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Adjacency Preserving Mappings of Rectangular Matrices

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Abstract. Let D be a division ring and let m, n be integers ≥ 2. Let $M_{m\times n}(D)$ be the space of $m \times n$ matrices. In the fundamental theorem of the geometry of rectangular matrices all bijective mappings φ of $M_{m\times n}(D)$ are determined such that both φ and φ^{-1} preserve adjacency. We show that if a bijective map φ of $M_{m\times n}(D)$ preserves the adjacency then also φ^{-1} preserves the adjacency. Thus the supposition that φ^{-1} preserves adjacency may be omitted in the fundamental theorem.

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1. Introduction

L.K. Hua initiated the study of the geometry of matrices in the mid fourties of the last century (cf. [3]). In this geometry, the points are a certain kind of matrices of a given size. The four kinds of matrices studied by Hua are rectangular matrices, symmetric matrices, skew-symmetric matrices and hermitian matrices. To each such space there is associated a group of motions. It is the aim to characterize the group of motions in the space by as few

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geometric invariants as possible. Two rectangular, symmetric, or hermitian matrices A, B are called *adjacent*, if $A-B$ has rank 1. Two skew-symmetric matrices A, B are called adjacent if $A-B$ has rank 2. Hua discovered that the invariant "adjacency" is sufficient to characterize the group of motions. He and his followers determined all bijections φ of the set of points which satisfy

$$A, B \text{ are adjacent} \qquad \Longleftrightarrow \qquad A^\varphi, B^\varphi \text{ are adjacent.} \tag{1.1}$$

This result is known as the *fundamental theorem of the geometry of matrices.*

In view of the fundamental theorems of affine and projective geometry, where all bijections of affine resp. projective spaces are determined which take lines to lines, it is a natural and important question posed in [4], whether in the geometries of matrices it is possible to replace the condition (1.1) by

$$A, B \text{ are adjacent} \qquad \Longrightarrow \qquad A^\varphi, B^\varphi \text{ are adjacent.} \tag{1.2}$$

It is shown in [2] that this is possible in the case of symmetric and hermitian matrices. In the present paper we answer this question for the space $M_{m\times n}(D)$ of $m \times n$ rectangular matrices over a division ring D.

Theorem 1.1. *Let D be a division ring. Let m, n be integers ≥ 2. If a bijective map φ from $M_{m\times n}(D)$ to itself preserves the adjacency in $M_{m\times n}(D)$ then also φ^{-1} preserves the adjacency.*

According to the fundamental theorem of the geometry of rectangular matrices [1, 5], when $m \neq n$, then φ is of the form

$$X^\varphi = PX^\sigma Q + R \qquad \text{for all } X \in M_{m\times n}(D), \tag{1.3}$$

where $P \in \mathrm{GL}_m(D)$, $Q \in \mathrm{GL}_n(D)$, $R \in M_{m\times n}(D)$, and σ is an automorphism of D. When $m = n$, then in addition to (1.3), φ might also be a mapping of the form

$$X^\varphi = P\,{}^t(X^\tau)Q + R \qquad \text{for all } X \in M_{m\times n}(D), \tag{1.4}$$

where τ is an anti-automorphism of D.

The space $M_{m\times n}(D)$ can be treated as a graph. We call the points of $M_{m\times n}(D)$ *vertices* and define two vertices A, B to be *adjacent* if $\mathrm{rank}(A-B) = 1$. Then we obtain the *graph of $m\times n$ matrices* over D, denoted by $\Gamma(M_{m\times n}(D))$. If D is infinite, then $\Gamma(M_{m\times n}(D))$ is an infinite graph. For finite graphs, a bijection which satisfies (1.2) is an automorphism. But there are counterexamples in the infinite case. For the graph $\Gamma(M_{m\times n}(D))$, the above theorem can be interpreted as follows.

Theorem 1.2. *Let D be a division ring. Let m, n be integers ≥ 2 and $\Gamma(M_{m\times n}(D))$ be the graph of $m \times n$ rectangular matrices over D. If φ is a bijective map from $\Gamma(M_{m\times n}(D))$ to itself for which*

$$A, B \text{ are adjacent} \qquad \Longrightarrow \qquad A^\varphi, B^\varphi \text{ are adjacent,}$$

is satisfied for any two vertices A, B of $\Gamma(M_{m\times n}(D))$ then φ is a graph automorphism of $\Gamma(M_{m\times n}(D))$.

2. Preliminaries

In this section we mention some definitions and propositions which are also contained in Wan's book [3]. We then define the notion of *covering radius* of a subset of $M_{m\times n}(D)$ and show some of its properties.

Definition 2.1. (Points, motions) *Let D be a division ring. Let m, n be integers ≥ 2. Denote by $M_{m\times n}(D)$ the space of $m\times n$ matrices over D. We call elements of $M_{m\times n}(D)$ the points of the space $M_{m\times n}(D)$. The group $G_{m\times n}(D)$ of motions of $M_{m\times n}(D)$ consists of transformations of the form*

$$X \mapsto PXQ + R \qquad \text{for all } X \in M_{m\times n}(D),$$

where $P \in \mathrm{GL}_m(D)$, $Q \in \mathrm{GL}_n(D)$, $R \in M_{m\times n}(D)$.

Proposition 2.1. ([3], Proposition 3.1) *The group $G_{m\times n}(D)$ acts transitively on $M_{m\times n}(D)$.*

Definition 2.2. (Adjacency) *Two points $A, B \in M_{m\times n}(D)$ are said to be adjacent if* $\mathrm{rank}(A - B) = 1$.

Definition 2.3. (Maximal set) *A maximal set $\mathcal{M}$ in $M_{m\times n}(D)$ is a subset of $M_{m\times n}(D)$ with the property that any two points in $\mathcal{M}$ are adjacent and there is no point in $M_{m\times n}(D) \setminus \mathcal{M}$ which is adjacent to any point in $\mathcal{M}$.*

Proposition 2.2. (cf. [3], Proposition 3.9) *There are two types of maximal sets of adjacent matrices,*

$$\text{Type 1:} \qquad \left\{ P \begin{pmatrix} x_{11} & \dots & x_{1n} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} Q + R \;\middle|\; x_{11}, \dots, x_{1n} \in D \right\},$$

$$\text{Type 2:} \qquad \left\{ P \begin{pmatrix} y_{11} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ y_{m1} & 0 & \dots & 0 \end{pmatrix} Q + R \;\middle|\; y_{11}, \dots, y_{m1} \in D \right\},$$

where $P \in \mathrm{GL}_m(D)$, $Q \in \mathrm{GL}_n(D)$, and $R \in M_{m\times n}(D)$. Any maximal set belongs to only one type. A maximal set of type 1 cannot be carried to a maximal set of type 2 under the group of motions $G_{m\times n}(D)$.

Proposition 2.3. ([3], Corollary 3.10) *For any pair of adjacent points $A, B \in M_{m\times n}(D)$ there are exactly one maximal set of type 1 and exactly one maximal set of type 2 containing both A and B.*

Proposition 2.4. *Let M_1, M_2 be two distinct maximal sets with $M_1 \cap M_2 \neq \emptyset$. Then*

$$|M_1 \cap M_2| \quad \begin{cases} = 1 & \text{when } M_1 \text{ and } M_2 \text{ are of the same type,} \\ > 1 & \text{when } M_1 \text{ and } M_2 \text{ are not of the same type.} \end{cases}$$

In the second case we call $M_1 \cap M_2$ a line.

Proposition 2.5. ([3], Corollary 3.13) *The parametric equation of a line in $M_{m\times n}(D)$ is*

$$\{{}^t pxq + R \mid x \in D\},$$

where p is a nonzero m-dimensional row vector over D, q is a nonzero n-dimensional row vector over D, and $R \in M_{m\times n}(D)$.

Proposition 2.6. ([3], Corollary 3.11 and Proposition 3.14)] *Two maximal sets which have only one point in common can be carried simultaneously under the group $G_{m\times n}(D)$ to*

$$\left\{ \begin{pmatrix} x_{11} & \dots & x_{1n} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \;\middle|\; x_{11}, \dots, x_{1n} \in D \right\}, \tag{2.1}$$

$$\left\{ \begin{pmatrix} 0 & \dots & 0 \\ x_{21} & \dots & x_{2n} \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \;\middle|\; x_{21}, \dots, x_{2n} \in D \right\} \tag{2.2}$$

or to

$$\left\{ \begin{pmatrix} y_{11} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ y_{m1} & 0 & \dots & 0 \end{pmatrix} \;\middle|\; y_{11}, \dots, y_{m1} \in D \right\}, \tag{2.3}$$

$$\left\{ \begin{pmatrix} 0 & y_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & y_{m2} & 0 & \dots & 0 \end{pmatrix} \;\middle|\; y_{12}, \dots, y_{m2} \in D \right\}. \tag{2.4}$$

Two intersecting maximal sets of different type 1 *and* 2 *can be carried simultaneously under the group $G_{m\times n}(D)$ to* (2.1) *and* (2.3)*.*

Proposition 2.7. ([3], Proposition 3.20) *Any maximal set $\mathcal{M}$ of type* 1 *has the structure of an n-dimensional left affine space, and any maximal set of type* 2 *has the structure of an m-dimensional right affine space, where the points and lines are defined above.*

Definition 2.4. (Distance) *The distance $d(A, B)$ between two distinct points $A, B \in M_{m\times n}(D)$ is defined to be the smallest nonnegative integer k with the property that there exists a sequence of consecutively adjacent points $A = A_0, A_1, \dots, A_k = B$. When $A = B$, we define $d(A, B) = 0$.*

We have $d(A, B) = d(B, A)$ and $d(A, B) = 0$ if, and only if, $A = B$. Furthermore, the distance satisfies the triangle inequality

$$d(A, C) \le d(A, B) + d(B, C) \quad \text{for all } A, B, C \in M_{m\times n}(D),$$

so $(M_{m\times n}(D), d)$ is a metric space. It was proved in [3] that for any two points $A, B \in M_{m\times n}(D)$,

$$d(A, B) = \operatorname{rank}(A - B).$$

Hence $0 \le d(A, B) \le \min\{m, n\}$.

Lemma 2.1. *For any maximal sets M and M' of type 1 resp. type 2, we can find a positive integer k and a sequence $M = M_0, \dots, M_k = M'$ of maximal sets of type 1 resp. type 2 satisfying $M_i \cap M_{i+1} \neq \emptyset$, $i = 0, \dots, k-1$.*

Proof. Choose $X \in M$, $Y \in M'$, $X \neq Y$, $k - 1 := d(X, Y)$. Then there is a sequence of consecutively adjacent points $X = X_0, \dots, X_{k-1} = Y$. For $i = 1, \dots, k-1$ define M_i to be the maximal set of type 1 resp. type 2 which contains X_{i-1} and X_i. Let $M_0 := M$ and $M_k := M'$. Then we have $M_i \cap M_{i+1} \neq \emptyset$, $i = 0, \dots, k-1$. □

Definition 2.5. (Covering radius) *Let $A \in M_{m\times n}(D)$ and let $M \subset M_{m\times n}(D)$. The distance between A and M is $d(A, M) := \min\{d(A, B) \mid B \in M\}$. The covering radius of M is*

$$\rho(M) := \max\{d(A, M) \mid A \in M_{m\times n}(D)\}.$$

The covering radius of $M \subset M_{m\times n}(D)$ is the smallest positive integer ρ with the property that the union of all balls

$$\bigcup_{A\in M} \{X \in M_{m\times n}(D) \mid d(A, X) \leq \rho\}$$

covers $M_{m\times n}(D)$. Clearly, for any two subsets $M, M' \subset M_{m\times n}(D)$, if there is an element $\psi \in G_{m\times n}(D)$ such that $M^\psi = M'$ then $\rho(M) = \rho(M')$.

Lemma 2.2. *Let $M = \left\{ \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\middle|\; x \in D^n \right\}$ and $P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \in M_{m\times n}(D)$ with* rank $\begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix} =$ *k. Then for all $X = \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in M$ we have*

$$d(X, P) \in \{k, k+1\} \qquad \textit{and} \qquad d(X, P) = k \;\Leftrightarrow\; x \in p_1 + \langle p_2, \dots, p_m\rangle \subset D^n,$$

where $\langle p_2, \dots, p_m\rangle$ denotes the subspace of D^n which is spanned by $\{p_2, \dots, p_m\}$.

Proof. We have $d(X, P) = \operatorname{rank} \begin{pmatrix} x-p_1 \\ -p_2 \\ \vdots \\ -p_m \end{pmatrix} \geq \operatorname{rank} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix}$. Furthermore,

$$\begin{aligned} \operatorname{rank} \begin{pmatrix} x-p_1 \\ -p_2 \\ \vdots \\ -p_m \end{pmatrix} = \operatorname{rank} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix} \quad &\Leftrightarrow \quad x - p_1 \in \langle p_2, \dots, p_m\rangle \\ &\Leftrightarrow \quad x \in p_1 + \langle p_2, \dots, p_m\rangle\,. \end{aligned}$$

□

Corollary 2.1. a) *In the case $m \leq n$ let M be a maximal set of type 1. Let $P \in M_{m\times n}(D)$ with $d(P, M) = m - 1$. Then $\{X \in M \mid d(X, P) = m - 1\}$ is an affine $(m-1)$-flat of M, if we consider M as an n-dimensional affine space.*
b) *In the case $n \leq m$ let M be a maximal set of type 2. Let $P \in M_{m\times n}(D)$ with $d(P, M) = n - 1$. Then $\{X \in M \mid d(X, P) = n - 1\}$ is an affine $(n-1)$-flat of M, if we consider M as an m-dimensional affine space.*

Lemma 2.3. *Let M be a maximal set of $M_{m\times n}(D)$. If M is of type* 1*,*

$$\rho(M) = \begin{cases} m-1 & \text{when } m \le n, \\ n & \text{when } m > n. \end{cases}$$

If M is of type 2*,*

$$\rho(M) = \begin{cases} n-1 & \text{when } m \ge n, \\ m & \text{when } m < n. \end{cases}$$

Proof. We only prove the case that M is of type 1. The case that M is of type 2 can be proved similarly. Since the covering radius is invariant under the group $G_{m\times n}(D)$, we can assume without loss of generality that M is of the form (2.1). Let $A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in M_{m\times n}(D)$ be any point. In the case $m \le n$ from Lemma 2.2 we have $d(A, M) = \operatorname{rank}\begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix} \le m-1$. Thus we have $\rho(M) = m-1$. In the case $m > n$, for any point A we have $d(A, M) = \operatorname{rank}\begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix} \le n$ and $d(A, M) = n$ if $\operatorname{rank}\begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix} = n$. □

Lemma 2.4. a) *Let $m \le n$. Let M be a maximal set of type* 1*. Consider M as an n-dimensional left affine space, then for any hyperplane H of M we have $\rho(H) = m$.*
b) *Let $n \le m$. Let M be a maximal set of type* 2*. Consider M as an m-dimensional right affine space, then for any hyperplane H of M we have $\rho(H) = n$.*

Proof. We prove a). Let M be a maximal set of type 1. Without loss of generality let M be of the form (2.1). Choose $p_1, \dots, p_m \in D^n$ with $\dim\langle p_2, \dots, p_m\rangle = m-1$ and $(p_1 + \langle p_2, \dots, p_m\rangle) \cap H = \emptyset$. Let $P := \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \in M_{m\times n}(D)$. Then by Lemma 2.2 we have $d(P, H) = m$ and $\rho(H) = m$. □

Lemma 2.5. *Let M_1, M_2 be two distinct maximal sets with $M_1 \cap M_2 \neq \emptyset$.*
a) *Let $m \le n$. If M_1 and M_2 are of type* 1*, or $m = n$ and M_1, M_2 are of different type, then $\rho(M_1 \cup M_2) = m-1$.*
b) *Let $n \le m$. If M_1 and M_2 are of type* 2*, or $m = n$ and M_1, M_2 are of different type, then $\rho(M_1 \cup M_2) = n-1$.*

Proof. a) Let $m \le n$. Without loss of generality we can assume that $0 \in M_1 \cap M_2$. Then for any $A \in M_{m\times n}(D)$ with $\operatorname{rank}(A) = m$ we have $d(A, M_1) = m-1 = d(A, M_2)$. Thus $d(A, M_1 \cup M_2) = m-1$ and $\rho(M_1 \cup M_2) = m-1$.
The case b) can be proved similarly. □

Lemma 2.6. a) *Let $m \leq n$,*

$$M_1 := \left\{ \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\Big|\; x_1 \in D^n \right\}, \quad M_2 := \left\{ \begin{pmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\Big|\; x_2 \in D^n \right\},$$

$$M_{12} := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\Big|\; \operatorname{rank}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 \right\}$$

and $\mathcal{S} := M_1 \cup M_2 \cup M_{12}$. Then $\rho(\mathcal{S}) = m-2$ if $m > 2$ and $\rho(\mathcal{S}) = m-1$ if $m = 2$.
b) *Let $n \leq m$,*

$$\begin{aligned} M_1 &:= \left\{ \begin{pmatrix} y_{11} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ y_{m1} & 0 & \dots & 0 \end{pmatrix} \;\Big|\; y_{11}, \dots, y_{m1} \in D \right\}, \\ M_2 &:= \left\{ \begin{pmatrix} 0 & y_{12} & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & y_{m2} & 0 & \dots & 0 \end{pmatrix} \;\Big|\; y_{12}, \dots, y_{m2} \in D \right\}, \\ M_{12} &:= \left\{ \begin{pmatrix} y_{11} & y_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ y_{m1} & y_{m2} & 0 & \dots & 0 \end{pmatrix} \;\Big|\; \operatorname{rank}\begin{pmatrix} y_{11} & y_{12} \\ \vdots & \vdots \\ y_{m1} & y_{m2} \end{pmatrix} = 2 \right\} \end{aligned}$$

and $\mathcal{S} := M_1 \cup M_2 \cup M_{12}$. Then $\rho(\mathcal{S}) = n-2$ if $n > 2$ and $\rho(\mathcal{S}) = n-1$ if $n = 2$.

Proof. Let $m \leq n$. Since $M_1 \subset \mathcal{S}$ and $d(X, \mathcal{S}) = m-2$ for all $X \in M_{m\times n}(D)$ with $\operatorname{rank}(X) = m$, we have $m - 2 \leq \rho(\mathcal{S}) \leq \rho(M_1) = m-1$. For any $X \in M_{m\times n}(D)$ we have $d(X, \mathcal{S}) \leq d(X, 0) = \operatorname{rank}(X)$. Thus $d(X, \mathcal{S}) = m-1$ implies $\operatorname{rank}(X) \in \{m, m-1\}$. In the case $m > 2$ let x_i denote the i^{th} row vector of X. If $\operatorname{rank}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2$, let $Y = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{S}$, then $d(X, Y) = \operatorname{rank}(X - Y) = \operatorname{rank}\begin{pmatrix} x_3 \\ \vdots \\ x_m \end{pmatrix} \leq m-2$. Now let $\operatorname{rank}(X) = m-1$ and $\operatorname{rank}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1$. There exists $v \in \langle x_3, \dots, x_m \rangle \setminus \langle x_1, x_2 \rangle$. Let $Y = \begin{pmatrix} x_1 \\ x_2 - v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{S}$, then $d(X, Y) = \operatorname{rank}\begin{pmatrix} 0 \\ v \\ x_3 \\ \vdots \\ x_m \end{pmatrix} = m-2$. Thus $\rho(\mathcal{S}) = m-2$ for $m > 2$. In the case $m = 2$, for any $X \in M_{m\times n}(D) \setminus \mathcal{S}$ we have $d(X, \mathcal{S}) = 1$, thus $\rho(\mathcal{S}) = 1 = m-1$. $\square$

Lemma 2.7. *Let M_1, M_2 and M_{12} be defined as in Lemma 2.6. Then for any $X \in M_{12}$ there is a maximal set M_3 of the same type as M_1, which contains X and satisfies $M_3 \cap M_1 \neq \emptyset \neq M_3 \cap M_2$.*

Proof. Let $m \leq n$. Let $X \in M_{12}$. For any $Y_i \in M_i$ which are adjacent to X, $i = 1, 2$, we have

$$Y_1 = \begin{pmatrix} x_1 + \lambda_1 x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 \\ \lambda_2 x_1 + x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and Y_1, Y_2 are adjacent if, and only if, $\lambda_1\lambda_2 = 1$. Choose $\lambda_1 \neq 0$, $\lambda_2 = \lambda_1^{-1}$ and let M_3 be a maximal set which contains X, Y_1 and Y_2. Then $M_3 \cap M_1 = \{Y_1\}$, $|M_3 \cap M_1| = 1$, and M_3 is of type 1. □

Lemma 2.8. *Let $m = n$. Let M_1 be a maximal set of type 1 and M_2 a maximal set of type 2 such that $M_1 \cap M_2 \neq \emptyset$. Then for any $A \in M_1 \cap M_2$ there exists $Q \in M_{m\times n}(D)$ such that $d(Q, M_1) = d(Q, M_2) = m - 1$, $d(Q, A) = m - 1$ and $H_1 \cap H_2 = \{A\} = H_1 \cap M_2 = H_2 \cap M_1$ where*

$$H_1 = \{X \in M_1 \mid d(Q, X) = m - 1\}, \qquad H_2 = \{Y \in M_2 \mid d(Q, Y) = m - 1\}.$$

Proof. Without loss of generality we can assume that M_1 is of the form (2.1) and M_2 is of the form (2.3). For any $A \in M_1 \cap M_2$, $A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$, let $Q := \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in M_{m\times n}(D)$. Then $d(Q, M_1) = d(Q, M_2) = m - 1$ and $d(Q, A) = m - 1$. Let $H_1 = \{X \in M_1 \mid d(Q, X) = m - 1\}$ and $H_2 = \{Y \in M_2 \mid d(Q, Y) = m - 1\}$, then we have $H_1 = \{X \in M_1 \mid x_{11} = a_{11}\}$, $H_2 = \{Y \in M_2 \mid y_{11} = a_{11}\}$ and $H_1 \cap H_2 = \{A\} = H_1 \cap M_2 = H_2 \cap M_1$. □

Lemma 2.9. *Let $n \geq 2$. Let V, V' be n-dimensional left or right affine spaces over a division ring D, $D \neq \mathbb{F}_2$. Let $f : V \to V'$ be an injective mapping which takes collinear points to collinear points, and let $f(V)$ not be contained in any affine hyperplane of V'. Then f induces injective mappings $f_t : V(t) \to V'(t)$ where $V(t)$, $V'(t)$ denote the sets of all affine t-flats of V resp. V', $0 \leq t \leq n$. Furthermore, assume there exists an integer k, $0 < k < n$ such that $f_k : V(k) \to V'(k)$ is bijective. Then f is bijective and f takes lines to lines.*

Proof. Since f takes collinear points to collinear points, we have

$$f(\langle a_0, a_1, \dots, a_t\rangle) \subseteq \langle f(a_0), f(a_1), \dots, f(a_t)\rangle$$

for all points $a_0, \dots, a_t \in V$, where $\langle a_0, \dots, a_t\rangle$ denotes the affine flat spanned by $a_0, \dots, a_t$. $f(V)$ is not contained in any affine hyperplane of V', thus f takes any affine basis of V to an affine basis of V'. This implies $\dim(v + U) = \dim(\langle f(v + U)\rangle)$ for any affine flats $v + U$ of V. Let $V(t)$, $V'(t)$ denote the sets of all affine t-flats of V resp. V', $0 \leq t \leq n$. Then f induces injective mappings $f_t : V(t) \to V'(t)$ for all $0 \leq t \leq n$ defined by $f_t(v + U) := \langle f(v + U)\rangle \in V(t')$ for any t-dimensional affine flat $v + U \in V(t)$. Now let k be an integer, $0 < k < n$ such that $f_k : V(k) \to V'(k)$ is bijective. We prove by induction that f_t is bijective for all $0 \leq t \leq k$. This is the assumption on f in the case $t = k$. Let f_t be bijective for some $0 \leq t \leq k$. Let $s + T$ be an affine $(t-1)$-flat of V'. Let $v_1' + U_1'$, $v_2' + U_2'$ be two distinct affine t-flats of V' with $(v_1' + U_1') \cap (v_2' + U_2') = s + T$. Since $f_t : V(t) \to V'(t)$ is bijective, there are $v_1 + U_1, v_2 + U_2 \in V(t)$ with $f_t(v_i + U_i) = v_i' + U_i'$, $i = 1, 2$. Since $v_1' + U_1'$ and $v_2' + U_2'$ are contained in an affine $(t+1)$-flat, also $v_1 + U_1$ and $v_2 + U_2$ are contained in an affine $(t+1)$-flat. Suppose $(v_1 + U_1) \cap (v_2 + U_2) = \emptyset$, i.e., $(v_1 + U_1) \parallel (v_2 + U_2)$. For any point $x \in v_1 + U_1$, its image $f(x) \notin (v_1' + U_1') \cap (v_2' + U_2')$ since otherwise the join $\{x\} \cup (v_2 + U_2)$ would be contained in $v_2' + U_2'$, and $f(V)$ would be contained in an affine hyperplane of V'. Let $v' + U'$ be any affine t-flat of V' such that $v' + U'$ is contained in the affine $(t + 1)$-flat $(v_1' + U_1') \cup (v_2' + U_2')$

and $(v' + U') \parallel (v_2' + U_2')$. Then $(v + U) \parallel (v_2 + U_2)$ implies $(v + U) \parallel (v_1 + U_1)$ where $v + U := f_t(v' + U')^{-1}$. Analogously we have $f(x) \notin (v' + U') \cap (v_1' + U_1')$ for all $x \in (v_1 + U_1)$. Thus

$$f(x) \notin v_1' + U_1' = \bigcup_{\substack{U'=U_2' \\ v' \in (v_1'+U_1') \cup (v_2'+U_2')}} \big((v' + U') \cap (v_1' + U_1')\big) \qquad \forall x \in v_1 + U_1,$$

a contradiction to $f_t(v_1 + U_1) = v_1' + U_1'$. So we have $(v_1 + U_1) \cap (v_2 + U_2) \neq \emptyset$ and $\dim\big((v_1 + U_1) \cap (v_2 + U_2)\big) = t - 1$. Hence $f = f_0$ is bijective. Let l be any line of V, then $f_1(l)$ is a line in V'. Let Q be any point of $f_1(l)$. Since f is bijective there is a point P with $f(P) = Q$. Assume $P \notin l$. Then the plane spanned by P and l is mapped by f to a subset of the line $f_1(l)$, a contradiction. So $f(l) = f_1(l)$ for any line $l \subset V$. □

Lemma 2.10. *Let $n \geq 2$. Let V, V' be n-dimensional left or right affine spaces over a division ring D. Let $f : V \to V'$ be an injective mapping which takes any line onto a line, i.e., for any line $l \in V$, its image $f(l)$ is a line in V'. Then f is a bijection.*

Proof. The assertion is true when D is finite. Now let D be infinite. For any $f(X) \neq f(Y) \in f(V)$ the line $\langle f(X), f(Y)\rangle = f(\langle X, Y\rangle)$ is contained in $f(V)$. Then $f(V)$ is an affine subspace of V'. V and $f(V)$ are isomorphic, so we have $n = \dim(V) = \dim(f(V))$. This implies $V' = f(V)$. □

3. Proof of Theorem 1.1

We will prove the theorem only in the case $m \leq n$. We can prove the theorem in the case $n < m$ analogously to the case $m < n$ by replacing maximal sets of type 1 by maximal sets of type 2 and vice versa.

We prove the theorem in several steps.

(i) *For any maximal set M, there is a maximal set M' containing M^φ.*

Proof. This follows immediately from the fact that φ preserves adjacency. □

(ii) *Let $A \in M_{m\times n}(D)$ and let $M \subset M_{m\times n}(D)$. Then we have $d(A, M) \geq d(A^\varphi, M^\varphi) \geq d(A^\varphi, M')$ and $\rho(M) \geq \rho(M')$ for all $M' \subset M_{m\times n}(D)$ with $M^\varphi \subset M'$.*

Proof. We prove that $\rho(M) \geq \rho(M')$. Let X be a point with $d(X, M') = \rho(M')$. Since φ is bijective, there is a point Y with $Y^\varphi = X$ and we have $d(Y, M) \geq d(X, M')$, thus $\rho(M) \geq d(Y, M) \geq d(X, M') = \rho(M')$. □

(iii) *Let M be a maximal set. Then there exists exactly one maximal set M' with $M' \supset M^\varphi$. If $m < n$ and M is of type 1, then also M' is of type 1.*

Proof. Assume there are two distinct maximal sets M_1' and M_2' with $M^\varphi \subset M_i'$, $i = 1, 2$. Since $|M| > 1$ we have $|M^\varphi| > 1$, which implies $|M_1' \cap M_2'| > 1$. By Proposition 2.3, M_1' and M_2' are not of the same type. By Proposition 2.6 we can assume that M_1' is of the form (2.1) and M_2' is of the form (2.3). Then $M_1' \cap M_2' = \left\{ x \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \;\middle|\; x \in D \right\}$.

Clearly, $\rho(M_1' \cap M_2') = m$. By Lemma 2.3, $\rho(M) = m-1$. But $M^\varphi \subset M_1' \cap M_2'$. By (ii), $\rho(M_1' \cap M_2') \le \rho(M)$, a contradiction. Therefore there is a unique maximal set $M' \supset M^\varphi$. Now let $m < n$. Since $M' \supset M^\varphi$, by (ii) we have $\rho(M') \le \rho(M) = m-1$. By Lemma 2.3, M' is of type 1. □

(iv) *Let M_1, M_2 be two distinct maximal sets of type* 1. *Let M_i' be two maximal sets with $M_i' \supset M_i^\varphi$. Then M_1' and M_2' are of the same type.*

Proof. In the case $m < n$, by (iii) the sets M_i' are of type 1.
In the case $m = n$ assume that M_1' and M_2' are not of the same type. Since M_1 and M_2 can be joined by a sequence of consecutively intersecting maximal sets of the same type, we may suppose that $M_1 \cap M_2 \neq \emptyset$. Let $\{X\} := M_1 \cap M_2$. By Proposition 2.6 we may assume without loss of generality that

$$X = 0, \qquad M_1 = \left\{ \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, v \in D^n \right\}, \qquad M_2 = \left\{ \begin{pmatrix} 0 \\ w \\ 0 \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, w \in D^n \right\}.$$

Let $Y := X^\varphi \in M_1' \cap M_2'$. From Lemma 2.8 there exists a point Q with $d(Q, M_1') = m-1 = d(Q, M_2')$ and $d(Q,Y) = m-1$ and such that $H_1' \cap M_2' = \{Y\}$, $H_2' \cap M_1' = \{Y\}$, where $H_i' := \big\{A \in M_i' \mid d(A,Q) = m-1\big\}$, $i = 1,2$. Let $P \in M_{m\times n}(D)$ with $P^\varphi = Q$. Since $m-1 = d(Q, M_i') \le d(P, M_i) \le m-1$, $d(P, M_i) = m-1$. Define $H_i := \big\{A \in M_i \mid d(A,P) = m-1\big\}$. We write $P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}$, then $p_1 \neq 0 \neq p_2$ and

$$H_1 = \left\{ \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, v \in p_1 + \langle p_2, \dots, p_m \rangle \right\}, \quad H_2 = \left\{ \begin{pmatrix} 0 \\ w \\ 0 \\ \vdots \\ 0 \end{pmatrix} \,\middle|\, w \in p_2 + \langle p_1, p_3, \dots, p_m \rangle \right\}.$$

If $p_1 = \lambda p_2$ for some $\lambda \in D^*$ then let $A := \begin{pmatrix} p_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in H_1 \setminus \{0\}$ and $B := \begin{pmatrix} 0 \\ p_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in H_2 \setminus \{0\}$, we have $d(A,B) = 1$. If p_1, p_2 are linearly independent, then for

$$A = \begin{pmatrix} p_1+p_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in H_1 \setminus \{0\}, \quad B = \begin{pmatrix} 0 \\ p_2+p_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in H_2 \setminus \{0\}$$

we have $d(A,B) = 1$. Let M be a maximal set containing 0, A and B. Since $|M \cap M_i| \ge 2$, M is of type 2. Let M' be the maximal set containing M^φ. Then $|M' \cap M_i'| \ge 2$, thus $M' = M_1'$ or $M' = M_2'$, and $B^\varphi \in M' \cap M_2' = M_1' \cap M_2'$ or $A^\varphi \in M_1' \cap M' = M_1' \cap M_2'$. But $B^\varphi \in H_2'$ and $A^\varphi \in H_1'$, so we have $B^\varphi = 0^\varphi$ or $A^\varphi = 0^\varphi$, a contradiction to the injectivity of φ. □

(v) *For any two distinct maximal sets M_1 and M_2 of type* 1 *with $M_1 \cap M_2 \neq \emptyset$ there is no maximal set M which contains $M_1^\varphi \cup M_2^\varphi$.*

Proof. Suppose there is a maximal set M which contains $M_1^\varphi \cup M_2^\varphi$. Without loss of generality let M_1, M_2, M_{12}, $\mathcal{S}$ be defined as in Lemma 2.6 a). Then for any $X \in M_{12}$, there is a maximal set M_3 of type 1 which contains X and intersects M_i with $M_3 \cap M_1 =: \{A\} \neq \{B\} := M_3 \cap M_2$. Then $A^\varphi, B^\varphi \in M_3^\varphi \cap M$, thus $|M_3^\varphi \cap M| \geq 2$. From (iv) we have that the maximal sets containing M_3^φ and M are of the same type. So $M_3^\varphi \subset M$, which implies $X^\varphi \in M$. Then $\mathcal{S}^\varphi \subset M$. In the case $m > 2$, from Lemma 2.6 we have $\rho(\mathcal{S}) = m - 2$. $\mathcal{S}^\varphi \subset M$ implies $\rho(M) \leq \rho(\mathcal{S}^\varphi) \leq \rho(\mathcal{S}) = m - 2$, a contradiction to $\rho(M) = m - 1$. In the case $m = 2$ let $Q \in M_{m\times n}(D)$ with $d(Q, 0^\varphi) = 2$, then $Q \notin M$. Let $P \in M_{m\times n}(D)$ with $P^\varphi = Q$ then $d(P, 0) \geq d(P^\varphi, 0^\varphi) = d(Q, 0^\varphi) = 2$. Thus $\operatorname{rank}(P) = 2$, i.e., $P \in \mathcal{S}$, and $Q = P^\varphi \in \mathcal{S}^\varphi \subset M$, a contradiction to $Q \notin M$. □

(vi) *Let M_1 and M_2 be two maximal sets of type* 1 *and type* 2 *respectively, such that $M_1 \cap M_2$ is a line. Then $M_1' \cap M_2'$ is a line, where M_i' is the maximal set with $M_i' \supset M_i^\varphi$, $i = 1, 2$. Thus φ takes any line to the subset of a line.*

Proof. Since $M_1' \cap M_2' \supset (M_1 \cap M_2)^\varphi$, $M_1' \cap M_2'$ is a line or $M_1' = M_2'$. Choose points $X \in M_1 \cap M_2$ and $Y \in M_2 \setminus M_1$. Let M_3 be the maximal set of type 1 with $X, Y \in M_3$. Then from (iv) and (v) we have that $M_3' \neq M_1'$ are of the same type where M_3' is the maximal set containing M_3^φ. Therefore $|M_1' \cap M_3'| = 1$. Since $X^\varphi, Y^\varphi \in M_2' \cap M_3'$, $X^\varphi \neq Y^\varphi$, we have $|M_2' \cap M_3'| \geq 2$ and thus $M_2' \neq M_1'$. It follows that $M_1' \cap M_2'$ is a line. □

(vii) *Let M be a maximal set of type* 1 *and let M' be the maximal set containing M^φ. Consider M and M' as affine spaces. Then M^φ is not contained in any affine hyperplane of M'.*

Proof. Assume M^φ is contained in a hyperplane H of M'. Then by Lemma 2.4 and (ii), we have $m = \rho(H) \leq \rho(M^\varphi) \leq \rho(M) = m - 1$, a contradiction. □

(viii) *Let M be a maximal set of type* 1 *and let M' be the maximal set containing M^φ. Then $\varphi : M \to M'$ is bijective and takes lines to lines.*

Proof. The assertion is true when D is finite. Now let D be infinite. Consider M, M' as affine spaces. Then $\varphi : M \to M'$ takes collinear points to collinear points, and by (vii), M^φ is not contained in a hyperplane of M'. By Lemma 2.9, φ induces an injective mapping $\varphi_{m-1} : M(m-1) \to M'(m-1)$, where $M(m-1)$, $M'(m-1)$ denote the sets of all affine $(m-1)$-flats of M resp. M'. Now let U' be an arbitrary affine $(m-1)$-flat of M'. There is a point P^φ of $M_{m\times n}(D)$ such that

$$d(P^\varphi, X) = m - 1 \quad \forall X \in U', \qquad d(P^\varphi, X) = m \quad \forall X \in M' \setminus U'.$$

By (ii) and Lemma 2.3, we have $m - 1 = d(P^\varphi, M') \leq d(P, M) \leq \rho(M) = m - 1$ and $d(P, M) = m - 1$. Let $U := \{X \in M \mid d(P, X) = m - 1\}$. By Corollary 2.1, U is an $(m-1)$-flat of M, and $U^\varphi \subset U'$, this implies that φ_{m-1} is bijective. By Lemma 2.9, $\varphi : M \to M'$ is bijective and takes lines to lines. □

(ix) *Let l be any line of $M_{m\times n}(D)$. Then l^φ is a line of $M_{m\times n}(D)$.*

Proof. The assertion is true when D is finite. Now let D be infinite. Let M be a maximal set of type 1 containing l. Let M' be the maximal set containing M^φ. Consider M, M' as affine spaces. Then by (viii) $\varphi : M \to M'$ takes lines to lines. □

(x) *Two points $A, B \in M_{m\times n}(D)$ are adjacent if A^φ, B^φ are adjacent.*

Proof. Choose maximal sets M_1, M_2 of type 1 with $M_1 \ni A$, $M_2 \ni B$. Let M_i' be the unique maximal set containing M_i^φ, $i = 1, 2$. By (iv), M_1' and M_2' are of the same type. Let M' be a maximal set containing A^φ and B^φ, which is not of the same type as M_1'. Then $A^\varphi \in M_1' \cap M'$, $B^\varphi \in M_2' \cap M'$, and $M_i' \cap M'$ is a line, $i = 1, 2$. There exist lines $g_1 \ni A$ and $g_2 \ni B$ in M_1 resp. M_2 with $g_i^\varphi = M_i' \cap M'$, $i = 1, 2$. Choose two maximal sets S_i of type 2 with $S_i \cap M_i = g_i$. Then $S_i^\varphi \subset M'$. Consider S_i as m-dimensional right affine space over D. If $m < n$ then M_1' is of type 1 and M' is of type 2. Then M' is also an m-dimensional right affine space over D. In the case $m = n$, if M' is of type 1 then M' can be considered as m-dimensional left affine space over D. The restriction $\varphi|_{S_i} : S_i \to M'$ is injective and takes lines to lines by (ix). Thus by Lemma 2.10, $S_i^\varphi = M'$ for $i = 1, 2$. This implies that $S_1 = S_2$ and A, B are adjacent. □

References

[1] Hua, L. K.: *A theorem on matrices over a sfield and its applications.* Acta Math. Sinica **1** (1951), 109–163.

[2] Huang, W.-l.; Höfer, R.; Wan, Z.-X.: *Adjacency preserving mappings of symmetric and hermitian matrices.* Aequationes Math., to appear.

[3] Wan, Z.-X.: *Geometry of matrices.* World Scientific, Singapore 1996. Zbl 0866.15008

[4] Wan, Z.-X.: *Geometry of matrices revisited.* In: K.-P. Shum, E. Taft, and Z.-X. Wan, editors: Algebra and Combinatorics, pp. 477–486, Hong Kong 1999. ICAC '97, Springer. An International Congress. Zbl 0952.15008

[5] Wan, Z.-X.; Wang, Y.: Discussions on *A theorem on matrices over a sfield and its applications.* Shuxue Jinzhan **5** (1962), 325–332. (In Chinese).

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