha_i, \beta_i, \gamma_i$ (Figure 4). Since $S_\omega = \cup_{i=1}^m S_{i\omega}$ we have to prove that

$$\alpha + \beta + \gamma - \pi = \sum_{i=1}^{m} (\alpha_i + \beta_i + \gamma_i - \pi).$$

The number of the vertices of the domains $S_{i\omega}$ is b, the number of those lying in the interior of the boundary of S_ω or of $S_{i\omega}$ is h.

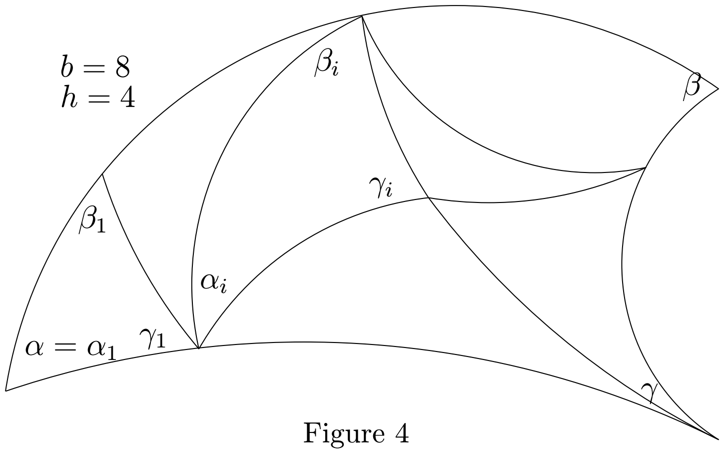

Figure 4

Between these masses there is a relation by Euler's theorem on polyhedra. The number of surfaces is $m+1$. The number of vertices is b. We get the number of edges twice if we add up the number of edges of the triangles $S_{i\omega}$ (this is $3m$) and the number of edges on the boundary of S_ω (this is $3+h$).

$$\text{We have} \quad m + 1 + b = \frac{3m+3+h}{2} + 2, \qquad \text{hence} \quad m = 2b - h - 5,$$

$$\sum_{i=1}^{m} (\alpha_i + \beta_i + \gamma_i - \pi) = \alpha + \beta + \gamma + \pi h + 2\pi(b - h - 3) - m\pi = \alpha + \beta + \gamma - \pi.$$

On the basis of the lemma we have the equalities

$$\sum_{i=1}^{r} M(S_i') = \sum_{i=1}^{r} \Big(\sum_{j,k} M(A_{ijk}) \Big) = \sum_{j=1}^{s} \Big(\sum_{i,k} M(A_{ijk}) \Big) = \sum_{j=1}^{s} M(S_i'')$$

and so Theorem 2 is proved.

Since the volume of the polyhedron is evidently isometry-invariant and additive, we have the following

Theorem 3. *If the polyhedra P, P' satisfy the relation $P \sim P'$ or $P \overset{+}{\sim} P'$ then $M(P) = M(P')$.*

Now we are going to consider the converse of this theorem.

Theorem 4. If the polyhedra P and P' satisfy $M(P) = M(P')$ then $P \overset{+}{\sim} P'$.

Proof. If the polyhedra P and P' are trihedra, then there exist moves g and g' so that the edges of the trihedra gP and $g'P'$ in our model are lines. Let the centre of the unit-sphere K and K' be the vertex of the trihedra gP and $g'P'$ respectively. Let G and G' be the spherical triangles of the trihedra gP and $g'P'$ on the spheres K and K' respectively.

If $M(P) = M(P')$, then $M(gP) = M(g'P')$ and $M(g) = M(G')$. By the theorem of Bolyai-Gerwin [1] $G \sim G'$. Hence $gP \sim g'P'$ and $P \sim P'$, the Theorem 4 is true for trihedra and evidently for bihedra and for halfspaces.

The definiton of the measure of the tetrahedron is also motivated by the following

Lemma. *For a tetrahedron T and a halfspace H the relation $T \cup H \sim H$ holds.*

Let L, E, F, G be the vertices of a non-asymptotic tetrahedron T, and let H be the halfspace $(EFG)_L$. It will be sufficient to prove the Lemma only for this halfspace H. If for an optional halfspace H' we have $int(T \cap H') = \emptyset$, then because of $H' \sim H$ we have $T \cup H' \sim T \cup H \sim H \sim H'$.

If $int(T \cap H') \neq \emptyset$, then $T \cup H' = (T \setminus H') \cup H'$,

$$T \setminus H' = \cup_{i=1}^{k} T_i \quad (T_i \text{ tetrahedron, } int(T_i \cup T_j) = \emptyset, \quad i \neq j) \quad \text{and}$$

$$T \cup H' = (\cup_{i=1}^{k} T_i) \cup H' = (\cup_{i=1}^{k-1} T_i) \cup (T_k \cup H') \sim$$
$$(\cup_{i=1}^{k-1} T_i) \cup H' \sim \cdots \sim (T_1 \cup H') \sim H' \sim H.$$

The halfspace given by the plane incident to the points E, F, G and containing (not containing) point L, will be denoted by $(EFG)_L$ $((EFG)_{\overline{L}})$.

$T \cup H$ is the union, in the sense of elementary geometry, of four disjoint trihedra (Figure 5).

The ideal point of the halfline of the line EF starting from E and containing (not containing) F will be denoted by E_F $(E_{\overline{F}})$. For the surface angle of the tetrahedron $T(LEFG)$ belonging to the edge EF we write $\sphericalangle EF$.

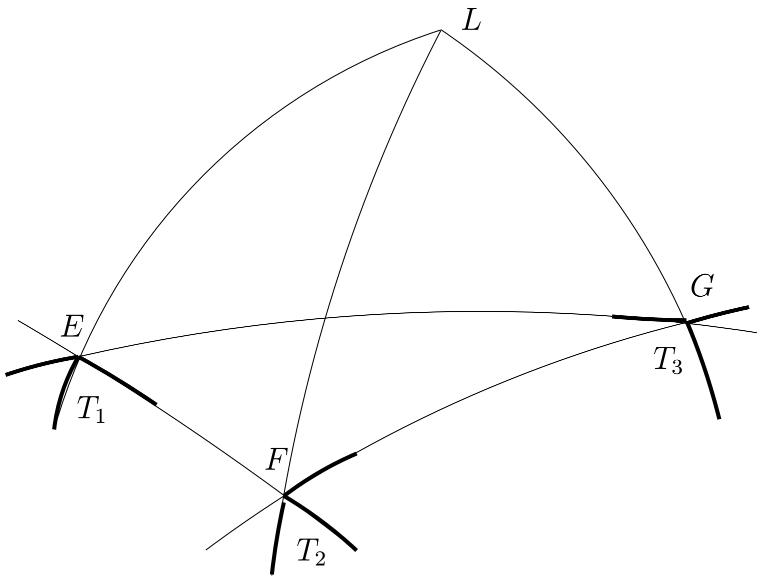

Figure 5

$$T_0 = TI(LL_EL_FL_G) = (LEF)_G \cap (LFG)_E \cap (LEG)_F,$$
$$T_1 = TI(EE_{\overline{L}}E_{\overline{G}}E_F) = (LEF)_{\overline{G}} \cap (EFG)_{\overline{L}} \cap (LEG)_F,$$
$$T_2 = TI(FF_{\overline{L}}F_{\overline{E}}F_G) = (LEF)_G \cap (LFG)_{\overline{E}} \cap (EFG)_{\overline{L}},$$
$$T_3 = TI(GG_{\overline{L}}D_{\overline{F}}G_E) = (LEG)_{\overline{F}} \cap (LFG)_E \cap (EFG)_{\overline{L}}.$$

$T_0 \cup T_1 \cup T_2 \cup T_3 = T \cup H \quad (int(T_i \cap T_j) = \emptyset \;\; i \neq j;\; 0 \leq i \leq 3,\; 0 \leq j \leq 3).$

Now the surface angles of T_1; T_2; T_3 given by the help of the surface angles of T are:

$$\sphericalangle EE_{\overline{L}} = \pi - \sphericalangle LE, \quad \sphericalangle EE_{\overline{G}} = \pi - \sphericalangle EG, \quad \sphericalangle EE_F = \sphericalangle EF;$$
$$\sphericalangle FF_{\overline{L}} = \pi - \sphericalangle LF, \quad \sphericalangle FF_{\overline{E}} = \pi - \sphericalangle EF, \quad \sphericalangle FF_G = \sphericalangle FG;$$
$$\sphericalangle GG_{\overline{L}} = \pi - \sphericalangle LG, \quad \sphericalangle GG_{\overline{F}} = \pi - \sphericalangle FG, \quad \sphericalangle GG_E = \sphericalangle EG,$$

$$\text{hence } \; M(T \cup H) = M(TI(LL_EL_FL_G)) + M(TI(EE_{\overline{L}}E_{\overline{G}}E_F)) +$$
$$+ M(TI(FF_{\overline{L}}F_{\overline{E}}F_G)) + M(TI(GG_{\overline{L}}G_{\overline{F}}G_E)) = 2\pi = M(H).$$

The Theorem 4 is true for trihedra, the statement of the Lemma follows.

The proof is similar if the tetrahedron is $1-$asymptotic and the vertex L is ideal point. Tetrahedron T which is $2-, 3-$, or $4-$asymptotic can be decomposed into non-asymptotic and $1-$asymptotic tetrahedra: $T = \cup_{i=1}^{n} T_i$.

$$\text{Then } \; T \cup H = (\cup_{i=1}^{n} T_i) \cup H = (\cup_{i=1}^{n-1} T_i) \cup (T_n \cup H) \sim (\cup_{i=1}^{n-1} T_i) \cup H.$$

Using this procedure repeatedly, we can see that $(\cup_{i=1}^{n} T_i) \cup H \sim H$.

Let us now prove the Theorem 4 itself.

Let $P = \cup_{i=1}^{n} S_i, \quad P' = \cup_{i=1}^{n'} S'_i$, where the S_i and the S'_i can be trihedron bases or tetrahedra.

Hence it will be sufficient to consider the case when $S_i\ (1 \leq i \leq m)$ and $S'_i\ (1 \leq i \leq m')$ are trihedron bases, $S_i\ (m+1 \leq i \leq n)$ and $S'_i\ (m'+1 \leq i \leq n')$ tetrahedra, and, say, $m \leq m'$.

By the Lemma, if the halfspaces H and H' satisfy $int(H \cap P) = \emptyset$, $int(H' \cap P') = \emptyset$, then

$$(\cup_{i=m+1}^{n} S_i) \cup H = (\cup_{i=m+1}^{n-1} S_i) \cup (S_n \cup H) \sim (\cup_{i=m+1}^{n-1} S_i) \cup H.$$

Using this procedure repeatedly, we can see that $(\cup_{i=m+1}^{n} S_i) \cup H \sim H$. Similarly, we obtain that $(\cup_{i=m'+1}^{n'} S'_i) \cup H' \sim H'$.

On the other hand it is clear that for the halfspaces H_{0i} and H'_{0i} of the trihedron bases S_i $(1 \leq i \leq m)$ and S'_i $(1 \leq i \leq m')$ each of the polyhedra $H_{0i} \setminus S_i$ and $H'_{0i} \setminus S'_i$ is the union of three bihedra.

Hence $M(S_i) = 2\pi - M(H_{0i} \setminus S_i)$ and $M(S'_i) = 2\pi - M(H'_{0i} \setminus S'_i)$, by $\sum_{i=1}^{m} M(S_i) = \sum_{i=1}^{m'} M(S'_i)$ we have

$$2m\pi - M(\cup_{i=1}^{m}(H_{0i} \setminus S_i)) = 2m'\pi - M(\cup_{i=1}^{m'}(H'_{0i} \setminus S'_i)),$$

and consequently, $M(\cup_{i=1}^{m'}(H'_{0i} \setminus S'_i)) = M(\cup_{i=1}^{m}(H_{0i} \setminus S_i)) + 2(m'-m)\pi$.

Using Theorem 4 for bihedra and for halfspaces we can see that

$$(\cup_{i=1}^{m'}(H'_{0i} \setminus S'_i)) \sim (\cup_{i=1}^{m}(H_{0i} \setminus S_i)) \cup (\cup_{i=1}^{m'-m} H_i),$$

where the H_i $(1 \leq i \leq m' - m,\ H_i \cap H_j = \emptyset\ i \neq j)$ are halfspaces and

$$(\cup_{i=1}^{m'-m} H_i) \cap (\cup_{i=1}^{m}(H_{0i} \setminus S_i)) = \emptyset.$$

By $\quad Q = H \cup (\cup_{i=1}^{m}(H_{0i} \setminus S_i)) \cup (\cup_{i=1}^{m'-m} H_i) \sim H' \cup (\cup_{i=1}^{m'}(H'_{0i} \setminus S'_i)) = Q'$

$$\begin{aligned} P \cup Q &= (\cup_{i=1}^{n} S_i) \cup H \cup (\cup_{i=1}^{m}(H_{0i} \setminus S_i)) \cup (\cup_{i=1}^{m'-m} H_i) \sim \\ &\sim (\cup_{i=1}^{m} S_i) \cup (\cup_{i=m+1}^{n} S_i) \cup H \cup (\cup_{i=1}^{m}(H_{0i} \setminus S_i)) \cup (\cup_{i=1}^{m'-m} H_i) \sim \\ &\sim (\cup_{i=1}^{m} S_i) \cup H \cup (\cup_{i=1}^{m}(H_{0i} \setminus S_i)) \cup (\cup_{i=1}^{m'-m} H_i) = \\ &= H \cup (\cup_{i=1}^{m}(H_{0i}) \cup (\cup_{i=1}^{m'-m} H_i) \sim H' \cup (\cup_{i=1}^{m'} H'_{0i}) = \\ &= H' \cup (\cup_{i=1}^{m'}(H'_{0i} \setminus S'_i)) \cup (\cup_{i=1}^{m'} S'_i) \sim \\ &\sim H' \cup (\cup_{i=m'+1}^{n'} S'_i) \cup (\cup_{i=1}^{m'}(H'_{0i} \setminus S'_i)) \cup (\cup_{i=1}^{m'} S'_i) = \\ &= (\cup_{i=1}^{n'} S'_i) \cup H' \cup (\cup_{i=1}^{m'}(H'_{0i} \setminus S'_i)) = P' \cup Q,' \end{aligned}$$

thus the Theorem 4 is true.

Theorem 5. *On the set of the polyhedra of the hyperbolic space there exists uniquely such kind of real valued function that satisfies the following four characteristics:*

1. $M(P) = M(P')$, *if* $P \cong P'$,
2. $M(P_1) + M(P_2) = M(P_1 \cup P_2)$, *if* $int(P_1 \cap P_2) = \emptyset$,
3. $M(H^3) = 4\pi$,
4. $M(A) \geq 0$, *if A is a bihedron (its angle α is positive).*

Proof. The existence of this function follows from the previous theorems.
If a function $M(P)$ satisfies the above mentioned characteristics, there the bihedron A, with angle α, $M(A) = f(\alpha)$ is such as follows, in the case of $0 < \alpha \leq 2\pi$ is positive valued, satisfied the Cauchy function-equality and $f(2\pi) = 4\pi$. It is known ([3], p. 61), that $f(\alpha) = 2\alpha$ is the unique proper function.
We calculate the value of the function $M(TB)$ belonging to the trihedron base TB, with angles α, β, γ, where we use the notations of the Figure 6 and we know that

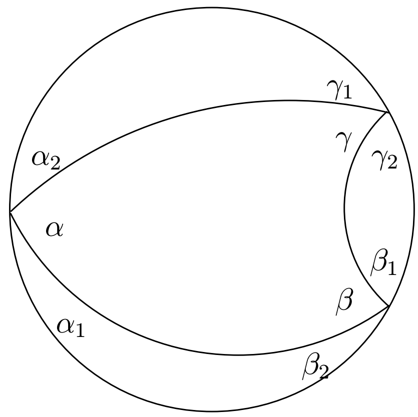

Figure 6

$\alpha_1 = \beta_2,\ \ \beta_1 = \gamma_2,\ \ \gamma_1 = \alpha_2$ and $\ \alpha_1 + \alpha + \alpha_2 = \beta_1 + \beta + \beta_2 = \gamma_1 + \gamma + \gamma_2 = \pi$:

$$\begin{aligned} M(TB) &= M(H) - f(\alpha_1) - f(\beta_1) - f(\gamma_1) = 2\pi - 2\alpha_1 - 2\beta_1 - 2\gamma_1 = \\ &= 2\pi - \alpha_1 - \beta_2 - \beta_1 - \gamma_2 - \gamma_1 - \alpha_2 = \alpha + \beta + \gamma - \pi. \end{aligned}$$

Remark. It is valid a similar theorem in the hyperbolic plane H^2.

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