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An Integral Geometric Theorem for Simple Valuations

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Abstract. We prove a translative mean value formula for simple valuations, taken at the intersection of a fixed and a translated convex body.

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1. Introduction

A real valued function on the space $\mathcal{K}^n$ of convex bodies (nonempty, compact, convex subsets) in n-dimensional Euclidean space $\mathbb{R}^n$ is called a *valuation* if it satisfies

$$\varphi(K \cup M) + \varphi(K \cap M) = \varphi(K) + \varphi(M)$$

whenever $K, M, K \cup M \in \mathcal{K}^n$. One extends the definition of φ by $\varphi(\emptyset) := 0$. For continuous valuations (where continuity refers to the Hausdorff metric on $\mathcal{K}^n$), Hadwiger has proved an integral geometric mean value formula for intersections of a fixed and a moving convex body. To formulate it, we recall that the intrinsic volumes $V_0, \dots, V_n$ are defined by the Steiner formula

$$V_n(K + \epsilon B^n) = \sum_{k=0}^{n} \epsilon^{n-k} \kappa_{n-k} V_k(K), \quad K \in \mathcal{K}^n,\ \epsilon \geq 0,$$

where V_n denotes the volume, B^n is the Euclidean unit ball, and κ_k is the volume of the k-dimensional unit ball. Each function V_k is a motion invariant continuous valuation. Hadwiger's characterization theorem says that every function $\varphi : \mathcal{K}^n \to \mathbb{R}$ with these properties is a linear combination of $V_0, \dots, V_n$ with constant coefficients. (A survey on valuations is given by McMullen [5].)

We denote by G_n the group of rigid motions of $\mathbb{R}^n$ and by μ its Haar measure, normalized so that $\mu(\{g \in G_n : gx \in B^n\}) = \kappa_n$ (for arbitrary $x \in \mathbb{R}^n$). The space $\mathcal{E}_q^n$ of q-dimensional planes in $\mathbb{R}^n$ is endowed with its motion invariant measure μ_q, normalized so that $\mu_q(\{E \in \mathcal{E}_q^n : E \cap B^n \neq \emptyset\}) = \kappa_{n-q}$.

Theorem 1. (Hadwiger) *Let $\varphi : \mathcal{K}^n \to \mathbb{R}$ be a continuous valuation. Then*

$$\int_{G_n} \varphi(K \cap gM)\, d\mu(g) = \sum_{k=0}^{n} \varphi_k(K) V_{n-k}(M)$$

for $K, M \in \mathcal{K}^n$, where the coefficients $\varphi_k(K)$ are given by

$$\varphi_k(K) = \int_{\mathcal{E}_{n-k}^n} \varphi(K \cap E)\, d\mu_{n-k}(E).$$

Hadwiger's proof is found in his book [2], p. 241, though with different notation and normalization. It is based on the characterization theorem quoted above. The latter is a convenient tool for proving classical integral geometric formulae for convex bodies, like the principal kinematic formula and the Crofton formula (see also Klain and Rota [4] for this approach). Whereas these results can be proved in several different ways, the only known proof for their abstract generalization in Theorem 1 is via Hadwiger's characterization theorem. It is interesting to note that recently Alesker [1] has proved new integral geometric formulae for real submanifolds in Hermitian spaces, generalizing classical formulae in Euclidean spaces, and that the proof is based on a characterization of unitarily invariant translation invariant continuous valuations.

Our aim in the present paper is in a similar spirit, but much more modest. Motivated by the increased interest that the integral geometry of the translation group has seen in recent years, we want to obtain a translative analogue of Hadwiger's general integral geometric theorem. We can do this for simple valuations, since a corresponding characterization theorem is known (a valuation on $\mathcal{K}^n$ is called simple if it vanishes on bodies of dimension less than n). The result is given by Theorem 2 below. It was already mentioned without proof in the survey article [3].

2. The result

For a convex body $K \in \mathcal{K}^n$, we denote by $h(K, \cdot)$ its support function and by $S_{n-1}(K, \cdot)$ its area measure (see [6] for notions from the theory of convex bodies that are not explained here). The area measure is a finite measure on the unit sphere $S^{n-1} := \{u \in \mathbb{R}^n : \langle u, u \rangle = 1\}$. Here $\langle \cdot, \cdot \rangle$ is the scalar product of $\mathbb{R}^n$. The set $H^-(u, t) := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq t\}$, where $u \in S^{n-1}$ and $t \in \mathbb{R}$, is a closed halfspace. By λ we denote Lebesgue measure on $\mathbb{R}^n$.

Theorem 2. *Let $\varphi : \mathcal{K}^n \to \mathbb{R}$ be a continuous simple valuation. Then*

$$\int_{\mathbb{R}^n} \varphi(K \cap (M+x))\, d\lambda(x) = \varphi(K)V_n(M) + \int_{S^{n-1}} f_{K,\varphi}(u) S_{n-1}(M, du)$$

for $K, M \in \mathcal{K}^n$, where the odd function $f_{K,\varphi} : S^{n-1} \to \mathbb{R}$ is given by

$$f_{K,\varphi}(u) = \int_{-h(K,-u)}^{h(K,u)} \varphi(K \cap H^-(u,t))\, dt - \varphi(K)h(K,u).$$

Proof. We fix a convex body $K \in \mathcal{K}^n$. For $M \in \mathcal{K}^n$, we define

$$\psi(M) := \int_{\mathbb{R}^n} \varphi(K \cap (M+x))\, d\lambda(x)$$

and

$$\overline{\varphi}(M) := \varphi(K)V_n(M) - V_n(K)\varphi(M), \tag{1}$$

so that $\overline{\varphi}(K) = 0$, further

$$\overline{\psi}(M) := \int_{\mathbb{R}^n} \overline{\varphi}(K \cap (M+x))\, d\lambda(x).$$

Then

$$\overline{\psi}(M) = V_n(K)[\varphi(K)V_n(M) - \psi(M)], \tag{2}$$

since

$$\int_{\mathbb{R}^n} V_n(K \cap (M+x))\, d\lambda(x) = V_n(K)V_n(M),$$

as follows from Fubini's theorem. The function $\overline{\psi} : \mathcal{K}^n \to \mathbb{R}$ is a continuous simple valuation and is, moreover, translation invariant. Hence, by the characterization theorem of [7], it is of the form

$$\overline{\psi}(M) = cV_n(M) + \int_{S^{n-1}} f(u) S_{n-1}(M, du) \tag{3}$$

for $M \in \mathcal{K}^n$, with a constant c and an odd continuous function $f : S^{n-1} \to \mathbb{R}$, both depending on K.

Let $P \in \mathcal{K}^n$ be a polytope, and let $r > 0$. By $T_n(r)$ we denote the set of all translation vectors $x \in \mathbb{R}^n$ for which $K \subset rP + x$, and by $T_{n-1}(r)$ the set of all $x \in \mathbb{R}^n$ for which K meets an $(n-1)$-face of $rP + x$, but no $(n-2)$-face. Finally, $T_{n-2}(r)$ is the set of all $x \in \mathbb{R}^n$ for which K meets an $(n-2)$-face of $rP + x$. Then

$$\begin{aligned} \overline{\psi}(rP) &= \sum_{k=0}^{2} \int_{T_{n-k}(r)} \overline{\varphi}(K \cap (rP + x))\, d\lambda(x) \\ &= \overline{\varphi}(K) r^n V_n(P) + O(r^{n-1}) \end{aligned} \tag{4}$$

as $r \to \infty$. From (3) we get

$$\overline{\psi}(rP) = cr^n V_n(P) + r^{n-1} \int_{S^{n-1}} f(u) S_{n-1}(P, du).$$

Letting $r \to \infty$, we conclude that $c = \overline{\varphi}(K) = 0$. Thus

$$\overline{\psi}(rP) = \int_{T_{n-1}(r)} \overline{\varphi}(K \cap (rP + x))\, d\lambda(x) + O(r^{n-2})$$

by (4) and

$$\overline{\psi}(rP) = r^{n-1} \int_{S^{n-1}} f(u) S_{n-1}(P, du) \tag{5}$$

by (3).

Let $F_1, \ldots, F_m$ be the facets of P, and let u_i be the outer unit normal vector of P at F_i. For $i \in \{1, \ldots, m\}$, let $T_{(i)}(r)$ be the set of all $x \in \mathbb{R}^n$ for which K meets $rF_i + x$, but no other face of $rP + x$. Then

$$\begin{aligned} &\int_{T_{(i)}(r)} \overline{\varphi}(K \cap (rP + x))\, d\lambda(x) \\ &= r^{n-1} V_{n-1}(F_i) \int_{-h(K,-u_i)}^{h(K,u_i)} \overline{\varphi}(K \cap H^-(u_i, t))\, dt + O(r^{n-2}). \end{aligned}$$

Putting

$$g(u) := \int_{-h(K,-u)}^{h(k,u)} \overline{\varphi}(K \cap H^-(u, t))\, dt, \tag{6}$$

we get

$$\begin{aligned} \overline{\psi}(rP) &= r^{n-1} \sum_{i=1}^{m} g(u_i) V_{n-1}(F_i) + O(r^{n-2}) \\ &= r^{n-1} \int_{S^{n-1}} g(u) S_{n-1}(P, du) + O(r^{n-2}). \end{aligned}$$

Together with (5), this shows that

$$\int_{S^{n-1}} f(u) S_{n-1}(P, du) = \int_{S^{n-1}} g(u) S_{n-1}(P, du)$$

for all convex polytopes P. By approximation, this extends to arbitrary convex bodies, due to the weak continuity of the area measures and the fact that the functions f and g are continuous. Thus,

$$\overline{\psi}(M) = \int_{S^{n-1}} g(u) S_{n-1}(M, du)$$

for $M \in \mathcal{K}^n$. By (6) and (1),

$$g(u) = \int_{-h(K,-u)}^{h(K,u)} \left[\varphi(K) V_n(K \cap H^-(u, t)) - V_n(K) \varphi(K \cap H^-(u, t))\right] dt,$$

and here the first term can be simplified by

$$\begin{aligned}
&\int_{-h(K,-u)}^{h(K,u)} V_n(K \cap H^-(u,t))\, dt \\
&= \int_{-h(K,-u)}^{h(K,u)} \int_K \mathbf{1}\{\langle x,u\rangle \le t\}\, d\lambda(x)\, dt \\
&= \int_K \int_{-h(K,u)}^{h(K,u)} \mathbf{1}\{t \ge \langle x,u\rangle\}\, dt\, d\lambda(x) \\
&= \int_K [h(K,u) - \langle x,u\rangle]\, d\lambda(x) \\
&= h(K,u)V_n(K) - \langle z_{n+1}(K),u\rangle,
\end{aligned}$$

where $z_{n+1}(K)$ denotes the moment vector of K. This yields

$$\begin{aligned}
\overline{\psi}(M) &= \varphi(K) \int_{S^{n-1}} [h(K,u)V_n(K) - \langle z_{n+1}(K),u\rangle]\, S_{n-1}(M,du) \\
&\quad -V_n(K) \int_{S^{n-1}} g_{K,\varphi}(u)\, S_{n-1}(M,du)
\end{aligned}$$

with

$$g_{K,\varphi}(u) := \int_{-h(K,-u)}^{h(K,u)} \varphi(K \cap H^-(u,t))\, dt.$$

Observing that

$$\int_{S^{n-1}} u\, S_{n-1}(M,du) = 0,$$

we now get from (2) the asserted representation.

That $f_{K,\varphi}$ is an odd function, follows from the fact that φ is a simple valuation: this gives

$$\varphi(K \cap H^-(-u,t)) = -\varphi(K \cap H^-(u,-t)) + \varphi(K)$$

and hence, for $u \in S^{n-1}$,

$$\begin{aligned}
&f_{K,\varphi}(u) + f_{K,\varphi}(-u) \\
&= \int_{-h(K,-u)}^{h(K,u)} \varphi(K \cap H^-(u,t))\, dt - \varphi(K)h(K,u) \\
&\quad + \int_{-h(K,u)}^{h(K,-u)} [-\varphi(K \cap H^-(u,-t)) + \varphi(K)]\, dt - \varphi(K)h(K,-u) \\
&= 0.
\end{aligned}$$

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