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Harmonic φ-Morphisms

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Abstract. By extending the main result of [6], we characterize the harmonicity of any φ-morphism $\Phi : TM \to TN$, covering a map $\varphi : M \to N$, between Riemannian manifolds, when the tangent bundles carry the complete lift metric. By following the pattern of (classical) harmonic morphisms [1], [3], we introduce in a natural way the notion of harmonic φ-morphism and give a characterization that corresponds to the one obtained in [4], [8]. One of the properties is that φ is a harmonic morphism if and only if $\mathrm{d}\varphi$ is a harmonic φ-morphism. We end with some examples and applications to (1,1)-tensor fields.

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Introduction

A distinguished class of harmonic maps is the class of harmonic morphisms, these are defined as maps between (semi)-Riemannian manifolds which pull back local harmonic functions to local harmonic functions. We refer to [1] as the first monograph on this topic.

Let $\varphi : (M, g) \to (N, h)$ be a map between Riemannian manifolds. From vector bundles category theory $\Phi : TM \to TN$ is a φ-morphism, provided the fibre restriction $\Phi_p : T_pM \to T_{\varphi(p)}N$ is linear at any $p \in M$. Thus Φ determines a 1-form $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ with values in the pull-back bundle $\varphi^{-1}TN$. We prove that $\Phi : (TM, g^c) \to (TN, h^c)$ is a harmonic map

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between semi-Riemannian manifolds (where c denotes the complete lift defined in [11]), if and only if $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ is coclosed w.r.t. the pull back connection $\nabla^{\varphi^{-1}TN}$. When we particularize Φ to a (1,1)-tensor field K on M, viewed as a map $K : (TM, g^c) \to (TN, h^c)$, then its harmonicity was characterized in [6]. Here we characterize the (1,1)-tensor field K of maximal rank which are harmonic morphisms and we find that K has to be the identity up to a non-zero constant factor. This restrictive condition is a reason to introduce *harmonic φ-morphisms,* by following the pattern of (classical) harmonic morphisms. If ∇ is a linear connection of $\varphi^{-1}TN$, compatible with Φ, then we call Φ a *harmonic φ-morphism* w.r.t. ∇, provided any harmonic local function f on N has the pull-back $\mathrm{d}f \circ \Phi$ coclosed on M. We prove that $\varphi : (M, g) \to (N, h)$ is a harmonic morphism if and only if $\mathrm{d}\varphi : TM \to TN$ is a harmonic φ-morphism w.r.t. $\nabla^{\varphi^{-1}TN}$. Different fom the behaviour of the harmonic maps, the composition of two harmonic morphisms is a harmonic morphism. We provide a class of connections w.r.t. which the same property is valid for φ-morphisms. Corresponding to the characterization of [4], [8] of harmonic morphisms as horizontally weakly conformal harmonic maps, we characterize here the harmonic φ-morphisms.

At the end, we apply the notion of harmonic φ-morphism to certain classes of (1,1)-tensor fields (almost complex and almost product structures and the Ricci (1,1)-tensor field).

Throughout the paper, all data are smooth and we assume the Einstein convention on the summing of repeated indices.

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1. Preliminaries

To fix notations, let $\varphi : (M, g) \to (N, h)$ be a map between Riemannian manifolds, with ∇^M and ∇^N the corresponding Levi-Civita connections and let $\Phi : TM \to TN$ be a φ-morphism.

A linear connection D of a vector bundle $E \to M$, defines the exterior derivative d and the coderivative δ of any bundle valued 1-form $\omega \in \mathcal{A}^1(E)$, respectively by

$$\mathrm{d}\omega(X, Y) = D\omega(X, Y) - D\omega(Y, X)\,, \tag{1.1}$$

where $D\omega(X, Y) = (D_X\omega)Y$, $\forall X, Y \in \Gamma(TM)$ and

$$\delta\omega = -\mathrm{div}\,\omega = -\mathrm{trace}\,D\omega\,. \tag{1.2}$$

ω is called *harmonic* if it is both closed ($\mathrm{d}\omega = 0$) and coclosed ($\delta\omega = 0$). Note that d and δ depend on D and therefore the closure, coclosure and harmonicity properties of ω also depend on D. In particular, when $E = \varphi^{-1}TN$ (resp. $E = M \times \mathbb{R}$) carries a linear connection ∇ (resp. standard connection on the trivial bundle), then $\mathrm{d}\Phi$ (resp. $\mathrm{d}(\theta \circ \Phi)$) denotes the exterior derivative of $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ (resp. $\theta \circ \Phi \in \mathcal{A}^1(M)$, for any $\theta \in \mathcal{A}^1(N)$). We distinguish between $\mathrm{d}\Phi \in \mathcal{A}^2(\varphi^{-1}TN)$ and the tangent map $\mathrm{d}\Phi : TTM \to TTN$. Which we use will be clear from the context.

Notation. Throughout this note, a pair $(\Phi,\ \nabla)$ will denote any φ-morphism $\Phi : TM \to TN$ and any linear connection ∇ on $\varphi^{-1}TN$.

Examples of Φ. (i) Obviously, $\mathrm{d}\varphi : TM \to TN$ is a φ-morphism; (ii) any (1,1)-tensor field on M determines a 1_M-morphism $K : TM \to TM$, where 1_M denotes the identity map of M.

Examples of ∇. (i) Let $\nabla^{\varphi^{-1}TN}$ be the unique (see [3]) linear connection on $\varphi^{-1}TN$, which satisfies

$$\nabla^{\varphi^{-1}TN}_X \varphi^* U = \varphi^* \nabla^N_{\mathrm{d}\varphi X} U \qquad \forall X \in \Gamma(TM), \qquad U \in \Gamma(TN), \tag{1.3}$$

where

$$\varphi^* U \in \Gamma(\varphi^{-1}TN), \qquad (\varphi^* U)_p = U_{\varphi(p)}, \qquad \forall p \in M. \tag{1.4}$$

(ii) Define ∇^Φ to be the linear connection on $\varphi^{-1}TN$ which satisfies

$$\nabla^\Phi_X \varphi^* U = \varphi^* \nabla^N_{\Phi X} U, \qquad \forall X \in \Gamma(TM), \qquad U \in \Gamma(TN). \tag{1.5}$$

The existence and uniqueness of ∇^Φ are proved as in [2, pp. 4] by replacing $\mathrm{d}\varphi$ with Φ.

Obviously (ii) generalizes (i) since $\nabla^{\varphi^{-1}TN} = \nabla^{\mathrm{d}\varphi}$.

From a straightforward calculation, we obtain:

Lemma 1.1.

(a) *The following conditions are equivalent for a pair* $(\Phi,\ \nabla)$:

$$\begin{aligned} &\nabla\Phi \quad \textit{is symmetric} \ \Leftrightarrow\ \Phi \in \mathcal{A}^1(\varphi^{-1}TN) \quad \textit{is closed} \ \Leftrightarrow \\ &\Leftrightarrow\ \nabla_X(\Phi Y) - \nabla_Y(\Phi X) = \Phi[X,\ Y], \qquad X,\ Y \in \Gamma(TM); \end{aligned} \tag{1.6}$$

(b) *The pair* $(\mathrm{d}\varphi,\ \nabla^{\varphi^{-1}TN})$ *satisfies* (1.6);

(c) *The pair* $(\Phi,\ \nabla^\Phi)$ *satisfies* (1.6) *if and only if* $[\Phi X,\ \Phi Y] = \Phi[X,\ Y],\ \forall X,\ Y \in \Gamma(TM)$.

Note that not every pair $(\Phi,\ \nabla)$ satisfies these conditions, for example if (M, g) is a Riemannian manifold with ∇ the Levi-Civita connection, then the pairs $(fI,\ \nabla)$ do not satisfy (1.6), where f is a non-constant function on M and $I : TM \to TM$ is the identity.

Formula. *For any φ-morphisms $\Phi,\ \Psi : TM \to TN$, the pair $(\Phi,\ \nabla^\Psi)$ satisfies:*

$$\nabla^\Psi(\theta \circ \Phi) = \theta \circ \nabla^\Psi \Phi + \nabla^N \theta(\Psi\cdot, \Phi\cdot), \qquad \forall \theta \in \mathcal{A}^1(N). \tag{1.7}$$

Proof.

$$\nabla^\Psi(\theta \circ \Phi)(X,\ Y) = (\nabla^\Psi_X(\theta \circ \Phi))Y = (\nabla^\Psi_X(\theta \circ \Phi(Y)) - \theta \circ \Phi(\nabla^M_X Y) =$$

$$= (\nabla^N_{\Psi X}\theta)\Phi Y + \theta(\nabla^\Psi_X(\Phi Y)) - \theta \circ \Phi(\nabla^M_X Y) = (\nabla^N_{\Psi X}\theta)\Phi Y + \theta(\nabla^\Psi_X \Phi)Y =$$

$$= \nabla^N \theta(\Psi X,\ \Phi Y) + \theta \circ \nabla^\Psi \Phi(X,\ Y), \qquad \forall X,\ Y \in \Gamma(TM).$$

2. Compatible pairs

This section is devoted to a certain class of pairs $(\Phi,\ \nabla)$, for which Φ and ∇ are related by a certain compatibility relation.

More precisely, the above Formula leads us to the following:

Definition 2.1. *A pair $(\Phi,\ \nabla)$ is called compatible if it satisfies the following compatibility relation*

$$\nabla(\theta\circ\Phi)=\theta\circ\nabla\Phi+\nabla^N\theta(\Phi\cdot,\ \Phi\cdot),\qquad\forall\theta\in\mathcal{A}^1(N). \tag{2.1}$$

Example. Let $\Phi,\ \Psi:TM\to TN$ be φ-morphisms. Then the pair $(\Phi,\ \nabla^\Psi)$ is compatible if and only if $\Phi=\Psi$. In particular, $(\mathrm{d}\varphi,\ \nabla^{\varphi^{-1}TN})$ is compatible.

Lemma 2.2. *Any compatible pair $(\Phi,\ \nabla)$ satisfies*

$$\mathrm{d}(\theta\circ\Phi)=\theta\circ\mathrm{d}\Phi+\mathrm{d}\theta(\Phi\cdot,\ \Phi\cdot),\qquad\forall\theta\in\mathcal{A}^1(N); \tag{2.2}$$

$$\nabla(\mathrm{d}f\circ\Phi)(X,Y)=[(\nabla_X\Phi)Y]f+\nabla^N\mathrm{d}f(\Phi X,\Phi Y),\ \forall X,Y\in\Gamma(TM),f\in\mathcal{F}(N); \tag{2.3}$$

$$\delta(\mathrm{d}f\circ\Phi)=(\delta\Phi)f+\mathrm{trace}\nabla^N\mathrm{d}f(\Phi\cdot,\ \Phi\cdot),\qquad\forall f\in\mathcal{F}(N). \tag{2.4}$$

Proof. (2.1) and (1.1) yield (2.2). From (2.1) applied to any exact form θ, it follows (2.3). We end the proof by obtaining (2.4) from (2.3) and (1.2).

Proposition 2.3. *Let $(\Phi,\ \nabla)$ be a compatible pair.*

(i) *Let $\Phi\in\mathcal{A}(\varphi^{-1}TM)$ be closed. Then $\theta\circ\Phi\in\mathcal{A}^1(M)$ is closed if and only if $\nabla^N\theta$ is symmetric in its two variables restricted to the image of Φ.*

In particular, if θ is closed, so is $\theta\circ\Phi$ and the converse holds if $\mathrm{rank}\ \Phi=\dim N$;

(ii) *$\Phi\in\mathcal{A}^1(\varphi^{-1}TN)$ is closed if and only if $\theta\circ\Phi\in\mathcal{A}^1(M)$ is closed whenever $\theta\in\mathcal{A}^1(N)$ is closed*;

(iii) *$\nabla\Phi$ is symmetric if and only if $\nabla(\mathrm{d}f\circ\Phi)$ is so.*

Proof. (i) follows from (2.2) and (1.1). We derive (ii) and (iii) from (2.3) and the symmetry of the Hessian $\nabla^N\mathrm{d}f,\ f\in\mathcal{F}(N)$, which complete the proof.

As a consequence of Proposition 2.3, (ii), we obtain:

Corollary 2.4. *The following assertions are equivalent for any compatible pair $(\Phi,\ \nabla)$ with closed $\Phi\in\mathcal{A}^{-1}(\varphi^{-1}TN)$:*

(i) *For any harmonic local function $f:\mathcal{U}\subset(N,h)\to\mathbb{R}$, the pull-back $\mathrm{d}f\circ\Phi$ is coclosed on M;*

(ii) *Φ pulls back any harmonic 1-form θ on N to a harmonic form $\theta\circ\Phi$ on M.*

3. Harmonic φ-morphisms

The main notion of this note is naturally introduced in this section, as being suggested by Corollary 2.4; note however that we do not require Φ to be closed.

Definition 3.1. *Let (Φ, ∇) be a compatible pair. Then we define Φ to be a harmonic φ-morphism (w.r.t. ∇), or briefly a ∇-harmonic φ-morphism, if φ pulls back any harmonic 1-form θ on N to a harmonic form $\theta \circ \varphi$ on M.*

An example of a large class of harmonic φ-morphisms is given by the following:

Theorem 3.2. *Any map $\varphi : (M, g) \to (N, h)$ is a harmonic morphism if and only if* $\mathrm{d}\varphi : TM \to TN$ *is a harmonic* φ-morphism *w.r.t.* $\nabla^{\varphi^{-1}TN}$.

The proof follows from Lemma 1.1 (b) and Corollary 2.4.

In order to characterize the harmonic φ-morphisms by analogy with the harmonic morphisms [4], [8], we state first the following:

Lemma 3.3 *Let $\varphi : (M^m, g) \to (N^n, h)$ and $\Phi : TM \to TN$ be as above. If $H_p = [\mathrm{Ker}\, \Phi_p]^{\perp}$ denotes the horizontal space at $p \in M$, then the following assertions are equivalent*:

(i) *For any $p \in M^m$, either $\Phi_p = 0$ or Φ_p is surjective and there exists a positive function $\lambda \in \mathcal{F}(M^m)$, called the dilation, such that*:

$$h(\Phi X, \Phi Y) = \lambda g(X, Y), \qquad \forall X, Y \in H_p\,; \tag{3.1}$$

(ii) *There exists a positive function $\lambda \in \mathcal{F}(M^m)$ such that*

$$g^{ij}\Phi_i^{\alpha}\Phi_j^{\beta} = \lambda h^{\alpha\beta} \tag{3.2}$$

for any local frames $\left\{\dfrac{\partial}{\partial x^i},\ i = \overline{1, m}\right\}$ and $\left\{\dfrac{\partial}{\partial x^{\alpha}},\ \alpha = \overline{1, n}\right\}$ on M^m and N^n, respectively.

The proof follows from the following algebraic result:

Fact. [3, pp. 41] Let $F : U \to W$ be a non-constant linear map between Euclidean spaces. By the identification $V^* = V$ and $W^* = W$, the adjoint $F^* : W \to V$ is given by $< F^*(w), v > = < F(v), w >$, $\forall v \in V$, $w \in W$. Then F satisfies Lemma 3.3 (i) if and only if F^* embeds W conformally in $(\mathrm{Ker}F)^{\perp} \subset V$.

Example. On any almost Hermitian (resp. Riemannian almost product) manifold, the almost complex (resp. almost product) structure satisfies the equivalent conditions of Lemma 3.3.

Properties. *Let $\Phi : TM \to TN$ be a φ-morphism which satisfies the equivalent conditions of Lemma* 3.3. *Then:*

(a) $\operatorname{rank}\Phi = \dim N \leq \dim M$ *on an open subset on* M;

(b) $\Phi_p = 0$ *at any* $p \in M$, *where rank* $\Phi_p < \dim N$.

Proof. From Lemma 3.3 (i), at any $p \in M$, either $\lambda(p) = 0$ and then rank $\Phi = 0$, or $\lambda(p) \neq 0$ and rank $\Phi = \dim N$.

The proof given in [3, pp. 42] to characterize the harmonic morphisms can be easily adapted here (by replacing $\mathrm{d}\varphi$ with Φ) such that from (2.4) and (3.2) we obtain

Theorem 3.4. *Let* (Φ, ∇) *be a compatible pair. Then* Φ *is a* ∇*-harmonic* φ*-morphism if and only if* Φ *satisfies the equivalent conditions of Lemma* 3.3 *and* $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ *is coclosed (w.r.t.* ∇*).*

Corresponding to the composition property of the harmonic morphisms, the harmonic φ-morphisms have the following behaviour:

Proposition 3.5. *For* $i = \overline{1,2}$, *let* $\varphi_i : M_i \to M_{i+1}$ *be a map between Riemannian manifolds and let* $\Phi_i : TM_i \to TM_{i+1}$ *be a harmonic* φ_i*-morphism with* $\Phi_i \in \mathcal{A}^2(\varphi^{-1}TM_{i+1})$ *closed w.r.t.* ∇^{Φ_i}. *Then* $\Phi_2 \circ \Phi_1$ *is a* $\nabla^{\Phi_2 \circ \Phi_1}$*-harmonic* $\varphi_2 \circ \varphi_1$*-morphism.*

Proof. From Lemma 1.1 (c), $\Phi_2 \circ \Phi_1$ is closed (since $\Phi_i \in \mathcal{A}^1(\varphi_i^{-1}TM_{i+1})$, $i = \overline{1,2}$ are so) We remark that both pairs $(\Phi_i, \nabla^{\Phi_i}), i = \overline{1,2}$, are compatible from the Example following Definition 2.1. Let θ be a harmonic 1-form, then by applying Corollary 2.4 (ii) to Φ_2, we see that $\theta \circ \Phi_2$ is harmonic; applying it to Φ_1 then shows that $\theta \circ \Phi_2 \circ \Phi_1$ is harmonic.

Due to Theorem 3.4, the main notion of this note, introduced by Definition 3.1 for compatible pairs, may be extended to arbitrary pairs, as follows:

Definition 3.6. *Let the pair* (Φ, ∇) *be arbitrary, then we define* Φ *to be a generalized harmonic* φ*-morphism (w.r.t.* ∇, *or briefly a* ∇*-harmonic* φ*-morphism), if* Φ *satisfies the equivalent conditions of Lemma* 3.3 *and* $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ *is coclosed w.r.t.* ∇.

The class of harmonic φ-morphisms is larger than the one provided by Theorem 3.2 and moreover, the pair (Φ, ∇) need not be compatible, as one can see from the following:

Example. Let $M = N = \mathbb{R}^2$ be the Euclidean space with the canonical coordinates (x^1, x^2) and let φ be the identity map of $\mathbb{R}^2$. If $\Phi : T\mathbb{R}^2 \to T\mathbb{R}^2$ is the φ-morphism defined such that $\Phi\left(\dfrac{\partial}{\partial x^1}\right) = x^1\dfrac{\partial}{\partial x^1} + x^2\dfrac{\partial}{\partial x^2}$ and $\Phi\left(\dfrac{\partial}{\partial x^2}\right) = -x^2\dfrac{\partial}{\partial x^1} + x^1\dfrac{\partial}{\partial x^2}$, then $\delta\Phi = 0$ w.r.t. the flat connection of $\mathbb{R}^2$ and Φ satisfies (3.1) with dilation $\lambda(x^1, x^2) = (x^1)^2 + (x^2)^2$. It follows that Φ is a generalized harmonic φ-morphism, which is not the tangent map of φ.

From Theorem 3.4 we note that the coclosure condition in Definition 3.6 is necessary for any harmonic φ-morphism. We characterize this condition in a special case, by giving a geometrical interpretation of it.

First we recall that some geometrical objects on a manifold M that can be lifted on TM by the vertical and complete lifts.

Definition 3.7. [11] *Let $\pi : TM \to M$ be the canonical projection. If $f \in \mathcal{F}(M)$, $X \in \Gamma(TM)$, $g \in \mathcal{T}_2^0(M)$ and ∇ denote respectively a real function, a vector field, a $(0,2)$-tensor field and a linear connection on M, then their vertical and complete lifts on TM are defined by:*

$$f^v,\, f^c \in \mathcal{F}(TM), \quad f^v = f \circ \pi, \quad f^c = \mathrm{d}f; \tag{3.3}$$

$$\begin{cases} X^v, X^c \in \Gamma(TTM), & X^v f^v = 0 \\ X^v f^c = (Xf)^v = X^c f^v, & X^c f^c = (Xf)^c; \end{cases} \tag{3.4}$$

$$\begin{cases} g^v, g^c \in \mathcal{T}_2^0(TM), \quad g^v(X^v, Y^v) = g^v(X^v, Y^c) = g^v(X^c, Y^v) = 0 \\ g^v(X^c, Y^c) = (g(X,Y))^v \quad \text{and} \quad g^c(X^v, Y^v) = 0, \\ g^c(X^v, Y^c) = (g(X,Y))^v = g^c(X^c, Y^v), \quad g^c(X^c, Y^c) = (g(X,Y))^c; \end{cases} \tag{3.5}$$

$$\begin{cases} \nabla^c_{X^v} Y^v = 0, & \nabla^c_{X^v} Y^c = (\nabla_X Y)^v = \nabla^c_{X^c} Y^v, \\ \nabla^c_{X^c} Y^c = (\nabla_X Y)^c, & \forall Y \in \Gamma(TM). \end{cases} \tag{3.6}$$

Remarks. (i) Any tensor field on TM may be expressed locally in terms of the vertical and complete lifts of some tensors on M;

(ii) $\Gamma(TTM) = \mathrm{span}\{X^v,\, X^c : X \in \Gamma(TM)\}$;

(iii) If $(M,\, g)$ is a Riemannian manifold of m-dimension and ∇ is its Levi-Civita connection, then g^c is a semi-Riemannian metric on TM of signature $(m,\, m)$ and ∇^c is its Levi-Civita connection.

Local coordinates 3.8. If $(M,\, g)$ is a Riemannian manifold of m-dimension, let (x^i) and $(x^i,\, y^i)$ be local coordinates which induce the local frames

$$\left\{ \frac{\partial}{\partial x^i} : i = \overline{1,m} \right\} \qquad \text{and} \qquad \left\{ \frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^i} \right)^c, \frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i} \right)^v : i = \overline{1,m} \right\}$$

on M and TM, respectively. With respect to the last local frame, we have the following local expression:

$$g^c = \begin{pmatrix} y^k \dfrac{\partial g_{ij}}{\partial x^k} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}, \qquad (g^c)^{-1} = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & y^k \dfrac{\partial g^{ij}}{\partial x^k} \end{pmatrix}. \tag{3.7}$$

Now we characterize the coclosure condition in a special case:

Theorem 3.9. *Let $\varphi : (M,g) \to (N,h)$ be a map between Riemannian manifolds. Then any φ-morphism $\Phi : (TM, g^c) \to (TN, h^c)$ is a harmonic map if and only if $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ is coclosed w.r.t. $\nabla^{\varphi^{-1}TN}$.*

This theorem is a consequence of the following

Formula. $\tau(\Phi) = 2(\delta\Phi)^v$.

Proof. The second fundamental form of Φ is given by, for all $U, V \in \Gamma(TTM)$:

$$\nabla \mathrm{d}\Phi(U,V) = \nabla^{\Phi^{-1}TTN}_U \mathrm{d}\Phi(V) - \mathrm{d}\Phi\left(\overset{M}{\bigtriangledown}{}^c_U V\right) = \overset{N}{\bigtriangledown}{}^c_{\mathrm{d}\Phi(U)}\, \mathrm{d}\Phi(V) - \mathrm{d}\Phi\left(\overset{M}{\bigtriangledown}{}^c_U V\right). \tag{3.8}$$

Let m and n be the dimensions of M and N, respectively. Similarly to the local coordinates which induce the local frames

$$\left\{\frac{\partial}{\partial u^\alpha} :\ \alpha = \overline{1,n}\right\} \quad \text{and} \quad \left\{\frac{\partial}{\partial u^\alpha} = \left(\frac{\partial}{\partial u^\alpha}\right)^c,\ \frac{\partial}{\partial v^\alpha} = \left(\frac{\partial}{\partial u^\alpha}\right)^v :\ i = \overline{1,m}\right\}$$

on N and TN, respectively. Then the map $\Phi : TM \to TN$ is given in local coordinates by

$$\Phi(x,y) = \left(\varphi^\alpha(x), \Phi^\alpha_i(x) y^i\right), \qquad \alpha = \overline{1,n}\,. \tag{3.9}$$

where $x = (x^1, \ldots, x^m),\ y = (y^1, \ldots, y^n)$. Then:

$$\begin{aligned} \mathrm{d}\Phi\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial}{\partial u^\alpha} + \frac{\partial \Phi^\alpha_k}{\partial x^i} y^k \frac{\partial}{\partial v^\alpha}\,; \\ \mathrm{d}\Phi\left(\frac{\partial}{\partial y^i}\right) &= \Phi^\alpha_i \frac{\partial}{\partial v^\alpha}, \qquad i = \overline{1,m}\,. \end{aligned} \tag{3.10}$$

From (3.8), (3.10) and (3.6), it follows:

$$\nabla \mathrm{d}\Phi\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \qquad i,\, j = \overline{1,m}\,. \tag{3.11}$$

By using (3.7), (3.8), (3.11) and the symmetry of $\nabla \mathrm{d}\Phi$, we obtain the local expression of the tensor field $\tau = \mathrm{trace}\, \nabla \mathrm{d}\Phi$:

$$\tau = 2g^{ij} \nabla \mathrm{d}\Phi\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^i}\right). \tag{3.12}$$

Let $\overset{M}{\Gamma}{}^k_{ij}$ and $\overset{N}{\Gamma}{}^\gamma_{\alpha\beta}$ denote the Christoffel symbols of $\overset{M}{\bigtriangledown}$ and $\overset{N}{\bigtriangledown}$, respectively. Then (3.8), (3.10) and (3.6) yield:

$$\tau = 2g^{ij}\left(\Phi^\alpha_j \frac{\partial \varphi^\beta}{\partial x^i}\, \overset{N}{\Gamma}{}^\gamma_{\alpha\beta} - \overset{M}{\Gamma}{}^k_{ij}\, \Phi^\alpha_k\right)\frac{\partial}{\partial v^\gamma}\,. \tag{3.13}$$

On the other hand, from (1.2) we infer the local expression:

$$\delta\Phi = g^{ij}\left(\nabla^{\varphi^{-1}TN}_{\frac{\partial}{\partial x^i}} \Phi\right)\frac{\partial}{\partial x^j}\,. \tag{3.14}$$

From (1.3), by a straightforward calculation, we obtain:

$$\delta\Phi = g^{ij}\left(\Phi^\alpha_j \frac{\partial \varphi^\beta}{\partial x^i}\, \overset{N}{\Gamma}{}^\gamma_{\alpha\beta} - \overset{M}{\Gamma}{}^k_{ij}\, \Phi^\alpha_k\right)\frac{\partial}{\partial u^\gamma}\,. \tag{3.15}$$

Then (3.13) and (3.15) yield the formula.

Remark. If in particular Φ is a (1,1)-tensor field on M (as in Section 1, Example of Φ), then the main theorem of [6] is obtained.

4. Applications to (1,1)-tensor fields

In the remaining part of the paper, the results of Section 3 are applied in the case when $(M, g) = (N, h)$ and φ is the identity map 1_M of M. Hence any φ-morphism Φ becomes a (1,1)-tensor field $K : TM \to TM$ which is given in the local coordinates (3.8) by

$$K(x, y) = (K^1, \ldots, K^m; K^{m+1}, \ldots, K^{2m}) = (x^1, \ldots, x^m; K^1_j y^j, \ldots, K^m_j y^j) \tag{4.1}$$

where $x = (x^1, \ldots, x^m)$, $y = (y^1, \ldots, y^m)$, $K\dfrac{\partial}{\partial x^j} = K^i_j \dfrac{\partial}{\partial x^j}$ and $K^i_j = K^i_j(x)$, $i, j = \overline{1, m}$.

As we mentioned in the last remark of Section 3, the class of all harmonic (1,1)-tensor fields $K : (TM, g^c) \to (TM, g^c)$ was studied in [6]. Here we determine the subclass of all (1,1)-tensor fields $K : (TM, g^c) \to (TM, g^c)$ which are harmonic morphisms. First we recall one of the equivalent definitions provided in [1].

Definition 4.1 *Let $F : (A, g) \to (B, h)$ be a map between semi-Riemannian manifolds and let $a \in A$. Then F is called:*

(i) *weakly conformal at a, if there is $\Lambda(a) \in R$ such that:*

$$h(\mathrm{d}F_a(U), \mathrm{d}F_a(V)) = \Lambda(a) g(U, V), \qquad \forall U, V \in T_a A\,, \tag{4.2}$$

(Λ(a) *is called the conformality factor*).

(ii) *horizontally weakly conformal at a if there is $\Lambda(a) \in R$ such that for any local frame $\{Z_\alpha : \alpha = \overline{1, n}\}$ on B:*

$$g(\operatorname{grad} F^\alpha, \operatorname{grad} F^\beta) = \Lambda(a) h^{\alpha\beta}, \qquad \alpha, \beta = \overline{1, n}\,, \tag{4.3}$$

where $n = \dim B$.

(iii) *(horizontally) weakly conformal on A if F is (horizontally) weakly conformal at all $a \in A$.*

Proposition 4.2. [1] *Let $F : (A, g) \to (B, h)$ be a map between semi-Riemannian manifolds and let $a \in A$. Then F is weakly conformal at a if and only if one of the following holds*:

(i) $\mathrm{d}F_a = 0$;

(ii) $\mathrm{d}F_a$ *maps $T_a A$ conformally onto its image, i.e. there exists $\Lambda(a) \neq 0$ such that* (4.2) *holds good*;

(iii) *the image of* $\mathrm{d}F_a$ *is non-zero and lightlike w.r.t. the semi-Riemannian metric h, i.e. h restricted to the image of* $\mathrm{d}F_a$ *is zero.*

From (4.1), Definition 4.1 and Proposition 4.2, we obtain:

Lemma 4.3. *Let $K : (TM, g^c) \to (TM, g^c)$ be a* (1-1)*-tensor field of maximal rank (i.e.* $\operatorname{rank} K = \dim M$*). Then the following assertions are equivalent*:

(a) *K is weakly conformal*;

(b) *K is horizontally conformal*;

(c) *at any $a \in A$, K satisfies condition* (ii) *of Proposition* 4.2.

Lemma 4.4. *Let K be a* (1,1)*-tensor field on M and let $m = \dim M$. Then $K : (TM, g^c) \to (TM, g^c)$ is horizontally weakly conformal of maximal rank and of conformality factor $\Lambda \in \mathcal{F}(TM)$ if and only if* (4.4) *and* (4.5) *are satisfied*:

$$K = \Lambda I, \tag{4.4}$$

where I is the identity tensor field and

$$\textit{either} \quad \Lambda = 1 \quad \textit{or} \quad \Lambda \in \mathcal{F}(M), \quad \Lambda(x) \neq 0 \tag{4.5}$$

and at any $x \in M$, $2\dfrac{\partial \ln|\Lambda - 1|}{\partial x^i} = \operatorname{trace}\left(\mathcal{L}_{\frac{\partial}{\partial x^i}} g\right)$, $i = \overline{1, m}$, where L denotes Lie derivative.

Remark. Actually, (4.5) says that Λ depends only on $x \in M$ and not on $(x, y) \in TM$.

Proof. Let K^α, $\alpha = \overline{1, 2m}$, be defined as in (4.1). Then:

$$g^c(\operatorname{grad} K^\alpha, U) = UK^\alpha, \qquad \forall U \in \Gamma(TTM), \qquad \alpha = \overline{1, 2m}. \tag{4.6}$$

Replacing all instances of U by $\left(\dfrac{\partial}{\partial x^i}\right)^v$, $\left(\dfrac{\partial}{\partial x^i}\right)^c$, $i = \overline{1, m}$, in turn we obtain:

$$\operatorname{grad} K^s = g^{sj}\frac{\partial}{\partial y^j}; \tag{4.7}$$

$$\operatorname{grad} K^{m+s} = K^s_i g^{ik}\frac{\partial}{\partial x^k} + y^k\left(g^{ij}\frac{\partial K^s_h}{\partial x^i} + \frac{\partial g^{jl}}{\partial x^h}K^s_l\right)\frac{\partial}{\partial y^j}, \qquad s = \overline{1, m}.$$

From (4.7) and (3.7) one obtains:

$$g^c(\operatorname{grad} K^t, \ \operatorname{grad} K^s) = 0; \tag{4.8}$$

$$g^c(\operatorname{grad} K^t, \ K^{m+s}) = K^s_i g^{it}, \qquad s, \, t = \overline{1, m}\,. \tag{4.9}$$

From (4.3), Lemma 4.3 and (3.7), K is horizontally weakly conformal if and only if there exists $\Lambda \in \mathcal{F}(TM)$ such that (4.8), (4.10) and (4.11) hold good, where

$$g^c(\operatorname{grad} K^t, \ \operatorname{grad} K^{m+s}) = \Lambda g^{st}; \tag{4.10}$$

$$g^c(\operatorname{grad} K^{m+t}, \ \operatorname{grad} K^{m+s}) = \Lambda y^k\frac{\partial g^{st}}{\partial x^k}, \qquad s, \, t = \overline{1, m}\,. \tag{4.11}$$

Note that (4.8) is always satisfied and that (4.10) is equivalent to (4.4) by virtue of (4.9). Hence the previous assertion combined with Lemma 4.3 and Proposition 4.2 ensure that K is horizontally weakly conformal of maximal rank and dilation $\Lambda \in \mathcal{F}(TM)$ if and only if (4.4) and (4.11) are satisfied for $\Lambda(x,y) \neq 0,\ \forall (x,y) \in TM$. Now, the lemma will be a consequence of the following:

Fact. *If* (4.4) *is satisfied for* $\Lambda(x,y) \neq 0,\ \forall (x,y) \in TM$, *then* (4.11) *is equivalent to* (4.5).

To show this fact, we assume (4.4) and then from (4.7) and (3.7) the following equivalence holds:

$$(4.11) \Leftrightarrow$$

$$\Leftrightarrow \Lambda^2 g^{tk} g^{sl} \frac{\partial g_{kl}}{\partial x^h} y^h + \Lambda g^{tk} y^h \left(g^{ij} \frac{\partial \Lambda}{\partial x^i} \delta^s_h + \Lambda \frac{\partial g^{js}}{\partial x^h} \right) g_{hj} +$$

$$+ \Lambda g^{sk} y^h \left(g^{ij} \frac{\partial \Lambda}{\partial x^i} \delta^t_h + \Lambda \frac{\partial g^{it}}{\partial x^h} \right) g_{kj} = \Lambda \frac{\partial g^{ts}}{\partial x^h} y^h \Leftrightarrow$$

$$\Leftrightarrow -\Lambda^2 \frac{\partial g^{ts}}{\partial x^h} y^h + \Lambda y^h \left(g^{it} \frac{\partial \Lambda}{\partial x^i} \delta^s_h + g^{is} \frac{\partial \Lambda}{\partial x^i} \delta^t_s + 2\Lambda \frac{\partial g^{ts}}{\partial x^h} \right) = \Lambda \frac{\partial g^{ts}}{\partial x^h} y^h \Leftrightarrow$$

$$\Leftrightarrow \Lambda^2 \frac{\partial g^{ts}}{\partial x^i} y^i + \Lambda \frac{\partial \Lambda}{\partial x^i} (g^{it} y^s + g^{is} y^t) = \Lambda \frac{\partial g^{ts}}{\partial x^i} y^i \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial \Lambda}{\partial x^i} (g^{it} y^s + g^{is} y^t) = (1 - \Lambda) y^i \frac{\partial g^{ts}}{\partial x^i} \ \ (\text{since } \Lambda \neq 0 \text{ at any point of } TM) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial \Lambda}{\partial x^i} 2y^i = (\Lambda - 1) y^i \frac{\partial g_{ts}}{\partial x^i} g^{ts} \Leftrightarrow (4.5),$$

which complete the above fact and hence the Lemma.

Example. Let $M = S^2 = \{x = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi, \cos\theta) \in \mathbb{R}^3 / \varphi \in [0, 2\pi],\ \theta \in [0, \pi]\}$ be the unit sphere endowed with the metric $g = (\mathrm{d}\theta)^2 + \sin^2\theta (\mathrm{d}\varphi)^2$ induced from $\mathbb{R}^3$. Then the identity I is the only one (1,1)-tensor field $K : (TM, g^c) \to (TM, g^c)$ which are horizontally weakly conformal and of maximal rank.

Theorem 4.5. *Any* (1,1)*-tensor field* $K : (TM, g^c) \to (TM, g^c)$ *of maximal rank is a harmonic morphism (of conformal factor* $\Lambda \in \mathcal{F}(TM)$*) if and only if* K *is the identity tensor field up to a non-zero constant factor* $\Lambda \in \mathbb{R}$, *which satisfies* (4.5)

Proof. As we mentioned in Introduction, the harmonic morphisms are characterized in the Riemannian case by [4], [8] and in the semi-Riemannian case by [4], as to be the maps which are horizontally weakly conformal and harmonic. From Theorem 3.9 and Lemma 4.4, K is a harmonic morphism of maximal rank and of conformal factor $\Lambda \in \mathcal{F}(TM)$ if and only if it satisfies three relations: $\delta K = 0$, (4.4) and (4.5). From (4.4) and $\delta K = 0$ one can see that Λ is constant. Therefore these relations are equivalent to $K = \Lambda I$, with Λ a non-zero constant satisfying (4.5) which complete the proof.

Proposition 4.6. [1] *Let $F : A \to B$ be a weakly conformal map between semi-Riemannian manifolds of the same dimension m which is non-degenerate on a dense subset. Then*

(i) *if $m = 2$, F is harmonic;*

(ii) *if $m \geq 3$, F is harmonic if and only if the conformality factor is constant.*

Remark. From Proposition 4.6 and Lemma 4.3, any (1,1)-tensor field $K : (TM, g^c) \to (TM, g^c)$ of maximal rank which is a harmonic morphism, has a constant conformality factor when $\dim TM \geq 3$ (i.e. $\dim M > 1$). Therefore Theorem 4.5 shows that Proposition 4.6 (ii) holds for any (1,1)-tensor field $K : (TM, g^c) \to (TM, g^c)$ of maximal rank which is a harmonic morphism, even when $\dim M = 1$.

Among the (1,1)-tensor fields of maximal rank, the class of (classical) harmonic morphisms determined by Theorem 4.5 is very restricted, so that we are motivated to study (1,1)-tensor fields which are harmonic φ-morphisms with φ the identity map.

5. Examples of harmonic φ-morphisms

Some of the examples of harmonic (1,1)-tensor fields obtained in [6] turn out to be of maximal rank and moreover turn out to be harmonic φ-morphisms (with φ the identity map) w.r.t. the canonical connection. We note that the notion of harmonic (1,1)-tensor field given in [6] is different of that used in [2]. As they are consequences of Definition 3.6 and Theorem 3.9 all the statements of this section are given without proof.

Proposition 5.1. *The identity tensor field of a Riemannian manifold M is a harmonic 1_M-morphism of dilation one.*

Proposition 5.2. *On any Einstein manifold M, the Ricci (1,1)-tensor field Ric: $TM \to TM$ is a harmonic 1_M-morphism.*

We recall that an almost Hermitian manifold (M, g, J) is called cosymplectic [10] or semi-Kähler [7], provided $\delta J = 0$.

Proposition 5.3. *An almost Hermitian manifold (M, g, J) is semi-Kähler if and only if J is a harmonic 1_M-morphism.*

Next, an almost product Riemannian manifold is defined as a Riemannian manifold (M, g) endowed with an almost product structure P (that is a (1,1)-tensor field $P \neq \pm 1_{TM}$, with $P^2 = 1_{TM}$) such that $g(PX, PY) = g(X, Y)$, $\forall X, Y \in \Gamma(TM)$. A classification of these manifolds is given in [9].

Proposition 5.4. *If (M, P, g) is an almost product Riemannian manifold lying in the class $W_1 \oplus W_2 \oplus W_4 \oplus W_5$, [9] then P is a harmonic 1_M-morphism.*

The needed property of being in the given class is $\delta P = 0$.

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