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Generalized Adjoint Semigroups of a Ring

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Abstract. In this paper, we introduce generalized adjoint semigroups (GA-semigroups) of a ring R. We construct generalized adjoint semigroups on a ring R by means of bitranslations of R. It is shown that GA-semigroups of a π-regular ring are π-regular. As an application we deduce that in any ring, idempotents can be lifted modulo π-regular ideals. GA-semigroups containing idempotents are described in terms of the ring of a Morita context.

1. Introduction

Let R be a ring not necessarily with identity. The composition defined by $a \circ b = a+b+ab$ for any $a, b \in R$ is usually called the circle or adjoint multiplication of R, which plays a role in the theory of Jacobson radical. It is well-known that $(R, \circ)$ is a monoid with identity 0, called the circle or adjoint semigroup of R. There are many interesting connections between a ring and its adjoint semigroup, which were studied in several papers, for example, [8, 13, 14, 16, 22, 23, 24, 30, 31]. Typical results are to describe the adjoint semigroup of a given ring and the ring with a given semigroup as its adjoint semigroup.

The circle multiplication of a ring satisfies the following generalized distributive laws:

$$a \circ (b + c - d) = a \circ b + a \circ c - a \circ d, \tag{1}$$
$$(b + c - d) \circ a = b \circ a + c \circ a - d \circ a, \tag{2}$$

or equivalently,

$$a \circ (b+c) = a \circ b + a \circ c - a \circ 0,$$
$$(b+c) \circ a = b \circ a + c \circ a - 0 \circ a,$$

which was observed in [1]. Thus as generalizations of the circle multiplication of a ring, a binary operation $\diamond$ (associative or nonassociative) on an Abelian group A satisfying the generalized distributive laws have been studied by several authors making use of different terminologies, for example, pseudo-ring in [33], weak rings in [10], quasirings in [11], prerings in [3, 4, 29]. In particular, the so-called (m, n)-distributive rings studied in [5, 26, 27, 36] also satisfy the generalized distributive laws (1) and (2). To such a system $(A, +, \diamond)$ there corresponds a unique associated ordinary ring. But, in general, even if A is a ring, there may exist no relation between the ring A and the associated ring of $(A, +, \diamond)$. In this paper, we are interested in a binary operation $\diamond$ on a ring R, satisfying the associative law, the generalized distributive laws as (1) and (2), and the compatibility:

$$xy = x \diamond y - x \diamond 0 - 0 \diamond y + 0 \diamond 0.$$

This is equivalent to say that $(R, +, \diamond)$ is a weak ring such that the ring R is exactly the associated ring of $(R, +, \diamond)$. Such a binary operation $\diamond$ is called a generalized adjoint multiplication on R and the semigroup $(R, \diamond)$ is called a generalized adjoint semigroup of R, abbreviated GA-semigroup, which is a generalization of the multiplicative semigroup and the adjoint semigroup of a ring R. Essentially, the multiplicative and adjoint semigroup of R are exactly generalized adjoint semigroups of R with zero and identity, respectively (cf. Theorem 2.14). The other generalization of adjoint multiplication was studied in [21].

The aim of this paper is to describe generalized adjoint semigroups of a ring R. In Section 2, we present a way to construct generalized adjoint multiplications on a ring R by means of bitranslations of R, characterize a GA-semigroup with identity or zero and describe GA-semigroups of a ring with 1.

In Section 3, we prove that GA-semigroups of a π-regular ring are π-regular.

In Section 4, we first prove that a GA-semigroup containing idempotents can be represented as a GA-semigroup of the ring of a Morita context. Then we present a sufficient condition and a necessary condition for a GA-semigroup to contain idempotents, in virtue of which we prove that in any ring, idempotents can be lifted modulo a π-regular ideal. This generalizes a classical result in ring theory which states that idempotents modulo a nil ideal can be lifted ([28]) and the ring-theoretic analogue of a result of Edwards ([19, Corollary 2]) which extends the well-known Lallement' lemma to eventually regular semigroups (i.e., π-regular semigroups). Finally, we prove that GA-semigroups of rings with DCC on principal right ideals contain idempotents.

In the forthcoming paper [17], we characterize the rings with a GA-semigroup having a property **P** and its such GA-semigroups, where **P** stands for orthodox, right inverse, inverse, pseudoinverse, E-unitary, and completely simple, respectively.

Although a ring R in this paper needs not contain identity, it is convenient to use a formal identity 1, which can be regarded as the identity of a unitary ring containing R, since R can be always embedded into a ring with identity 1; for example, we can write $a \circ b = (1+a)(1+b) - 1$ for any $a, b \in R$ and write $x^0 = 1$ for any $x \in R$ by making use of a formal 1.

For $x \in R$ and a positive integer n we denote by $x^{[n]}$ the n-th power of x with respect to a generalized adjoint multiplication $\diamond$, and $x^{[0]}$ stands for an empty word.

A radical ring means a Jacobson radical ring.

For the algebraic theory and terminology on semigroups we will refer to [9, 20, 25].

2. A construction of GA-semigroups

Definition 2.1. *Let R be a ring. A binary operation $\diamond$ on R is called a generalized adjoint multiplication on R, if it satisfies the following conditions:*

(i) *the associative law: $x \diamond (y \diamond z) = (x \diamond y) \diamond z$;*

(ii) *the generalized distributive laws:*

$$x \diamond (y+z) = x \diamond y + x \diamond z - x \diamond 0,$$
$$(y+z) \diamond x = y \diamond x + z \diamond x - 0 \diamond x;$$

(iii) *the compatibility: $xy = x \diamond y - x \diamond 0 - 0 \diamond y + 0 \diamond 0$.*

The semigroup $(R, \diamond)$ is called a generalized adjoint semigroup of R, abbreviated GA-semigroup and denoted by $R^\diamond$.

We now remark that for a binary operation $\diamond$ on R, the generalized distributive laws are equivalent to

$$w \diamond (x+y-z) = w \circ x + w \diamond y - w \diamond z,$$
$$(x+y-z) \diamond w = x \circ w + y \diamond w - z \diamond w.$$

Example 2.2. The multiplicative semigroup $R^\bullet$ of a ring R is a GA-semigroup of R with zero 0. The adjoint semigroup R° of R is a GA-semigroup of R with identity 0.

Lemma 2.3. *For any $x_i, y_j \in R$, and $p_i, q_j \in \mathbb{Z}$ with $\sum p_i = \sum q_j = 0$, we have*

$$\left(\sum p_i x_i\right)\left(\sum q_j y_j\right) = \sum p_i q_j (x_i \diamond y_j).$$

Proof. Set $p = \sum p_i$, and $q = \sum q_j$. Then we have that

$$\begin{aligned}
&\left(\sum p_i x_i\right)\left(\sum q_j y_j\right)\\
&= \sum p_i q_j (x_i y_j)\\
&= \sum p_i q_j (x_i \diamond y_j) - \sum p_i q_j (x_i \diamond 0)\\
&\quad - \sum p_i q_j (0 \diamond y_j) + \sum p_i q_j (0 \diamond 0) \qquad \text{(by the compatibility)}\\
&= \sum p_i q_j (x_i \diamond y_j) - q \sum p_i (x_i \diamond 0) - p \sum q_j (0 \diamond y_j) + pq(0 \diamond 0)\\
&= \sum p_i q_j (x_i \diamond y_j),
\end{aligned}$$

as desired. □

Corollary 2.4. *If $x \diamond y = y \diamond x$, then $(x-y)^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x^{[i]} \diamond y^{[n-i]}$.*

Proof. As the usual binomial theorem, the corollary can be proved by use of an induction on n and Lemma 2.3. □

Recall that a bitranslation is a pair $(\lambda, \rho) \in End(R_R) \times End({}_RR)$ such that $x\lambda(y) = \rho(x)y$ for any $x, y \in R$. The set $\Omega(R)$ of all bitranslations of R is a subring of $End(R_R) \times End({}_RR)$ with identity $(1_R, 1_R)$, called the translational hull of R. For $a \in R$, let λ_a and ρ_a be the left and right multiplications by a, respectively. Then (λ_a, ρ_a) is a bitranslation of R, denoted by π_a, and $\pi : a \longmapsto \pi_a$ defines a ring homomorphism form R into $\Omega(R)$ such that the image $\pi(R)$ is an ideal of $\Omega(R)$ and the kernel is $Ann(R) = \{x \in R \mid xR = Rx = 0\}$. Hence we can identify $a \in R$ with π_a and R with $\pi(R)$ whenever $Ann(R) = 0$. A bitranslation $\theta = (\lambda, \rho)$ will be considered as a double operator on R defined by $\theta x = \lambda(x)$ and $x\theta = \rho(x)$ for any $x \in R$. Then $\theta = \theta'$ if and only if $\theta x = \theta' x$ and $x\theta = x\theta'$ for any $x \in R$. A bitranslation θ is called self-permutable if $(\theta x)\theta = \theta(x\theta)$ for any $x \in R$ ([32, 34, 35]).

For a self-permutable bitranslation θ, there is no ambiguity if we write $\theta x y \theta^2 z$, for example.

By an associated pair of R we mean a pair $(\theta, \vartheta) \in \Omega(R) \times R$ satisfying the following conditions:

(i) $\theta\vartheta = \vartheta\theta$;

(ii) θ is self-permutable;

(iii) $\theta^2 = \theta + \pi_\vartheta$.

Theorem 2.5. *Let (θ, ϑ) be an associated pair of a ring R and define*

$$x \diamond y = xy + x\theta + \theta y + \vartheta \tag{3}$$

for any $x, y \in R$. Then $\diamond$ is a generalized adjoint multiplication on R (called one induced by (θ, ϑ)). Conversely, every generalized adjoint multiplication $\diamond$ on R can be obtained in this fashion by setting $\vartheta = 0 \diamond 0$, $\theta x = 0 \diamond x - 0 \diamond 0$ and $x\theta = x \diamond 0 - 0 \diamond 0$. Moreover, the correspondence $(\theta, \vartheta) \to \diamond$ is a 1-1 correspondence between the associated pairs of R and generalized adjoint multiplications on R.

Proof. Suppose that (θ, ϑ) is an associated pair of R and the operation $\diamond$ is given by (3). Then the associative law is verified as follows:

$$\begin{aligned}
&(x \diamond y) \diamond z \\
&= (xy + x\theta + \theta y + \vartheta) \diamond z \quad \text{(by (3))} \\
&= xyz + x\theta z + \theta yz + \vartheta z + xy\theta + x\theta^2 + \theta y\theta + \vartheta\theta + \theta z + \vartheta \\
&= xyz + xy\theta + x\theta z + x\vartheta + x\theta + \theta yz + \theta y\theta + \theta z + \vartheta z + \theta\vartheta + \vartheta \\
&= xyz + xy\theta + x\theta z + x\vartheta + x\theta + \theta yz + \theta y\theta + \theta^2 z + \theta\vartheta + \vartheta \\
&= x \diamond (yz + y\theta + \theta z + \vartheta) \quad \text{(by (3))} \\
&= x \diamond (y \diamond z).
\end{aligned}$$

For the generalized distributive laws, we have that

$$\begin{aligned}
&x \diamond (y + z) \\
&= xy + xz + x\theta + \theta y + \theta z + \vartheta \quad \text{(by (3))} \\
&= (xy + x\theta + \theta y + \vartheta) + (xz + x\theta + \theta z + \vartheta) - (x\theta + \vartheta) \\
&= x \diamond y + x \diamond z - x \diamond 0, \quad \text{(by (3))}
\end{aligned}$$

and similarly $(y + z) \diamond x = y \diamond x + z \diamond x - 0 \diamond x$. The compatibility follows from

$$\begin{aligned}
&x \diamond y - x \diamond 0 - 0 \diamond y + \vartheta \\
&= (xy + x\theta + \theta y + \vartheta) - (x\theta + \vartheta) - (\theta y + \vartheta) + \vartheta \quad \text{(by (3))} \\
&= xy.
\end{aligned}$$

Thus $\diamond$ is a generalized circle multiplication on R.

Conversely, suppose $\diamond$ is a generalized adjoint multiplication on R. Set $\vartheta = 0 \diamond 0$, $\lambda(x) = 0 \diamond x - 0 \diamond 0$, $\rho(x) = x \diamond 0 - 0 \diamond 0$ and $\theta = (\lambda, \rho)$. For any $a, x, y \in R$, we have that

$$\lambda(x + y) = 0 \diamond (x + y) - 0 \diamond 0 = 0 \diamond x + 0 \diamond y - 2\vartheta = \lambda(x) + \lambda(y),$$

$$\begin{aligned}
\lambda(x)a &= (0 \diamond x - 0 \diamond 0)(a - 0) \\
&= 0 \diamond x \diamond a - 0 \diamond x \diamond 0 - 0 \diamond 0 \diamond a + 0 \diamond 0 \diamond 0 \qquad \text{(by Lemma 2.3)} \\
&= 0 \diamond (x \diamond a - x \diamond 0 - 0 \diamond a + 0 \diamond 0) - 0 \diamond 0 \\
&= 0 \diamond (xa) - 0 \diamond 0 \\
&= \lambda(xa),
\end{aligned}$$

which imply that $\lambda \in End(R_R)$. Symmetrically, $\rho \in End({}_RR)$. Note that

$$\begin{aligned}
x\lambda(y) &= (x - 0)(0 \diamond y - 0 \diamond 0) \\
&= x \diamond 0 \diamond y - x \diamond 0 \diamond 0 - 0 \diamond 0 \diamond y + 0 \diamond 0 \diamond 0 \qquad \text{(by Lemma 2.3)} \\
&= (x \diamond 0 - 0 \diamond 0)(y - 0) \qquad \text{(by Lemma 2.3)} \\
&= \rho(x)y.
\end{aligned}$$

Thus θ is a bitranslation of R such that $\theta x = 0 \diamond x - 0 \diamond 0$ and $x\theta = x \diamond 0 - 0 \diamond 0$. Hence $\theta\vartheta = 0 \diamond \vartheta - 0 \diamond 0 = \vartheta \diamond 0 - 0 \diamond 0 = \vartheta\theta$. Since

$$\begin{aligned}(\theta x)\theta &= (0 \diamond x - 0 \diamond 0) \diamond 0 - 0 \diamond 0 = 0 \diamond x \diamond 0 - 0 \diamond 0 \diamond 0,\\ \theta(x\theta) &= 0 \diamond (x \diamond 0 - 0 \diamond 0) - 0 \diamond 0 = 0 \diamond x \diamond 0 - 0 \diamond 0 \diamond 0,\end{aligned}$$

we have that $(\theta x)\theta = \theta(x\theta)$, that is, θ is self-permutable. Observing that

$$\begin{aligned}&(\theta + \pi_\vartheta)x = \theta x + \vartheta x\\ &= 0 \diamond x - 0 \diamond 0 + \vartheta \diamond x - \vartheta \diamond 0 - 0 \diamond x + \vartheta\\ &= \vartheta \diamond x - 0 \diamond 0 - 0 \diamond \vartheta + 0 \diamond 0\\ &= \theta(0 \diamond x) - \theta\vartheta\\ &= \theta(0 \diamond x - \vartheta)\\ &= \theta^2 x,\end{aligned}$$

and similarly $x(\theta + \pi_\vartheta) = x\theta^2$, we see that $\theta^2 = \theta + \pi_\vartheta$. It follows that (θ, ϑ) is an associated pair of R. Since

$$x \diamond y = xy + x \diamond 0 + 0 \diamond y - \vartheta = xy + x\theta + \theta y + \vartheta$$

we see that $\diamond$ is induced by (θ, ϑ).

If two associated pairs (θ, ϑ) and (θ', ϑ') of R induce the same generalized adjoint multiplication on R, then for any $x, y \in R$ we have

$$xy + x\theta + \theta y + \vartheta = xy + x\theta' + \theta' y + \vartheta',$$

and so we have $\vartheta = \vartheta'$ by taking $x = y = 0$, $x\theta = x\theta'$ by taking $y = 0$, and $\theta y = \theta' y$ by taking $x = 0$, whence $(\theta, \vartheta) = (\theta', \vartheta')$. Thus the correspondence $(\theta, \vartheta) \to \diamond$ is a 1-1 correspondence. □

Theorem 2.5 is an analogue of results in [26, 27].

Corollary 2.6. *If $Ann(R) = 0$, then any generalized adjoint multiplication on R is induced by a bitranslation θ of R such that $\theta^2 - \theta \in R$, and further there exists a* 1-1 *correspondence between the set of bitranslations being idempotent modulo $\pi(R)$ and generalized adjoint multiplications on R.*

Proof. If $Ann(R) = 0$, then $\Omega(R)$ is an ideal extension of R. Let $\diamond$ be the generalized adjoint multiplication on R induced by an associated pair (θ, ϑ). Then $\theta^2 - \theta \in R$, and $\theta^2 = \theta + \pi_\vartheta$ implies $\vartheta = \theta^2 - \theta$ since $Ann(R) = 0$. It is clear that $x \diamond y = (x + \theta)(y + \theta) - \theta$. From Theorem 2.5 the correspondence $\theta \to \diamond$ is a 1-1 correspondence. □

The following corollary will be used freely throughout the rest of this paper.

Corollary 2.7. *For any $x_i, y_j \in R$, and $p_i, q_j \in \mathbb{Z}$ with $\sum p_i = \sum q_j = 1$, we have*

$$\Big(\sum p_i x_i\Big) \diamond \Big(\sum q_j y_j\Big) = \sum p_i q_j (x_i \diamond y_j).$$

Proof. For any $x_i, y_j \in R$, and $p_i, q_j \in \mathbb{Z}$ with $\sum p_i = \sum q_j = 1$, we have

$$\begin{aligned}
&\sum p_i q_j (x_i \diamond y_j)\\
&= \sum p_i q_j (x_i y_j) + \sum p_i q_j (x_i \theta) + \sum p_i q_j (\theta y_j) + \sum p_i q_j \vartheta\\
&= \left(\sum p_i x_i\right)\left(\sum q_j y_j\right) + \left(\sum p_i x_i\right)\theta + \theta\left(\sum q_j y_j\right) + \vartheta\\
&= \left(\sum p_i x_i\right) \diamond \left(\sum q_j y_j\right),
\end{aligned}$$

as desired. □

Corollary 2.8. *If $x, y \in R^\diamond$ such that $x \diamond y = y \diamond x$ and $p, q \in \mathbb{Z}$ such that $p+q=1$, then*

$$(px+qy)^{[n]} = \sum_{i=0}^{n} p^i q^{n-i} \left(\begin{array}{c} n \\ i \end{array}\right) x^{[i]} \diamond y^{[n-i]}.$$

Proof. As the usual binomial theorem, the corollary can be proved by use of an induction on n and Corollary 2.7. □

By an affine subsemigroup of $R^\diamond$ we mean a subsemigroup M of $R^\diamond$ such that $x+y-z \in S$ for any $x, y, z \in M$.

For example, for an ideal extension $\tilde{R}$ of R (i.e., $\tilde{R}$ is a ring containing R as an ideal) and $a \in \tilde{R}$ such that $a^2 - a \in R$, then $(R+a, \bullet)$ is an affine subsemigroup of $\tilde{R}^\bullet$. The semigroup $(R+a, \bullet)$ was studied in [18] to deal with lifting idempotents.

Definition 2.9. *Let M and N be affine subsemigroups of GA-semigroups $R^\diamond$ and $S^\diamond$ of rings R and S, respectively. If there exists a bijection ϕ from M onto N such that*

$$\phi(x+y-z) = \phi(x) + \phi(y) - \phi(z) \quad \text{and} \quad \phi(x \diamond y) = \phi(x) \diamond \phi(y)$$

for any $x, y, z \in M$, then M and N are called affinely isomorphic, notationally $M \simeq N$.

Corollary 2.10. *Let $\tilde{R}$ be an ideal extension of R. Then any $a \in \tilde{R}$ such that $a^2 - a \in R$ induces a generalized adjoint multiplication on R given by*

$$x \diamond y = (x+a)(y+a) - a$$

for $x, y \in R$, and $R^\diamond$ is affinely isomorphic to the affine subsemigroup $(R+a, \bullet)$ of $\tilde{R}^\bullet$.

Proof. It is clear that a induces a bitranslation θ of R by $\theta x = ax$ and $x\theta = xa$. If $a^2 - a \in R$, then $(\theta, a^2 - a)$ is an associated pair of R and the induced generalized adjoint multiplication on R given by $x \diamond y = xy + xa + ay + \vartheta = (x+a)(y+a) - a$. Let ϕ be a map from R into $R+a$ given by $\phi(x) = x+a$ for any $x \in R$. Then it is easy to check that ϕ is an affine isomorphism from $R^\diamond$ onto the affine subsemigroup $(R+a, \bullet)$ of $\tilde{R}^\bullet$. □

Lemma 2.11. *Let M be an affine subsemigroup of $R^\diamond$. Then*

$$M - M = M - a = \left\{ \sum p_i s_i \,\middle|\, s_i \in M, \text{ and } p_i \in \mathbb{Z} \text{ with } \sum p_i = 0 \right\}$$

for any $a \in M$, and $M - M$ is a subring of R.

Proof. The proof is a routine computation. □

Theorem 2.12. *Let M and N be affine subsemigroups of GA-semigroups $R^\diamond$ and $S^\diamond$ of rings R and S, respectively. If $M \simeq N$, then the rings $M - M$ and $N - N$ are isomorphic to each other. In particular, if $R^\diamond \simeq S^\diamond$, then $R \cong S$.*

Proof. Suppose ϕ is an affine isomorphism from M onto N. Take a fixed $a \in M$ and let ϕ^* be the mapping from M into N defined by $\phi^*(x-a) = \phi(x) - \phi(a)$ for any $x \in M$. Then we see that ϕ^* is a bijection. Since for any $x, y \in M$,

$$\begin{aligned} &\phi^*\left((x-a)-(y-a)\right) \\ &= \phi^*((x-y+a)-a) \\ &= \phi(x-y+a)-\phi(a) \\ &= \phi(x)-\phi(y)+\phi(a)-\phi(a) \\ &= \phi^*(x-a)-\phi^*(y-a), \end{aligned}$$

$$\begin{aligned} &\phi^*\left((x-a)(y-a)\right) \\ &= \phi^*(x\diamond y - x\diamond a - a\diamond y + a\diamond a) \\ &= \phi(x\diamond y - x\diamond a - a\diamond y + a\diamond a + a) - \phi(a) \\ &= \phi(x\diamond y) - \phi(x\diamond a) - \phi(a\diamond y) + \phi(a\diamond a) + \phi(a) - \phi(a) \\ &= \phi(x)\diamond\phi(y) - \phi(x)\diamond\phi(a) - \phi(a)\diamond\phi(y) + \phi(a)\diamond\phi(a) \\ &= (\phi(x)-\phi(a))(\phi(y)-\phi(a)) \\ &= \phi^*(x-a)\phi^*(y-a), \end{aligned}$$

we have that ϕ^* is a ring isomorphism from the ring $M - M$ onto $N - N$ by Lemma 2.11. □

Lemma 2.13. *Let M be an affine subsemigroup of $R^\diamond$.*

(i) *If M has identity, then $M \simeq (M - M, \circ)$;*

(ii) *If M has zero, then $M \simeq (M - M, \bullet)$.*

Proof. Given $e \in (M, \diamond)$, we define $\phi : M \to M - M$ by $\phi(x) = x - e$. It is clear that $\phi(x+y-z) = \phi(x)+\phi(y)-\phi(z)$. Note that for any $x, y \in M$

$$(x-e)(y-e) = x\diamond y - x\diamond e - e\diamond y + e\diamond e. \tag{4}$$

Thus, if e is identity of M, then

$$\begin{aligned} \phi(x\diamond y) &= x\diamond y - e \\ &= (x-e)(y-e) + x + y - 2e \quad \text{(by (4))} \\ &= (x-e)\circ(y-e) \\ &= \phi(x)\circ\phi(y); \end{aligned}$$

while if e is zero of M, then by (4),

$$\phi(x \diamond y) = x \diamond y - e = (x - e)(y - e) = \phi(x)\phi(y).$$

Hence ϕ is an affine isomorphism if e is identity or zero of M. $\square$

Theorem 2.14. *Let $R^\diamond$ be a GA-semigroup of a ring R. Then*

(i) *$R^\diamond$ has identity if and only if $R^\diamond \simeq R^\circ$;*

(ii) *$R^\diamond$ has zero if and only if $R^\diamond \simeq R^\bullet$;*

(iii) *if R has identity, then $R^\diamond \simeq R^\bullet \simeq R^\circ$.*

Proof. (i) and (ii) are immediate results of Lemma 2.13. If R has 1, then $R = \Omega(R)$ and so by Corollary 2.10 there is $a \in R$ such that $x \diamond y = (x+a)(y+a) - a$ for any $x, y \in R$. Clearly, $-a$ is zero of $R^\diamond$. Thus $R^\diamond \simeq R^\bullet$ by (ii), and $R^\circ \simeq R^\bullet$ under the affine isomorphism $x \to 1 + x$ from R° onto $R^\bullet$, proving (iii). $\square$

3. GA-semigroups of π-regular rings

Recall that a semigroup S is (left, right, completely) π-regular if and only if for any $x \in S$ there exists a positive integer n such that ($x^n \in Sx^{n+1}$, $x^n \in x^{n+1}S$, $x^n \in Sx^{n+1} \cap x^{n+1}S$) $x^n \in x^nSx^n$.

For a positive integer n, a semigroup S is called (left, right, completely) π_n-regular if ($x^n \in Sx^{n+1}$, $x^n \in x^{n+1}S$, $x^n \in Sx^{n+1} \cap x^{n+1}S$) $x^n \in x^nSx^n$ for any $x \in S$. By a (left, right, completely) π_0-regular semigroup we mean a (left, right, completely) π-regular semigroup.

For a non-negative integer n, a ring is called (left, right, completely) π_n-regular if its multiplicative semigroup is (left, right, completely) π_n-regular.

In [15] we proved that the adjoint semigroup of a π-regular ring is π-regular and in [16], we proved further that the adjoint semigroup of a (left, right, completely) π_n-regular ring is (left, right, completely) π_n-regular. In this section, we will prove that this is true for GA-semigroups.

Lemma 3.1. *For any $a, b, x, y, z \in R$, we have*

$$(a - a \diamond x)z(b - y \diamond b) \in a \diamond R \diamond b - a \diamond R \diamond b.$$

Proof. Noting that $a \diamond R \diamond b$ is an affine subsemigroup of $R^\diamond$, we see that

$$\begin{aligned}
&(a - a \diamond x)z(b - y \diamond b) \\
&= (a - a \diamond x)(z - 0)(b - y \diamond b) \\
&= a \diamond z \diamond b - a \diamond z \diamond y \diamond b - a \diamond 0 \diamond b + a \diamond 0 \diamond y \diamond b - a \diamond x \diamond z \diamond b \\
&\quad + a \diamond x \diamond z \diamond y \diamond b + a \diamond x \diamond 0 \diamond b - a \diamond x \diamond 0 \diamond y \diamond b \quad \text{(by Lemma 2.3)} \\
&\in a \diamond R \diamond b - a \diamond R \diamond b, \quad \text{(by Lemma 2.11)}
\end{aligned}$$

completing the proof. $\square$

Lemma 3.2. *Let $\mathcal{A} = b \diamond R \diamond c - b \diamond R \diamond c$. If x commutes with c in $R^\diamond$, then $a - a \diamond x \in \mathcal{A}$ implies $a - a \diamond x^{[n]} \in \mathcal{A}$ for any positive integer n.*

Proof. To prove the lemma, we proceed with an induction on n. It is trivial for $n = 1$. Assume $n > 1$ and $a - a \diamond x^{[n-1]} \in \mathcal{A}$. Let $a - a \diamond x^{[n-1]} = b \diamond y \diamond c - b \diamond z \diamond c$. Then multiplication (with respect to $\diamond$) by x on the right shows that

$$a \diamond x - a \diamond x^{[n]} = b \diamond y \diamond x \diamond c - b \diamond z \diamond x \diamond c,$$

whence by Lemma 2.11

$$\begin{aligned} a - a \diamond x^{[n]} &= a - a \diamond x + a \diamond x - a \diamond x^{[n]} \\ &= a - a \diamond x + b \diamond y \diamond x \diamond c - b \diamond z \diamond x \diamond c \\ &\in \mathcal{A}, \end{aligned}$$

as desired. □

Lemma 3.3. *Let a and x commute with each other in $R^\diamond$. Then for any positive integers m and n we have that*

$$(a - a^{[m]} \diamond x)^n = a^{[n]} - a^{[n+m-1]} \diamond y,$$

for some y commuting with a and x in $R^\diamond$.

Proof. By Corollary 2.4,

$$\begin{aligned} &(a - a^{[m]} \diamond x)^n \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a^{[i]} \diamond (a^{[m]} \diamond x)^{[n-i]} \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a^{[i+m(n-i)]} \diamond x^{[n-i]} \\ &= a^{[n]} - \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} a^{[n+m-1]} \diamond a^{[(m-1)(n-1-i)]} \diamond x^{[n-i]} \\ &= a^{[n]} - a^{[n+m-1]} \diamond \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} \left(a^{[(m-1)(n-1-i)]} \diamond x^{[n-i]}\right), \end{aligned}$$

since $\sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} = 1$. Let

$$y = \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} \left(a^{[(m-1)(n-1-i)]} \diamond x^{[n-i]}\right).$$

Then $(a - a \diamond x)^n = a^{[n]} - a^{[n+m-1]} \diamond y$ and it is clear that y commutes with both a and x. □

Lemma 3.4. *Let $a, w, x, y, z \in R$ such that x, y and z commute with a in $R^\diamond$, and let n be a positive integer, and k and m be non-negative integers not all zero. If*

$$(a - a \diamond x \diamond a)^n = (a - a \diamond y)^k w (a - z \diamond a)^m,$$

then $a^{[n]} = a^{[k]} \diamond u \diamond a^{[m]}$ for some $u \in R$.

Proof. Let $\mathcal{A} = a^{[k]} \diamond R \diamond a^{[m]} - a^{[k]} \diamond R \diamond a^{[m]}$. Then by Lemma 3.3 and Lemma 3.1, we have

$$a^{[n]} - a^{[n+1]} \diamond r = (a^{[k]} - a^{[k]} \diamond s) w (a^{[m]} - t \diamond a^{[m]}) \in \mathcal{A}$$

for some $r, s, t \in R$ commuting with a. By Lemma 3.2, $a^{[n]} - a^{[n+k+m]} \diamond r \in \mathcal{A}$. Let $a^{[n]} - a^{[n+k+m]} \diamond r = a^{[k]} \diamond b \diamond a^{[m]} - a^{[k]} \diamond c \diamond a^{[m]}$. Then we have that

$$a^{[n]} = a^{[n+k+m]} \diamond r + a^{[k]} \diamond b \diamond a^{[m]} - a^{[k]} \diamond c \diamond a^{[m]} = a^{[k]} \diamond (a^{[n]} \diamond r + b - c) \diamond a^{[m]},$$

as desired. □

Theorem 3.5. *For a non-negative integer n, if a ring R is (left, right, completely) π_n-regular, then so is its any GA-semigroup.*

Proof. Let $R^\diamond$ be a GA-semigroup of R. If R be a right π_n-regular ring for $n \geq 1$, then for any $x \in R$, there exist $y \in R$ such that $(x - x^{[3]})^n = (x - x^{[3]})^{n+1} y$. From Lemma 3.4, we deduce that $x^{[n]} = x^{[n+1]} \diamond z$ for some $z \in R$, whence $(R, \diamond)$ is a right π_n-regular semigroup. The remainder can be proved similarly. □

4. GA-semigroups with idempotents

Let $R^\diamond$ be a GA-semigroup of R. Then $R^\diamond$ is called (centrally) 0-idempotent if the additive 0 of R is an (central) idempotent in $R^\diamond$. Let $R^\diamond$ be a 0-idempotent GA-semigroup induced by the associated pair (θ, ϑ). Then it is clear that $\vartheta = 0$ and so θ is idempotent. One should note that (centrally) 0-idempotent is not an affine isomorphism invariant.

Lemma 4.1. *Every GA-semigroup containing (central) idempotents is affinely isomorphic to a (centrally) 0-idempotent one.*

Proof. Suppose $R^\diamond$ is a GA-semigroup containing an (central) idempotent e. Let $R_e = (R, \boxplus, *)$ with

$$\begin{aligned} x \boxplus y &= x + y - e, \\ x * y &= (x - e)(y - e) + e, \end{aligned}$$

for any $x, y \in R$. Then R_e is a ring in which e acts as additive zero and $*$ is clearly an associative binary operation on R_e. Denote by $\boxminus$ the minus in R_e. Noting that

$x+y-z = x \boxplus y \boxminus z$ for any $x, y, z \in R$, we see that the operation $\diamond$ satisfies the generalized distributive laws in R_e, and further we have that

$$\begin{aligned} x * y &= (x-e)(y-e)+e \\ &= x \diamond y - x \diamond e - e \diamond y + e \diamond e + e \\ &= x \diamond y \boxminus x \diamond e \boxminus e \diamond y \boxplus e \diamond e \boxplus e \\ &= x \diamond y \boxminus x \diamond e \boxminus e \diamond y \boxplus e \diamond e. \end{aligned}$$

Thus $\diamond$ is a GA-multiplication on the ring R_e such that $R_e^\diamond$ is (centrally) 0-idempotent. It is easy to see that the identity mapping of R is an affine isomorphism from $R^\diamond$ onto $R_e^\diamond$. $\square$

Given two rings S and T, two bimodules ${}_SU_T$ and ${}_TV_S$, an S-S-homomorphism $\phi : U \otimes_T V \to S$ and a T-T-homomorphism $\psi : V \otimes_S U \to T$ (write uv for $\phi(u \otimes v)$ and vu for $\psi(v \otimes u)$) such that $u(vu') = (uv)u'$ and $v(uv') = (vu)v'$ for any $u, u' \in U$ and $v, v' \in V$. Let $R = \begin{pmatrix} S & U \\ V & T \end{pmatrix}$ be the set of formal matrices. Then R is a ring with the usual matrix operations, called the ring of the Morita context, or a Morita ring, and denoted by $\mathcal{M}(S,T,U,V)$. Denote by $\tilde{S}$ and $\tilde{T}$ the Dorroh extension of S and T, respectively. Then ${}_{\tilde{S}}U_{\tilde{T}}$ and ${}_{\tilde{T}}V_{\tilde{S}}$ are unitary bimodules in a natural way. Let $\tilde{R} = \begin{pmatrix} \tilde{S} & U \\ V & \tilde{T} \end{pmatrix}$. Then $\tilde{R}$ is a unitary ring with the usual matrix operations and R is an ideal of $\tilde{R}$. Let $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \tilde{R}$. Then the generalized adjoint multiplication induced by E_{11} is given by

$$\begin{aligned} A \diamond B &= AB + AE_{11} + E_{11}B \\ &= (A + E_{11})(B + E_{11}) - E_{11} \\ &= \begin{pmatrix} s \circ s' + uv' & (1+s)u + ut' \\ u(1+s') + tv' & uu' + tt' \end{pmatrix} \end{aligned}$$

for any $A = \begin{pmatrix} s & u \\ v & t \end{pmatrix}, B = \begin{pmatrix} s' & u' \\ v' & t' \end{pmatrix} \in R$. The semigroup $R^\diamond$ is called the E_{11}-GA-semigroup of R, denoted by $\mathcal{M}_{11}^\diamond(S,T,U,V)$. It is clear that the E_{11}-GA-semigroup $\mathcal{M}_{11}^\diamond(S,T,U,V)$ is 0-idempotent.

Lemma 4.2. *Let $R^\diamond$ be a 0-idempotent GA-semigroup induced by an idempotent self-permutable bitranslation θ, and let $R_{11} = \theta R \theta$, $R_{10} = \theta R(1-\theta)$, $R_{01} = (1-\theta)R\theta$, and $R_{00} = (1-\theta)R(1-\theta)$. Then*

(i) *$R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00}$ as additive groups;*

(ii) *$R_{ij}R_{kl} \subset \delta_{jk}R_{jl}$, where δ_{jk} is the Kronecker delta, $i, j, k, l = 0, 1$;*

(iii) *if we write $x = \sum x_{ij}$, $y = \sum y_{ij}$, where $x_{ij}, y_{ij} \in R_{ij}$, $i, j = 0, 1$, then*

$$\begin{aligned} x \diamond y = &(x_{11} \circ y_{11} + x_{10}y_{01}) + (x_{10} + x_{11}x_{10} + x_{10}y_{00}) \\ &+ (x_{01} + x_{01}y_{11} + x_{00}y_{01}) + (x_{01}y_{10} + x_{00}y_{00}); \end{aligned}$$

(iv) R_{ij}, $i,j=0,1$, *are subrings of* R *such that* $R_{11}^{\diamond}=R_{11}^{\circ}$, $R_{00}^{\diamond}=R_{00}^{\bullet}$, $R_{10}^{\diamond}$ *is a right zero semigroup, and* $R_{01}^{\diamond}$ *is a left zero semigroup.*

Proof. Since θ is idempotent, the proof of (i) and (ii) is essentially similar to that of Pierce decomposition of a ring. For $x=\sum x_{ij}$, $y=\sum y_{ij}$, where $x_{ij},y_{ij}\in R_{ij}$, $i,j=0,1$, we have by (ii) that

$$\begin{aligned}
x\diamond y &= \left(\sum x_{ij}\right)\left(\sum y_{ij}\right)+\theta\left(\sum x_{ij}\right)+\left(\sum y_{ij}\right)\theta\\
&= \left(\sum x_{ij}y_{kl}\right)+x_{11}+x_{10}+y_{11}+y_{01}\\
&= (x_{11}\circ y_{11}+x_{10}y_{01})+(y_{10}+x_{11}y_{10}+x_{10}y_{00})\\
&\quad+(x_{01}+x_{01}y_{11}+x_{00}y_{01})+(x_{01}y_{10}+x_{00}y_{00}),
\end{aligned}$$

proving (iii). If $x,y\in R_{11}$, then

$$x\diamond y=xy+x\theta+\theta y=xy+x+y=x\circ y,$$

whence $R_{11}^{\diamond}=R_{11}^{\circ}$, and similarly, $R_{00}^{\diamond}=R_{00}^{\bullet}$. For any $x,y\in R_{10}$, we have by (ii) that

$$x\diamond y=xy+x\theta+\theta y=y,$$

which implies that $R_{10}^{\diamond}$ is a right zero semigroup, and similarly $R_{01}^{\diamond}$ is a left zero semigroup, proving (iv). □

Theorem 4.3. *Let* $R^{\diamond}$ *be a GA-semigroup of* R. *If* $R^{\diamond}$ *contains idempotents, then there exists a Morita ring* $\mathcal{M}(S,T,U,V)$ *such that* $R\cong\mathcal{M}(S,T,U,V)$ *and* $R^{\diamond}\simeq\mathcal{M}_{11}^{\diamond}(S,T,U,V)$.

Proof. Let $R^{\diamond}$ be a GA-semigroup induced by the associated pair (θ,ϑ). If $R^{\diamond}$ contains idempotents, then by Lemma 4.1, without loss of generality, we may assume that $R^{\diamond}$ is 0-idempotent. By Lemma 4.2, it is a routine matter to verify that $\mathcal{M}(R_{11},R_{00},R_{10},R_{01})$ is a Morita ring in a natural way. By Lemma 4.2 straightforward computation shows that the mapping $\phi: R\to\mathcal{M}(R_{11},R_{00},R_{10},R_{01})$ defined by

$$\phi(x)=\begin{pmatrix}\theta x\theta & \theta x(1-\theta)\\ (1-\theta)x\theta & (1-\theta)x(1-\theta)\end{pmatrix}$$

is a ring isomorphism. Noting that

$$\begin{aligned}
\phi(x\diamond y) &= \phi(xy+x\theta+\theta y)\\
&= \phi(x)\phi(y)+\phi(x\theta)+\phi(\theta y)\\
&= \phi(x)\phi(y)+\begin{pmatrix}\theta x\theta & 0\\ (1-\theta)x\theta & 0\end{pmatrix}+\begin{pmatrix}\theta y\theta & (1-\theta)y\theta\\ 0 & 0\end{pmatrix}\\
&= \phi(x)\diamond\phi(y),
\end{aligned}$$

we see that ϕ is an affine isomorphism from $R^{\diamond}$ onto the E_{11}-GA-semigroup of $\mathcal{M}(R_{11},R_{00},R_{10},R_{01})$. □

Corollary 4.4. *A GA-semigroup $R^\diamond$ is (centrally) 0-idempotent if and only if there exists an ideal extension $\tilde{R}$ with 1 of R and an idempotent $\varepsilon \in \tilde{R}$ (commuting with elements of R) such that $x \diamond y = (x+\varepsilon)(y+\varepsilon) - \varepsilon$ for any $x, y \in R$.*

Proof. It follows from Theorem 4.3, the definition of the E_{11}-GA-semigroup and taking $\varepsilon = E_{11}$. □

Lemma 4.5. *If $(a - a^{[2]})^2 = 0$, then there exists an idempotent $e = \sum p_i a^{[i]}$ with $\sum p_i = 1$ such that $a^{[2]} = e \diamond a^{[2]}$.*

Proof. By Corollary 2.4, $(a - a^{[2]})^2 = a^{[2]} - 2a^{[3]} + a^{[4]}$, and so

$$a^{[2]} = 2a^{[3]} - a^{[4]} = a^{[2]} \diamond (2a - a^{[2]}) = a^{[2]} \diamond (2a - a^{[2]})^{[2]} = a^{[2]} \diamond (2a - a^{[2]})^{[3]}.$$

Note that by Corollary 2.8,

$$(2a - a^{[2]})^{[3]} = 8a^{[3]} - 12a^{[4]} + 6a^{[5]} - a^{[6]} = a^{[2]} \diamond (8a - 12a^{[2]} + 6a^{[3]} - a^{[4]}).$$

Let $b = 8a - 12a^{[2]} + 6a^{[3]} - a^{[4]}$. Then b commutes with a and $a^{[2]} = a^{[2]} \diamond b \diamond a^{[2]}$. Let $e = a^{[2]} \diamond b$. Then it is clear that e is an idempotent of $R^\diamond$ such that $a^{[2]} = e \diamond a^{[2]}$. □

Let $\Gamma(R) = \{\theta \in \Omega(R) \,|\, \theta x = x\theta \text{ for any } x \in R\}$.

Lemma 4.6. *A GA-semigroup of R induced by (θ, ϑ) has (central) idempotents if and only if θ can be lifted to an idempotent of $\Omega(R)$ (contained in $\Gamma(R)$).*

Proof. Assume semigroup $R^\diamond$ has an idempotent e. Then

$$e = e \diamond e = e^2 + e\theta + \theta e + \vartheta,$$

whence $\pi_e = \pi_e^2 + \pi_e\theta + \theta\pi_e + \pi_\vartheta = \pi_e^2 + \pi_e\theta + \theta\pi_e + \theta^2 - \theta = (\pi_e + \theta)^2 - \theta$. Thus $\pi_e + \theta$ is idempotent. Moreover, if e is central in $R^\diamond$, then $e \diamond x = x \diamond e$ for any $x \in R$, that is, $ex + e\theta + \theta x + \vartheta = xe + x\theta + \theta e + \vartheta$, and particularly, $e\theta = \theta e$ by taking $x = 0$. Thus $(\pi_e + \theta)x = ex + \theta x = xe + x\theta = x(\pi_e + \theta)$, yielding $\pi_e + \theta \in \Gamma(R)$.

Assume θ can be lifted to an idempotent of $\Omega(R)$. Then $\pi_a + \theta$ is idempotent for some $a \in R$, whence $\pi_a = \pi_a^2 + \pi_a\theta + \theta\pi_a + \theta^2 - \theta = \pi_a^2 + \pi_a\theta + \theta\pi_a + \pi_\vartheta$. Thus we have $ax = a^2x + (a\theta)x + (\theta a)x + \vartheta x = a^{[2]}x$, forcing $(a - a^{[2]})R = 0$. In particularly, $(a - a^{[2]})^2 = 0$, whence $R^\diamond$ contains an idempotent $e = \sum p_i a^{[i]}$ with $\sum p_i = 1$ by Lemma 4.5. Further, if $\pi_a + \theta$ is an idempotent contained in $\Gamma(R)$. Then for any $x \in R$, $(\pi_a + \theta)x = x(\pi_a + \theta)$, that is, $ax + \theta x = xa + x\theta$, and particularly $\theta a = a\theta$ by taking $x = a$, whence

$$a \diamond x = ax + \theta x + a\theta + \vartheta = xa + x\theta + \theta a + \vartheta = x \diamond a.$$

Hence $e \diamond x = x \diamond e$, that is, e is a central idempotent of $R^\diamond$. □

Theorem 4.7. *Consider the following conditions:*

(i) *every GA-semigroup of R contains (central) idempotents;*

(ii) *in any ideal extension* $\tilde{R}$ *of* R*, idempotents of* $\tilde{R}/R$ *can be lifted to idempotents of* $\tilde{R}$ *(contained in the centralizer of* R *in* $\tilde{R}$*);*

(iii) *idempotents of* $\Omega(R)/\pi(R)$ *can be lifted to idempotents of* $\Omega(R)$ *(contained in* $\Gamma(R)$*). Then* (iii)⇒(i)⇒(ii)*. Moreover, if* $Ann(R) = 0$*, then* (i), (ii) *and* (iii) *are equivalent.*

Proof. (iii)⇒(i) follows from Lemma 4.6.
(i)⇒(ii): If $a \in \tilde{R}$ and $a^2 - a \in R$, then the pair (θ, ϑ) defined by

$$\theta x = ax, x\theta = xa, \text{ and } \vartheta = a^2 - a$$

is an associated pair and so $x \diamond y = xy + xa + ay + a^2 - a$ defines a GA-multiplication on R. If e is an idempotent of $R^\diamond$, then $e = e^2 + ea + ae + a^2 - a = (e + a)^2 - a$, and so $e + a$ is an idempotent of $\tilde{R}$. Further if e is a central idempotent of $R^\diamond$, then $e \diamond x = x \diamond e$ for any $x \in R$, that is

$$ex + ea + ax + \vartheta = xe + xa + ae + \vartheta,$$

and particularly, $ea = ae$ by taking $x = 0$. Thus $(e + a)x = ex + ax = xe + xa = x(e + a)$, which implies that $e + a$ is contained in the centralizer of R in $\tilde{R}$.

The remainder is clear. □

The following corollary is independently interesting, which is a generalization of a classical result in ring theory which states that idempotents modulo a nil ideal can be lifted ([28]) and is a generalization of ring-theoretic analogue of a result of Edwards ([19, Corollary 2]) which extends the well-known Lallement's lemma to eventually regular semigroups (i.e., π-regular semigroups).

Theorem 4.8. *In any ring, idempotents modulo a* π*-regular ideal can be lifted.*

Proof. By Theorem 3.5, any GA-semigroup of a π-regular ring contains idempotent, and so by Theorem 4.7 idempotents modulo a π-regular ideal can be lifted. □

If R is a ring with ECI, then idempotents can be lifted from $\Omega(R)/R$ to $\Omega(R)$ ([7, Corollary 3.6]), and so any GA-semigroup of R contains idempotents by Theorem 4.7. Particularly, every GA-semigroup of a biregular ring contains idempotents. On the other hand, there is a ring such that idempotents modulo the radical cannot be lifted. Hence a GA-semigroup of a radical ring need not contain idempotents.

A semigroup S is called completely primitive if the left ideal Se and the right ideal eS are minimal for every idempotent e of S ([6]). A completely primitive semigroup S has kernel which is completely simple and contains all of idempotents of S ([9]).

Lemma 4.9. *Let* $R^\diamond$ *be a GA-semigroup of a radical ring* R*. If* $R^\diamond$ *contains idempotents, then* $R^\diamond$ *is completely primitive.*

Proof. Let e be an idempotent of $R^\diamond$. Then it is sufficient to prove that $e \diamond R \diamond e$ is a group. Since $e \diamond R \diamond e \simeq (e \diamond R \diamond e - e \diamond R \diamond e, \circ)$ by Lemma 2.11 and Lemma 2.13, we have to prove that $e \diamond R \diamond e - e \diamond R \diamond e$ is a radical ring. By Corollary 4.4, there are an ideal extension $\tilde{R}$ of R and an idempotent $\varepsilon \in \tilde{R}$ such that $x \diamond y = (x + \varepsilon)(y + \varepsilon) - \varepsilon$ for any $x, y \in R$. Thus $e \diamond R \diamond e - e \diamond R \diamond e = (e+\varepsilon)(R+\varepsilon)(e+\varepsilon) - (e+\varepsilon)(R+\varepsilon)(e+\varepsilon) = (e+\varepsilon)R(e+\varepsilon)$. Since $e \diamond e = e$, we have that $e+\varepsilon$ is an idempotent of $\tilde{R}$ and so it is easy to see that $(e+\varepsilon)R(e+\varepsilon)$ is a radical ring since R is a radical ring. □

Lemma 4.9 is a GA-semigroup version of [18, Theorem 1 (b)–(c)]. Actually, many results in [18] can be reexplained in terms of GA-semigroup.

Theorem 4.10. *Any GA-semigroup of a nil ring is a completely primitive π-regular semigroup.*

Proof. It follows from Theorem 3.5 and Lemma 4.9. □

Theorem 4.11. *Let R be a ring with descending chain condition for principal right ideals. Then any GA-semigroup of R is completely π-regular. Particularly, any GA-semigroup of a right Artinian ring is completely π-regular.*

Proof. If R is a ring with descending chain condition for principal right ideals, then R is completely π-regular by Dischinger [12, Theorem 1] and Azumaya [2, Lemma 1]. □

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