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Classification of Five Dimensional Hypersurfaces with Affine Normal Parallel Cubic Form

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Abstract. We consider and classify those five dimensional hypersurfaces with affine normal parallel cubic form. The problem is firstly reduced to the classification of a certain class of solutions to the equation of Monge-Ampère type $\det(\partial_{ij} f) = \pm 1$. Then, it is used the so-called "method of algorithmic sequence of coordinate changes", in order to achieve the latter.

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1. Introduction

The problem of classifying hypersurfaces with affine normal parallel cubic form, which are not hyperquadrics, was first considered, and solved for dimension $n = 2$, by Nomizu and Pinkall in [5]. See also the book by Nomizu and Sasaki, [6], where a different method of proof is presented. We summarize their result as follows:

Theorem A. *Let $X : M^2 \to E^3$ be a nondegenerate surface with parallel cubic form, $\nabla C = 0$, which is not a quadric, i.e., C does not vanish identically on M. Then $X(M)$ is affinely congruent to the Cayley Surface, i.e., expressible as the graph function $t_3 = t_1 t_2 + t_1^3$.*

The next known result on the topic is the article by L. Vrancken (for dimension $n = 3$), exposed in [7], and stated here as:

Theorem B. *Let $X : M^3 \to E^4$ be a nondegenerate hypersurface with parallel cubic form, $\nabla C = 0$, which is not a hyperquadric, C does not vanish identically on the hypersurface. Then $X(M)$ is affinely congruent to one of the following graph immersions*

a) $t_4 = t_1t_2 + t_3^2 + t_1^3$.
b) $t_4 = t_1t_2 + t_1^2t_3 + t_3^2$.

In a previous article [4] we introduced a new method of approaching the solution to the problem, different to the ones previously used by the other mentioned authors. We called it "algorithmic sequence of coordinate changes" and it consists, basically, in referring the hypersurface to a suitable linear coordinate system of the ambient space, and then making algorithmic adjustments into the Hessian matrix so that this can be integrated, fairly easily, to obtain the graph function representing the hypersurface. With this approach the classification depends strongly on two integer constants, that we labeled k and r, with $1 \leq k \leq n/2$, $1 \leq r \leq n-1$, where $n =$ dimension of the immersed manifold. Moreover, in that article we presented new proofs of the previously stated results and, then, extended the classification to dimension $n = 4$, by proving:

Theorem C. *Let $X : M^4 \to E^5$ be a nondegenerate hypersurface with parallel cubic form, $\nabla C = 0$, which is not a hyperquadric, C does not vanish identically on the hypersurface. Then $X(M)$ is affinely congruent to one of the following graph immersions:*

a) *For $k = r = 1$:* $t_5 = t_1t_2 + t_1^3 + t_3^2 + t_4^2$.
b) *For $k = 1$, $r = 2$:* $t_5 = t_1t_2 + t_1^2t_3 + t_3^2 + t_4^2$.
c) *The case where $k = 1$, $r = 3$ is not possible, i.e., it does not exist any non degenerate hypersurface immersion with the required geometrical properties in the case where $k = 1$, $r = 3$.*
d) *For $k = 2$, $r = 1$:* $t_5 = t_1t_2 + t_1^3 + t_3t_4$.
e) *For $k = r = 2$ we have the following subcases:*
   e$_{11}$) $t_5 = t_1t_2 + \frac{a}{6}t_1^3 + \frac{b}{2}t_1^2t_3 + \frac{c}{2}t_1t_3^2 + t_3t_4 + \frac{d}{6}t_3^3$.
   e$_{12}$) $t_5 = t_1t_2 + \frac{c}{2}t_1t_3^2 + \beta\frac{c}{2}t_2t_3^2 + t_3t_4 + \frac{d}{6}t_3^3$.
f) *For $k = 2$, $r = 3$:* $t_5 = t_1t_2 + \frac{a}{2}t_2t_3^2 + t_3t_4 + \frac{b}{2}t_1^2t_3 + \frac{c}{2}t_1t_3^2 + \frac{d}{6}t_1^3 + \frac{e}{6}t_3^3$.

In each of cases e) *and* f) *the constants must be related among them in order to fulfill the condition that the maximal rank of the complementary matrix equals respectively* 2 *and* 3, *i.e., $r = 2, 3$.*

It is the object of this paper to further extend the classification to the case of dimension $n = 5$, by proving the following:

Main Theorem. *Let $X : M^5 \to E^6$ be a nondegenerate hypersurface with parallel cubic form, $\nabla C = 0$, which is not a hyperquadric, C does not vanish identically on the hypersurface. Then $X(M)$ is affinely congruent to one of the following graph immersions:*

a) *For $k = r = 1$:* $t_6 = t_1t_2 + \frac{1}{6}t_1^3 + \frac{1}{2}(t_3^2 + t_4^2 + t_5^2)$.

b) *For $k = 1$, $r = 2$:* $t_6 = t_1t_2 + t_1^2t_3 + \frac{1}{2}(t_3^2 + t_4^2 + t_5^2)$.

c) *The case where $k = 1$, $r = 3$ is not possible, i.e., it does not exist any non degenerate hypersurface immersion with the required geometrical properties in the case where $k = 1$, $r = 3$.*

d) *The same situation happens with the case where $k = 1$, $r = 4$, i.e., it does not exist any non degenerate hypersurface immersion with the required geometrical properties in the case where $k = 1$, $r = 4$.*

e) *For $k = 2$, $r = 1$:* $t_6 = t_1t_2 + \frac{1}{6}t_1^3 + t_3t_4 + \frac{1}{2}t_5^2$.

f) *For $k = r = 2$ we have the following subcases:*

f$_1$) $t_6 = t_1t_2 + \frac{1}{6}a_1t_1^3 + \frac{1}{2}a_3t_1^2t_3 + \frac{1}{2}b_3t_1t_3^2 + t_3t_4 + \frac{1}{6}d_3t_3^3 + \frac{1}{2}t_5^2$.

f$_2$) $t_6 = t_1t_2 + \frac{1}{2}a_3t_1t_3^2 - \frac{1}{2}b_2a_3t_2t_3^2 + t_3t_4 + \frac{1}{2}t_5^2$.

g) *For $k = 2$, $r = 3$ we have the following subcases:*

g$_1$) $t_6 = t_1t_2 + \frac{1}{2}at_2t_3^2 + \frac{1}{6}bt_1^3 + \frac{1}{2}ct_1^2t_3 + \frac{1}{2}dt_1t_3^2 + t_3t_4 + \frac{1}{2}t_5^2$.

g$_2$) $t_6 = t_1t_2 + \frac{1}{6}a_1t_1^3 + \frac{1}{2}a_3t_1^2t_3 + \frac{1}{2}a_5t_1^2t_5 + c_5t_1t_3t_5 + t_3t_4 + \frac{1}{6}f_3t_3^3 + \frac{1}{2}f_5t_3^2t_5 + \frac{1}{2}t_5^2$.

g$_3$) $t_6 = t_1t_2 + \frac{1}{2}a_3t_1^2t_3 + \frac{1}{2}a_4t_1^2t_4 + \frac{1}{2}c_3t_1t_3^2 + t_3t_4 + \frac{1}{6}f_3t_3^3 + \frac{1}{2}t_5^2$.

h) *For $k = 2$, $r = 4$:* $t_6 = t_1t_2 + \frac{1}{2}c_1t_1^2t_4 + e_1t_1t_3t_5 + t_3t_4 + \frac{1}{2}e_4t_3^2t_5 + \frac{1}{2}g_1t_1t_3^2 + \frac{1}{6}g_4t_3^3 + \frac{1}{2}h_1t_1^2t_3 + \frac{1}{2}t_5^2$.

In each of cases f), g) *and* h) *the constants must be related among them in order to fulfill the condition that the maximal rank of the complementary matrix equals respectively* 2, 3 *and* 4, *i.e., $r = 2, 3, 4$.*

Remark. In previous papers we also referred to the cubic form C, alternatively, as the *second fundamental form II_{ua}*, even though it is indeed a $(0,3)$-tensor. The reason for this is because this form, together with the *first fundamental form I_{ua}* (pseudo-Riemannian metric), determine the geometry of the immersed manifold. In particular, the fundamental existence and uniqueness theorems [1, 2]. However, in the current work no question regarding those matters arises. Thus, we refer presently to that (fundamental) geometrical object as just the *cubic form C*.

2. Summary of auxiliary results

The following properties of the class under consideration were proved in [6]. They can also be proved by a different method [1, 2, 3, 4]. We include their statement here for the sake of completeness:

Proposition 2.1. *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface with parallel cubic form, $\nabla C = 0$, which is not a hyperquadric, $C \neq 0$. Then the following properties hold:*

1) *$X(M)$ is an improper affine hypersphere.*

2) *$X(M)$ is expressible in the form of Monge, i.e., a graph immersion and with respect to a suitable affine system of coordinates the graph function f satisfies a Monge-Ampère type equation* $\det(f_{ij}) = \pm 1$. *Moreover, it is representable as a polynomial function of degree exactly equal to three.*
3) *The following geometrical objects associated with $X(M)$ are all vanishing: $III_{ga} = 0$, $\hat{R}ic = 0$, $R = L = J = 0$.*
4) *The first fundamental form I_{ua} is indefinite.*

It turns out, too, that the conditions expressed by property 2) in the above Proposition are characterizing. In fact, we also proved, in [4], the following complementary result:

Proposition 2.2. *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface which is expressible in the form of Monge, i.e., a graph immersion with respect to some affine system of coordinates in the ambient space, such that the graph function f is a polynomial function of degree exactly equal to three and satisfies the Monge-Ampère type equation* $\det(f_{ij}) = \pm 1$. *Then, $X(M)$ is an improper affine hypersphere with parallel cubic form, $\nabla C = 0$, which is not a hyperquadric, i.e., with $C \neq 0$.*

We summarize next the method of *algorithmic sequence of coordinate changes.* To begin with, we apply the characterizing properties described by Propositions 2.1 and 2.2. Thus, by means of a translation, if necessary, we may assume that a linear system of coordinates has been chosen in the ambient space in such a way that the origin of coordinates lies in the hypersurface $X(M)$, that the hyperplane on which $X(M)$ is projectable is precisely the tangent hyperplane $T_0(X(M))$ to $X(M)$ at that point, and that the last coordinate is chosen in the (constant) direction of the affine normal vector field e_{n+1}. We denote by $(t_1, t_2, \ldots, t_n, t_{n+1})$ such an affine system of the vector space E, and represent the immersed hypersurface by equation

$$X(t_1, t_2, \ldots, t_n) = (t_1, t_2, \ldots, t_n, f(t_1, t_2, \ldots, t_n)),$$

with the point $(t_1, t_2, \ldots, t_n)$ varying in an open, connected subset $U \subset T_0(X(M))$, this last being obviously identifiable with $\mathbb{R}^n$. By the choices made we have that

$$f(0, \ldots, 0) = f_1(0, \ldots, 0) = \cdots = f_n(0, \ldots, 0) = 0 \tag{2.1}$$

All of the remaining affine changes of coordinates shall occur in the tangent hyperplane and be of a linear nature, i.e., given by a system of linear equations like

$$t_i^* = \sum a_i^k t_k, \; t_{n+1}^* = t_{n+1}.$$

Most usually the change shall be unimodular, i.e., with $\det\left(a_i^k\right) = 1$, although we may allow, occasionally, a rescaling in order to make the exposition less involved.

Once such a change is made, in the new coordinate system, conditions expressed by equations (2.1) remain unchanged, and the Hessian matrix $H(f)$ changes to

$$H^*(f) = PH(f)P^t, \tag{2.2}$$

where the matrix P is nonsingular and P^t denotes the transpose of P. Now, it is well-known that, since P is expressible as a product of elementary matrices, the product to the left by P is equivalent to performing the corresponding row elementary operations to $H(f)$, and the product to the right by P^t is obtained by performing the equivalent kinds of column elementary operations, both in the same order of execution. Thus, to obtain $H^*(f)$ from $H(f)$ we may do so by means of the following row and column elementary operations, which we define next:

1) R_{ij} interchanges rows i and j. C_{ij} interchanges columns i and j.
2) $R_i+\sum a_{ij}R_j$, with $j \neq i$, substitutes the i-th row by the linear combination as indicated. Similarly, the notation for columns shall be indicated by $C_i + \sum a_{ij}C_j$.

Obviously, these two kinds of elementary operation are unimodular. The third kind consists of multiplying a row, and the corresponding column by a nonzero constant. This produces a rescaling.

In [4], we also proved the following procedural result:

Lemma 2.3. *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface with parallel cubic form, $\nabla C = 0$, which is not a hyperquadric, $C \neq 0$. Then there exists an affine coordinate system in the ambient space such that $X(M)$ is expressible in the form of Monge (i.e., by means of a graph function f) and such that the corresponding Hessian matrix is given by*

$$H(f) = (f_{ij}) = J_k + (x_{ij})$$

where J_k is a matrix with k (≥ 1) blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, in diagonal position, occupying the first $2k$ diagonal entries the (possible) remaining diagonal elements are equal to 1, and with all of the rest of entries equal 0; while all of the entries of the matrix (x_{ij}) are linear functions of the (domain) coordinates $t_1, t_2, \dots, t_n$, i.e., $x_{ij} = \sum a_{ijk}t_k$. Moreover, the matrix of linear functions (x_{ij}) is everywhere singular, whose maximal rank r is attained on an open, dense subset of the domain, and we have $1 \leq r \leq n-1$.

The two positive integers k, with $1 \leq k \leq n/2$, and r, with $1 \leq r \leq n-1$, are characteristic of, and determined by, each hypersurface with the required geometrical properties of having parallel second fundamental form with respect to the affine normal connection, i.e., $\nabla C = 0$, and not being a hyperquadric, i.e., with $C \neq 0$. Thus, we emphasize again here that these two numbers play an essential role in the classification procedure.

3. Proof of the main theorem

From Lemma 2.3 we have two possible values for $k = 1, 2$; and four for $r = 1, 2, 3, 4$.

Cases a) $k = r = 1$; and b) $k = 1$, $r = 2$; are quite similar to the same labeled cases in Theorem C. Thus we omit their proofs.

c) The third possible case corresponds to the values $k = 1$, $r = 3$. By Lemma 2.3 we can first reduce the Hessian matrix to

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & x_{14} & x_{15} \\ 1+x_{12} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{13} & x_{23} & 1+x_{33} & x_{34} & x_{35} \\ x_{14} & x_{24} & x_{34} & 1+x_{44} & x_{45} \\ x_{15} & x_{25} & x_{35} & x_{45} & 1+x_{55} \end{bmatrix} \tag{3.1}$$

We define the vectors $X_i := (x_{1i}, x_{2i}, x_{3i}, x_{4i}, x_{5i})$. Then, it is easy to see, by means of suitable elementary operations, that the present case can be reduced to three subcases, labeled as: c_1), c_2) and c_3).

c_1) Let us assume, first, that the vectors X_1, X_2 and X_3 are linearly independent on an open, dense subset of the domain, represent the remaining ones by $X_i = a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3$, $i = 3, 4$, and perform into the Hessian matrix defined by expression (3.1) the elementary operations suggested by the latter equality, i.e., $R_i - a_{i1}R_1 - a_{i2}R_2 - a_{i3}R_3$ and $C_i - a_{i1}C_1 - a_{i2}C_2 - a_{i3}C_3$, $i = 3, 4$, to obtain

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & -a_{42} & -a_{52} \\ 1+x_{12} & x_{22} & x_{23} & -a_{41} & -a_{51} \\ x_{13} & x_{23} & 1+x_{33} & -a_{43} & -a_{53} \\ -a_{42} & -a_{41} & -a_{43} & b_{44} & b_{45} \\ -a_{52} & -a_{51} & -a_{53} & b_{45} & b_{55} \end{bmatrix},$$

where $b_{44} = 1 + 2a_{41}a_{42} + a_{43}^2$, $b_{45} = a_{41}a_{52} + a_{42}a_{51} + a_{43}a_{53}$, $b_{55} = 1 + 2a_{51}a_{52} + a_{53}^2$. Besides, the 2×3 submatrix (a_{ij}) must be different from zero (otherwise the Hessian determinant vanishes), and (the determinant of the 2×2 submatrix) $|b_{ij}| = 0$. From this we have two subcases c_{11}) $(b_{ij}) = 0$ or c_{12}) $(b_{ij}) \neq 0$.

In the first subcase, c_{11}), we must also have $a_{41} \neq 0$, $a_{42} \neq 0$, $a_{51} \neq 0$, $a_{52} \neq 0$. Then, we perform into the latter expression of the Hessian matrix the elementary operations $R_4 + \frac{a_{42}}{a_{52}}R_5$, $C_4 + \frac{a_{42}}{a_{52}}C_5$; $R_2 + \frac{a_{51}}{a_{52}}R_1$, $C_2 + \frac{a_{51}}{a_{52}}C_1$; $R_3 + \frac{a_{53}}{a_{52}}R_1$, $C_3 + \frac{a_{53}}{a_{52}}C_1$; the entries $-a_{41}$, $-a_{43}$ transformed into a'_{41}, a'_{43}, and one of these must be different from zero. It is easy to see that, by suitable further elementary operations, we can get the Hessian transformed, first into

$$\begin{bmatrix} x_{11} & 1+x_{12} & b_{13}+x_{13} & 0 & 1 \\ 1+x_{12} & b_{22}+x_{22} & b_{23}+x_{23} & 1 & 0 \\ b_{13}+x_{13} & b_{23}+x_{23} & b_{33}+x_{33} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and then, by also using the fact that the determinant of the latter $\triangle = b_{33} + x_{33}$, and hence

$x_{33} = 0$, $b_{33} = -1$, into

$$\begin{bmatrix} x_{11} & 1 & 0 & x_{14} & x_{15} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_{35} \\ x_{14} & 0 & 1 & x_{44} & 0 \\ x_{15} & 0 & x_{35} & 0 & 1 \end{bmatrix}.$$

This is a contradiction, since we are assuming $k = 1$, so that this subcase is not possible.
In the second subcase c$_{12}$), we may assume $b_{55} \neq 0$, write $(b_{44}, b_{45}) = c\,(b_{45}, b_{55})$ and perform elementary operations so as to transform the Hessian matrix into

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & a'_{42} & 0 \\ 1+x_{12} & x_{22} & x_{23} & a'_{41} & 0 \\ x_{13} & x_{23} & 1+x_{33} & a'_{43} & 0 \\ a'_{42} & a'_{41} & a'_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, some of the a'_{4i} must be nonvanishing and we can further reduce to two subcases: c$_{121}$) $a'_{43} \neq 0$, c$_{122}$) $a'_{42} \neq 0$.
In the first of these we perform further elementary operations to get the Hessian equal to

$$\begin{bmatrix} b_{11}+x_{11} & b_{12}+x_{12} & x_{13} & 0 & 0 \\ b_{12}+x_{12} & b_{22}+x_{22} & x_{23} & 0 & 0 \\ x_{13} & x_{23} & x_{33} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

but since the determinant of the latter equals minus the determinant of the 2×2 submatrix $(b_{ij} + x_{ij})$, we must have $|b_{ij} + x_{ij}| = 1$, so that we can further diagonalize the latter to get the Hessian

$$\begin{bmatrix} 1 & 0 & x_{13} & 0 & 0 \\ 0 & 1 & x_{23} & 0 & 0 \\ x_{13} & x_{23} & x_{33} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is not possible, since we are assuming $r = 3$.
In subcase c$_{122}$) we perform elementary operations so as to get the Hessian transformed into

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & 1 & 0 \\ x_{12} & b_{22}+x_{22} & b_{23}+x_{23} & 0 & 0 \\ x_{13} & b_{23}+x_{23} & b_{33}+x_{33} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, again, we obtain that $|b_{ij}+x_{ij}|=1$ and arrive, all the same, to a contradiction.

c$_2$) If we assume, second, that the vectors X_2, X_3 and X_4 are linearly independent, we find that this case is quite similar to the previous one and, therefore, omit its proof.

c$_2$) Finally, let us assume that the vectors X_3, X_4 and X_5 are linearly independent, write the other ones as $X_i = a_{i3}X_3 + a_{i4}X_4 + a_{i5}X_5$, $i = 1, 2$, and perform into the Hessian matrix defined by equation (3.1) the elementary operations $R_i - a_{i3}R_3 - a_{i4}R_4 - a_{i5}R_5$ and $C_i - a_{i3}C_3 - a_{i4}C_4 - a_{i5}C_5$, $i = 1, 2$, to obtain

$$\begin{bmatrix} b_{11} & b_{12} & -a_{13} & -a_{14} & -a_{15} \\ b_{12} & b_{22} & -a_{23} & -a_{24} & -a_{25} \\ -a_{13} & -a_{23} & 1+x_{33} & x_{34} & x_{35} \\ -a_{14} & -a_{24} & x_{34} & 1+x_{44} & x_{45} \\ -a_{15} & -a_{25} & x_{35} & x_{45} & 1+x_{55} \end{bmatrix},$$

where $b_{11} = a_{13}^2 + a_{14}^2 + a_{15}^2$, $b_{12} = 1 + a_{13}a_{23} + a_{14}a_{24} + a_{15}a_{25}$, $b_{22} = a_{23}^2 + a_{24}^2 + a_{25}^2$. Then, we must have the determinant $|b_{ij}| = 0$, but at the same time the 2×2 submatrix $(b_{ij}) \neq 0$, the vectors $(a_{13}, a_{14}, a_{15}) \neq 0$, $(a_{23}, a_{24}, a_{25}) \neq 0$; and we may assume that $(b_{12}, b_{22}) = c\,(b_{11}, b_{12})$, for some $c \in \mathbb{R}$. Thus, by performing the operations $R_2 - cR_1$, $C_2 - cC_1$, if necessary, we may further assume that $b_{12} = b_{22} = 0$, $b_{11} \neq 0$. Next, it is easy to see that we can perform operations to transform the Hessian matrix into

$$\begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & a'_{23} & a'_{24} & a'_{25} \\ 0 & a'_{23} & 1+x_{33} & x_{34} & x_{35} \\ 0 & a'_{24} & x_{34} & 1+x_{44} & x_{45} \\ 0 & a'_{25} & x_{35} & x_{45} & 1+x_{55} \end{bmatrix},$$

where we must have the vector $(a'_{23}, a'_{24}, a'_{25}) \neq 0$. Thus, further elementary operations allow to write the Hessian

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & x_{33} & x_{34} & x_{35} \\ 0 & 0 & x_{34} & 1 & 0 \\ 0 & 0 & x_{35} & 0 & 1 \end{bmatrix}.$$

However, here we would have $r < 3$, so that this subcase is also not possible, as were the previous ones. As a consequence, the whole case c) cannot happen.

d) The fourth case corresponds to the values $k = 1$, $r = 4$; and it is easy to see that this reduces to two subcases: d$_1$) X_2, X_3, X_4 and X_5 are linearly independent; d$_2$) The linearly independent vectors are X_1, X_2, X_3 and X_4.

The proof of both of these follow a pattern similar to the previous case and to case c) in Theorem C. Thus, we omit it.

e) If we consider now the case $k = 2$, $r = 1$, it is easy to see that this also reduces to two subcases: e$_1$) $x_{11} \neq 0$, on an open, dense subset of the domain; e$_2$) $x_{55} \neq 0$.

Here again, the proof is quite similar to that in Theorem C, case d), and shall also be omitted.

f) The sixth possible case corresponds to $k = 2$, $r = 2$; which can be reduced to three subcases: f$_1$), f$_2$), f$_3$):

f$_1$) Let us assume, first, that the vectors X_1 and X_3 are linearly independent, represent the remaining ones by $X_i = a_{i1}X_1 + a_{i3}X_3$, $i = 2, 4, 5$; and perform into the Hessian matrix for this case

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & x_{14} & x_{15} \\ 1+x_{12} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{13} & x_{23} & x_{33} & 1+x_{34} & x_{35} \\ x_{14} & x_{24} & 1+x_{34} & x_{44} & x_{45} \\ x_{15} & x_{25} & x_{35} & x_{45} & 1+x_{55} \end{bmatrix} \tag{3.2}$$

the elementary operations $R_i - a_{i1}R_1 - a_{i3}R_3$ and $C_i - a_{i1}C_1 - a_{i3}C_3$, $i = 2, 4, 5$, to obtain

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 & 0 \\ 1 & b_{22} & 0 & b_{24} & b_{25} \\ x_{13} & 0 & x_{33} & 1 & 0 \\ 0 & b_{24} & 1 & b_{44} & b_{45} \\ 0 & b_{25} & 0 & b_{45} & 1 \end{bmatrix},$$

where we must have that the 3×3 submatrix formed with rows and columns 2, 4 and 5 is singular, i.e., with vanishing determinant. Then, there are two subcases that we label f$_{11}$) and f$_{12}$):

f$_{11}$) Suppose that $(b_{24}, b_{44}, b_{45}) = c\,(b_{22}, b_{24}, b_{25}) + d\,(b_{25}, b_{45}, 1)$, for some $c, d \in \mathbb{R}$. Then, by making suitable elementary operations we can transform the above so as to have the vector $(b_{24}, b_{44}, b_{45}) = 0$, with the rest of entries the same as before. Next, by computing the determinant we obtain $\Delta = 1 + (b_{25}^2 - b_{22})\,x_{11}$, so that we have two possibilities: f$_{111}$) $b_{25}^2 - b_{22} = 0$, or f$_{112}$) $x_{11} = 0$.

f$_{111}$) We have $b_{25}^2 - b_{22} = 0$, and perform into the Hessian matrix the elementary operations $R_2 - b_{25}R_5$, $C_2 - b_{25}C_5$, to obtain

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

with the condition $x_{11}x_{33} - x_{13}^2 \neq 0$. Then, straightforward integration allows to write the solution as

$$t_6 = t_1t_2 + \tfrac{1}{6}a_1t_1^3 + \tfrac{1}{2}a_3t_1^2t_3 + \tfrac{1}{2}b_3t_1t_3^2 + t_3t_4 + \tfrac{1}{6}d_3t_3^3 + \tfrac{1}{2}t_5^2.$$

f$_{112}$) If we assume now that $x_{11} = 0$, we may perform into the Hessian matrix the elementary operations $R_2 - b_{25}R_5$, $C_2 - b_{25}C_5$, to obtain

$$\begin{bmatrix} 0 & 1 & x_{13} & 0 & 0 \\ 1 & 2b_2 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

with $2b_2 = b_{22} - b_{25}^2$. Finally, it is easy to see that the Hessian matrix can be reduced to

$$\begin{bmatrix} 0 & 1 & x_{13} & 0 & 0 \\ 1 & 0 & -b_2x_{13} & 0 & 0 \\ x_{13} & -b_2x_{13} & x_{33} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The latter can be integrated straightforwardly to obtain the solution

$$t_6 = t_1t_2 + \tfrac{1}{2}a_3t_1t_3^2 - \tfrac{1}{2}b_2a_3t_2t_3^2 + t_3t_4 + \tfrac{1}{2}t_5^2.$$

f_{12}) If we assume now that $(b_{22}, b_{24}, b_{25}) = c\,(b_{24}, b_{44}, b_{45}) + d\,(b_{25}, b_{45}, 1)$, for some $c, d \in \mathbb{R}$, it can be shown, by a quite similar procedure, that this subcase produces the same kind of solutions as before.

f_2) We assume, next, that the vectors X_1 and X_2 are the linearly independent ones, write $X_i = a_{i1}X_1 + a_{i2}X_2$, $i = 3, 4, 5$, and perform the operations $R_i - a_{i1}R_1 - a_{i2}R_2$ and $C_i - a_{i1}C_1 - a_{i2}C_2$, $i = 3, 4, 5$, to obtain

$$\begin{bmatrix} x_{11} & 1 + x_{12} & -a_{32} & -a_{42} & -a_{52} \\ 1 + x_{12} & x_{22} & -a_{31} & -a_{41} & -a_{51} \\ -a_{32} & -a_{31} & b_{33} & b_{34} & b_{35} \\ -a_{42} & -a_{41} & b_{34} & b_{44} & b_{45} \\ -a_{52} & -a_{51} & b_{35} & b_{45} & b_{55} \end{bmatrix},$$

where we must have, on the indicated submatrices, the conditions: $(a_{ij}) \neq 0, (b_{ij}) \neq 0, |b_{ij}| = 0$. Then, we may diagonalize the submatrix (b_{ij}), and execute further elementary operations so as to transform the Hessian into three possible forms labeled f_{21}), f_{22}), f_{23}). The careful analysis of these subcases gives rise to solutions which are the same as the ones quoted before, with some of the constants vanishing.

f_3) Let us assume, finally, that the vectors X_4 and X_5 are linearly independent. Then again, this subcase repeats the same solutions as those in f_2).

g) The seventh possible case corresponds to the values $k = 2$, $r = 3$, and again we can initially reduce this to three subcases: g_1), g_2), g_3).

g_1) We assume, first, that the vectors X_1, X_2 and X_3 are linearly independent, represent $X_i = a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3$, $i = 4, 5$, and perform into the Hessian matrix, represented by expression (3.2), the operations $R_i - a_{i1}R_1 - a_{i2}R_2 - a_{i3}R_3$, and $C_i - a_{i1}C_1 - a_{i2}C_2 - a_{i3}C_3$, $i = 4, 5$, to get

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & -a_{42} & -a_{52} \\ 1+x_{12} & x_{22} & x_{23} & -a_{41} & -a_{51} \\ x_{13} & x_{23} & x_{33} & 1 & 0 \\ -a_{42} & -a_{41} & 1 & b_{44} & b_{45} \\ -a_{52} & -a_{51} & 0 & b_{45} & b_{55} \end{bmatrix}.$$

Then, it is easy to determine that we must have the 2×2 submatrix $(b_{ij}) \neq 0$, $|b_{ij}| = 0$. Hence, by means of suitable operations we can transform the latter expression into

$$\begin{bmatrix} a_{11}+x_{11} & a_{12}+x_{12} & x_{13} & 0 & 0 \\ a_{12}+x_{12} & a_{22}+x_{22} & x_{23} & 0 & 0 \\ x_{13} & x_{23} & x_{33} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next, we proceed to diagonalize the 2×2 submatrix (a_{ij}) which, by the conditions of the problem, can be made in only one possible way, so that the above Hessian takes the form

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & 0 & 0 \\ 1+x_{12} & x_{22} & x_{23} & 0 & 0 \\ x_{13} & x_{23} & x_{33} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

but then, by evaluating the determinant, we come to the conclusion that we must have $x_{12} = 0$, $x_{11}x_{22} = 0$, and we may assume that $x_{22} = 0$. Hence, we integrate to obtain the solution

$$t_6 = t_1t_2 + \tfrac{1}{2}at_2t_3^2 + \tfrac{1}{6}bt_1^3 + \tfrac{1}{2}ct_1^2t_3 + \tfrac{1}{2}dt_1t_3^2 + t_3t_4 + \tfrac{1}{6}et_3^3 + \tfrac{1}{2}t_5^2.$$

Moreover, by the conditions of the problem we can perform further operations in order to make $e = 0$ in the above. Thus, we obtain the solution quoted as g_1) in the statement of the theorem.

g_2) We assume, second, that the vectors X_1, X_3 and X_5 are linearly independent, represent $X_i = a_{i1}X_1 + a_{i3}X_3 + a_{i5}X_5$, $i = 2, 4$, and perform into the Hessian the operations $R_i - a_{i1}R_1 - a_{i3}R_3 - a_{i5}R_5$, and $C_i - a_{i1}C_1 - a_{i3}C_3 - a_{i5}C_5$, $i = 2, 4$, to obtain

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 & x_{15} \\ 1 & b_{22} & 0 & b_{24} & -a_{25} \\ x_{13} & 0 & x_{33} & 1 & x_{35} \\ 0 & b_{24} & 1 & b_{44} & -a_{45} \\ x_{15} & -a_{25} & x_{35} & -a_{45} & 1+x_{55} \end{bmatrix},$$

where we must have for the 2×2 submatrix (b_{ij}) the condition $|b_{ij}| = 0$, and from this we obtain two subcases: g_{21}) $(b_{ij}) = 0$, g_{22}) $(b_{ij}) \neq 0$, that we analyze next:

g_{21}) Here, further elementary operations allow to transform the Hessian matrix into

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 & x_{15} \\ 1 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 & x_{35} \\ 0 & 0 & 1 & 0 & 0 \\ x_{15} & 0 & x_{35} & 0 & 1+x_{55} \end{bmatrix},$$

and, since the determinant of the latter equals $\triangle = 1+x_{55}$, it follows that $x_{55}=0$. Thus, by integrating, we obtain the solution quoted as g_2) in the statement of the theorem, i.e.,

$$t_6 = t_1t_2 + \tfrac{1}{6}a_1t_1^3 + \tfrac{1}{2}a_3t_1^2t_3 + \tfrac{1}{2}a_5t_1^2t_5 + c_5t_1t_3t_5 + t_3t_4 + \tfrac{1}{6}f_3t_3^3 + \tfrac{1}{2}f_5t_3^2t_5 + \tfrac{1}{2}t_5^2.$$

g_{22}) We may assume $(b_{22}, b_{24}) = c\,(b_{24}, b_{44})$, $c \in \mathbb{R}$, and transform the Hessian matrix into

$$\begin{bmatrix} x_{11} & 1 & x_{13} & 0 & x_{15} \\ 1 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 & x_{35} \\ 0 & 0 & 1 & b_{44} & 0 \\ x_{15} & 0 & x_{35} & 0 & 1+x_{55} \end{bmatrix}.$$

By evaluating the determinant of the latter we obtain $\triangle = 1+x_{55}-b_{44}x_{33}-b_{44}\left(x_{33}x_{55}-x_{35}^2\right)$, from which we get the conditions $x_{55} = b_{44}x_{33}$, $x_{33}x_{55}-x_{35}^2 = 0$. Now, if $b_{44} < 0$, we arrive at a contradiction since in such a case we would also have $x_{33} = x_{35} = x_{55} = 0$, so that $r < 3$. Hence, we must have $b_{44} > 0$, and it follows that $x_{35} = \sqrt{b_{44}}x_{33}$. Thus, by a couple of elementary operations we reduce the Hessian matrix to

$$\begin{bmatrix} x_{11} & 1 & x_{13} & x_{14} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 & 0 \\ x_{14} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The latter expression can be integrated immediately to give the solution

$$t_6 = t_1t_2 + \tfrac{1}{6}a_1t_1^3 + \tfrac{1}{2}a_3t_1^2t_3 + \tfrac{1}{2}a_4t_1^2t_4 + \tfrac{1}{2}c_3t_1t_3^2 + t_3t_4 + \tfrac{1}{6}f_3t_3^3 + \tfrac{1}{2}t_5^2.$$

Then, by the conditions of the problem, we can make further operations in order to have $a_1 = 0$ in the above. Therefore, this gives the solution labeled as g_3) in the statement of the theorem.

g_3) If we assume, finally, that the vectors X_3, X_4 and X_5 are linearly independent and repeat the procedure we arrive at the same kinds of solutions as before. So we omit this part of the argument.

h) The eighth and last possible case corresponds to the values $k=2$, $r=4$. We can initially reduce this to two subcases: h_1), h_2).

h_1) We assume, first, that the vectors X_1, X_2, X_3 and X_4 are linearly independent, write $X_5 = a_{51}X_1+a_{52}X_2+a_{53}X_3+a_{54}X_4$, and perform into the Hessian, represented by expression (3.2), suitable operations to obtain

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & x_{14} & -a_{52} \\ 1+x_{12} & x_{22} & x_{23} & x_{24} & -a_{51} \\ x_{13} & x_{23} & x_{33} & 1+x_{34} & -a_{54} \\ x_{14} & x_{24} & 1+x_{34} & x_{44} & -a_{53} \\ -a_{52} & -a_{51} & -a_{54} & -a_{53} & b \end{bmatrix},$$

where $b = 1 + 2\,(a_{51}a_{52} + a_{53}a_{54}) = 0$, so that we must have $a_{51}a_{52} \neq 0$ or $a_{53}a_{54} \neq 0$, and we can assume $a_{52} \neq 0$. Hence, by performing further operations we can express the Hessian matrix as

$$\begin{bmatrix} x_{11} & 1+x_{12} & x_{13} & x_{14} & -a_{52} \\ 1+x_{12} & b_{22}+x_{22} & b_{23}+x_{23} & b_{24}+x_{24} & 0 \\ x_{13} & b_{23}+x_{23} & x_{33} & 1+x_{34} & 0 \\ x_{14} & b_{24}+x_{24} & 1+x_{34} & x_{44} & 0 \\ -a_{52} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, by evaluating the determinant, we conclude that we may write $a_{52} = -1$, and that the 3×3 submatrix formed by rows and columns 2, 3, 4 has determinant equal to -1. Thus, by making suitable operations we may bring the above Hessian to

$$\begin{bmatrix} x_{11} & 1 & x_{13} & x_{14} & x_{15} \\ 1 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 & x_{35} \\ x_{14} & 0 & 1 & 0 & 0 \\ x_{15} & 0 & x_{35} & 0 & 1 \end{bmatrix},$$

which can be integrated to obtain the solution

$$t_6 = t_1t_2 + \tfrac{1}{2}c_1t_1^2t_4 + e_1t_1t_3t_5 + t_3t_4 + \tfrac{1}{2}e_4t_3^2t_5 + \tfrac{1}{2}f_1t_1^2t_5 + \tfrac{1}{2}g_1t_1t_3^2 + \tfrac{1}{6}g_4t_3^3 + \tfrac{1}{2}h_1t_1^2t_3 + \tfrac{1}{6}k_1t_1^3 + \tfrac{1}{2}t_5^2.$$

Finally, from the conditions of the problem we can make a couple of further operations in order to get $f_1 = k_1 = 0$ in the above equation, so to obtain the result quoted as h) in the statement of the theorem.

h$_2$) We assume, next, that the vectors X_2, X_3, X_4 and X_5 are linearly independent, express $X_1 = a_{12}X_2 + a_{13}X_3 + a_{14}X_4 + a_{15}X_5$, and perform into the Hessian appropriate operations to get

$$\begin{bmatrix} b_{11} & 1 & -a_{14} & -a_{13} & -a_{15} \\ 1 & x_{22} & x_{23} & x_{24} & x_{25} \\ -a_{14} & x_{23} & x_{33} & 1+x_{34} & x_{35} \\ -a_{13} & x_{24} & 1+x_{34} & x_{44} & x_{45} \\ -a_{15} & x_{25} & x_{35} & x_{45} & 1+x_{55} \end{bmatrix},$$

where we must have $b_{11} = 0$. Then, by performing further, easily determined elementary operations we can get the Hessian matrix expressed by

$$\begin{bmatrix} x_{11} & 1 & x_{13} & x_{14} & x_{15} \\ 1 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1+x_{34} & x_{35} \\ x_{14} & 0 & 1+x_{34} & x_{44} & x_{45} \\ x_{15} & 0 & x_{35} & x_{45} & 1+x_{55} \end{bmatrix}.$$

Here, by evaluating the determinant we obtain the condition that the 3×3 principal submatrix formed by rows and columns 3, 4, 5 has determinant equal to -1. Thus, by the classifying procedure for dimension $n = 3$, we can reduce the above Hessian matrix to

$$\begin{bmatrix} x_{11} & 1 & x_{13} & x_{14} & x_{15} \\ 1 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & x_{33} & 1 & x_{35} \\ x_{14} & 0 & 1 & 0 & 0 \\ x_{15} & 0 & x_{35} & 0 & 1 \end{bmatrix},$$

which is the same as the previous case h_1). The proof is concluded.

We remark finally that, since some of the solutions listed in the main theorem contain constant parameters, i.e., cases f), g) and h), and taking into consideration the conditions to be fulfilled in each case, one could perform further reductions in every one of them. However, we have preferred to leave the solutions expressed as stated, because they represent the most general form.

Aside from this fact, and the further possibility of performing rescalings, the solutions obtained are inequivalent for different values of the parameters, i.e., they do belong to different classes under the action of the unimodular affine group $ASL\,(n+1,\mathbb{R})$.

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