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Biharmonic Anti-invariant Submanifolds in Sasakian Space Forms *

Kadri Arslan Ridvan Ezentas Cengizhan Murathan Toru Sasahara

Department of Mathematics, Faculty of Science, Uludag University, Bursa, Turkey
e-mail: arslan@uludag.edu.tr rezentas@uludag.edu.tr cengiz@uludag.edu.tr

Department of Mathematics, Oita National College of Technology
1666 Maki, Oita, 870-0150, Japan
e-mail: t-sasa@oita-ct.ac.jp

Abstract. We obtain some classification results and the stability conditions of nonminimal biharmonic anti-invariant submanifolds in Sasakian space forms.

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1. Introduction

The study of submanifolds in contact metric manifolds from Riemannian geometric point of view was initiated in 1970's and it is a very active field during the last quarter of century. In contact metric manifolds there are two polar submanifolds tangent to Reeb vector field: invariant submanifolds and anti-invariant submanifolds [18]. Invariant submanifolds are minimal (see, [1]) and hence automatically critical points of the 2-energy functional, that is, biharmonic (2-harmonic) in the sense of Eells and Sampson [8]. However, anti-invariant submanifolds are not so in general. Thus, it is natural and interesting to investigate the class of nonminimal biharmonic anti-invariant submanifolds in contact metric manifold.

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A Sasakian space form is regarded as an odd dimensional analogue of a complex space form, and therefore it is among the most important contact metric manifolds. Many interesting results on submanifolds in a Sasakian space form have been obtained by many differential geometers. The purpose of this paper is to obtain the following:

(i) the existence and uniqueness theorems of nonminimal biharmonic anti-invariant submanifolds in Sasakian space forms of low dimension,

(ii) the stability conditions of nonminimal biharmonic anti-invariant submanifolds in Sasakian space forms of general dimension.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold N^{2n+1} is called a *contact manifold* if there exists a globally defined 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. On a contact manifold there exists a unique global vector field ξ satisfying

$$d\eta(\xi, X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

for all $X \in TN^{2n+1}$. The vector field ξ is called *Reeb vector field*.

Moreover it is well-known that there exist a tensor field ϕ of type $(1,1)$, a Riemannian metric g which satisfy

$$\begin{gathered} \phi^2 = -I + \eta \otimes \xi, \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(\xi, X) = \eta(X), \\ d\eta(X,Y) = g(X, \phi Y), \end{gathered} \tag{2.2}$$

for all $X, Y \in TN^{2n+1}$ (see, for instance, [1]).

The structure (ϕ, ξ, η, g) is called *contact metric structure* and the manifold N^{2n+1} with an contact metric structure is said to be a *contact metric manifold*.

A contact metric manifold is said to be a *Sasakian manifold* if it satisfies $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on N^{2n+1}, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. On Sasakian manifolds, we have

$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.3}$$

$$\bar{\nabla}_X \xi = -\phi X, \tag{2.4}$$

for any vector fields X and Y, where $\bar{\nabla}$ is the Levi-Civita connection of N^{2n+1}. The tangent planes in T_pN^{2n+1} which is invariant under ϕ are called *ϕ-section* (see, [1]). The sectional curvature of ϕ-section is called ϕ-sectional curvature.

If the ϕ-sectional curvature is constant on N^{2n+1}, then N^{2n+1} is said to be of *constant ϕ-sectional curvature.*

Complete and connected Sasakian manifolds of constant ϕ-sectional curvature are called *Sasakian space forms.* Denote Sasakian space forms of constant ϕ-sectional curvature c by $N^{2n+1}(c)$. The curvature tensor $\bar{R}$ of $N(c)$ is given by

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(Z,X)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad +g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\}. \qquad (2.5)\end{aligned}$$

Let M^m be a submanifold tangent to ξ. If $\phi(TM^m) \subset T^{\perp}M^m$, then M^m is called an *anti-invariant* submanifold. If $\phi(TM^m) \subset TM^m$, then M^m is said to be an *invariant* submanifold.

If η restricted to M^m vanishes, then M^m is called an *integral submanifold*, in particular if $m = n$, it is called a *Legendre submanifold*.

Let $x : M^m \to N^{2n+1}$ be an isometric immersion. Denote the Levi-Civita connection of N^{2n+1} (resp. M^m) by $\bar{\nabla}$ (resp. ∇). The formulas of Gauss and Weingarten are given respectively by

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + h(X,Y), \\ \bar{\nabla}_X V &= -A_V X + D_X V,\end{aligned} \qquad (2.6)$$

where $X, Y \in TM^m$, $V \in T^{\perp}M^m$. Here h, A and D are the second fundamental form, the shape operator and the normal connection, respectively. The following relation holds:

$$\langle A_V X, Y\rangle = \langle h(X,Y), V\rangle, \qquad (2.7)$$

where $\langle , \rangle := g(,)$.

The mean curvature vector H is given by $H = \frac{1}{m}\mathrm{trace}\, h$. If $H = 0$ at any point, M^m is called *minimal*. The allied mean curvature vector is defined by $a(H) = \sum_{r=m+1}^{2n+1} \mathrm{tr}(A_H A_{V_r})V_r$, where $\{V_r\}$ are mutually orthogonal normal vector fields. If M^m satisfies $a(H) \equiv 0$, then it is called *Chen submanifold*.

Denote by R the Riemann curvature tensor of M^m. Then the equations of Gauss, Codazzi and Ricci are given respectively by

$$\langle R(X,Y)Z, W\rangle = \langle A_{h(Y,Z)}X, W\rangle - \langle A_{h(X,Z)}Y, W\rangle + \langle \bar{R}(X,Y)Z, W\rangle, \quad (2.8)$$

$$(\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \qquad (2.9)$$

$$\langle R^D(X,Y)V_1, V_2\rangle = \langle \bar{R}(X,Y)V_1, V_2\rangle + \langle [A_{V_1}, A_{V_2}](X), Y\rangle, \qquad (2.10)$$

where X, Y, Z, W (resp. V_1 and V_2) are vectors tangent (resp. normal) to M^m, $R^D(X,Y) = [D_X, D_Y] - D_{[X,Y]}$, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \qquad (2.11)$$

Hereafter, submanifolds and immersions mean isometrically immersed manifolds and isometric immersions, respectively.

The dimension of anti-invariant submanifolds in contact metric $(2n+1)$-manifolds is less than or equal to $n+1$ (see, [18]). In general, the study of anti-invariant submanifolds is difficult. However in case the dimension is maximum, we do have some good properties as the study of Legendre submanifolds. In fact, we have the following existence and uniqueness theorems for anti-invariant $(n+1)$-submanifolds in Sasakian space form $N^{2n+1}(c)$ (see, [2]):

Theorem 1. *Let $(M^{n+1}, \langle\cdot,\cdot\rangle)$ be an $(n+1)$-dimensional simply connected Riemannian manifold. Suppose that there exist an unit global vector field ξ on M and a symmetric bilinear TM-valued form α on M such that for X, Y, Z, $W \in TM$, we have*

$$\langle\alpha(X,Y),\xi\rangle = 0, \quad \nabla_X\xi = 0, \tag{2.12}$$

and the equations

$$\alpha(X,\xi) = X - \eta(X)\xi, \tag{2.13}$$

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \langle\alpha(X,W),\alpha(Y,Z)\rangle - \langle\alpha(X,Z),\alpha(Y,W)\rangle \\ &+\frac{c+3}{4}\{\langle Y,Z\rangle\langle X,W\rangle - \langle Z,X\rangle\langle Y,W\rangle\} + \frac{c-1}{4}\{\eta(X)\eta(Z)\langle Y,W\rangle \\ &-\eta(Y)\eta(Z)\langle X,W\rangle + \eta(Y)\eta(W)\langle X,Z\rangle - \eta(X)\eta(W)\langle Y,Z\rangle\},\end{aligned} \tag{2.14}$$

$$\langle\alpha(X,Y),Z\rangle - \langle\alpha(X,Z),Y\rangle + \langle X,Y\rangle\eta(Z) - \langle X,Z\rangle\eta(Y) = 0, \tag{2.15}$$

$$(\nabla_X\alpha)(Y,Z) = (\nabla_Y\alpha)(X,Z) \tag{2.16}$$

are satisfied, where η denotes the dual 1-form of ξ. Then there exists an anti-invariant immersion into a Sasakian space form $x : (M^{n+1}, \langle\cdot,\cdot\rangle) \to N^{2n+1}(c)$ whose second fundamental form h satisfies $h(X,Y) = -\phi\alpha(X,Y)$.

Theorem 2. *Let $x^1, x^2 : M^{n+1} \to N^{2n+1}(c)$ be two anti-invariant immersions of a connected Riemannian $(n+1)$-manifold into a Sasakian manifold $N^{2n+1}(c)$ with second fundamental form h^1 and h^2. If there is a vector field $\bar{\xi}$ on M^{n+1} such that $x^i_{*p}(\bar{\xi}) = \xi_{x^i(p)}$ for any i and $p \in M^{n+1}$ and that*

$$\langle h^1(X,Y),\phi x^1_*Z\rangle = \langle h^2(X,Y),\phi x^2_*Z\rangle$$

for all vector fields X, Y, Z tangent to M^{n+1}, there exists an isometry A of $N^{2n+1}(c)$ such that $x^1 = A \circ x^2$.

3. Biharmonic maps

Let (M^m, g) and $(N^n, \tilde{g})$ be Riemannian manifolds and $f : M^m \to N^n$ a smooth map. The *tension field* $\tau(f)$ of f is a section of the vector bundle f^*TN^n defined by

$$\tau(f) := \mathrm{tr}(\nabla^f df) = \sum_{i=1}^{m}\{\nabla^f_{e_i}df(e_i) - df(\nabla_{e_i}e_i)\},$$

where ∇^f, ∇ and $\{e_i\}$ denote the induced connection, the connection of M^m and a local orthonormal frame field of M^m respectively.

A smooth map f is said to be a *harmonic map* if its tension field vanishes. It is well-known that f is harmonic if and only if f is a critical point of the *energy*:

$$E(f|_\Omega) = \int_\Omega \sum_{i=1}^{m} \tilde{g}(df(e_i), df(e_i))dv_g$$

over every compact domain Ω of M^m. Here dv_g denotes the volume form of g.

J. Eells and J. H. Sampson [8] suggested to study 2-harmonic maps which are critical points of 2-energy E_2:

$$E_2(f|_\Omega) = \int_\Omega \tilde{g}(\tau(f), \tau(f))dv_g.$$

If f is an isometric immersion, the functional E_2 is given by

$$E_2(f|_\Omega) = m^2 \int_\Omega \tilde{g}(H, H)dv_g,$$

where H is the mean curvature vector field.

The Euler-Lagrange equation of the functional E_2 was computed by Jiang [11] as follows:

$$\mathcal{J}_f(\tau(f)) = 0. \tag{3.1}$$

Here the operator $\mathcal{J}_f$ is the *Jacobi operator of harmonic maps* defined by

$$\mathcal{J}_f(V) := \bar{\Delta}_f V - \mathcal{R}_f(V), \quad V \in \Gamma(f^*TN^n), \tag{3.2}$$

$$\bar{\Delta}_f := -\sum_{i=1}^{m}(\nabla^f_{e_i}\nabla^f_{e_i} - \nabla^f_{\nabla_{e_i}e_i}), \mathcal{R}_f(V) := \sum_{i=1}^{m} R^{N^n}(V, df(e_i))df(e_i), \tag{3.3}$$

where R^{N^n} is the curvature tensor of N^n.

In particular, if N^n is the Euclidean n-space $\mathbf{E}^n$ and $f = (x_1, \ldots, x_n)$ is an isometric immersion, then

$$\mathcal{J}_f(\tau(f)) = (-\Delta_M\Delta_M x_1, \ldots, -\Delta_M\Delta_M x_n),$$

where Δ_M is the Laplace operator acting on $C^\infty(M^m)$. Thus the 2-harmonicity for an isometric immersion into Euclidean space is equivalent to the biharmonicity in the sense of Chen (see [6]). For this reason, 2-harmonic maps are frequently called *biharmonic maps*. Nonharmonic biharmonic maps are said to be *proper*.

Remark 3. It is natural and interesting to investigate isometric immersions which attain the least value of E_2 for given two Riemannian manifolds M and N. B.-Y. Chen [6] introduced new Riemannian invariants and established inequalities between the new invariants and $|H|^2$. Isometric immersions satisfying the equality case of Chen's inequalities are the ones which attain the least value of E_2. In [13], the fourth author studied CR-immersions into complex hyperbolic spaces satisfying the equality case of Chen's inequalities.

Here we would like to exhibit a known result on biharmonic Legendre submanifolds in the unit sphere.

Consider the complex Euclidean $(n+1)$-space $\mathbf{C}^{n+1}$ and identify $z = (x_1 + iy_1, \ldots, x_{n+1} + iy_{n+1}) \in \mathbf{C}^{n+1}$ with $(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) \in \mathbf{E}^{2n+2}$. Let J be its usual almost complex structure. It is well-known that a Sasakian space form $N^{2n+1}(1)$ is isomorphic to $S^{2n+1}(1)$ endowed with the Sasakian structure induced by J of $\mathbf{C}^{n+1}$. (For example, see [1].)

There are no proper biharmonic Legendre curves in $S^3(1)$ (cf. [3], [9]). On the other hand, in [17] the author explicitly determined biharmonic Legendre surfaces in $S^5(1)$.

Proposition 4. [17] *Let $f : M^2 \to S^5(1) \subset \mathbf{C}^3$ be a proper biharmonic Legendre immersion. Then the position vector $f = f(x,y)$ of M^2 in $\mathbf{C}^3$ is given by*

$$f(x,y) = \frac{1}{\sqrt{2}}(e^{ix}, ie^{-ix}\sin\sqrt{2}y, ie^{-ix}\cos\sqrt{2}y). \tag{3.4}$$

Let $f : M \to \mathbf{E}^n$ be an isometric immersion. If the position vector f can be written as

$$f = f_1 + f_2, \quad \Delta_M f_1 = \lambda_1 f_1, \quad \Delta_M f_2 = \lambda_2 f_2,$$

for two different constants λ_1 and λ_2, then f is said to be of 2-*type* (see, [5]). We put

$$\begin{aligned} f_1(x,y) &:= \frac{1}{\sqrt{2}}(e^{ix}, 0, 0), \\ f_2(x,y) &:= \frac{1}{\sqrt{2}}(0, ie^{-ix}\sin\sqrt{2}y, ie^{-ix}\cos\sqrt{2}y). \end{aligned}$$

Then we have $f = f_1 + f_2$, $\Delta_M f_1 = f_1$ and $\Delta_M f_2 = 3f_2$. Thus (3.4) is of 2-type.

Now, put $g_1(x) = (\cos x, \sin x)$ and $g_2(y) = \frac{1}{\sqrt{2}}(1, \sin\sqrt{2}y, \cos\sqrt{2}y) \in S^2(1)$. Then we see that $f(x,y)$ can be written as $f(x,y) = g_1 \otimes g_2$ (for more details on the tensor product immersions, see [7]). We remark that g_2 is proper biharmonic in $S^2(1)$ (see, [4]).

We shall show how to construct new examples of proper biharmonic submanifolds from proper biharmonic submanifolds and minimal submanifolds in the unit sphere by using tensor product immersions.

Proposition 5. *Let $g_1 : M^m \to S^{p-1}(1)$ and $g_2 : N^n \to S^{q-1}(1)$ be isometric immersions. The tensor product immersion $g_1 \otimes g_2 : M^m \times N^n \to S^{pq-1}(1)$ is proper biharmonic if and only if one of g_1 and g_2 is proper biharmonic and the other is minimal.*

Proof. Let $\bar{H}_1$, $\bar{H}_2$, $\bar{H}$ be the mean curvature vector fields of g_1, g_2, $g_1 \otimes g_2$ in $\mathbf{E}^p$, $\mathbf{E}^q$, $\mathbf{E}^{pq}$, respectively. We put

$$BIH_1 = \bar{\Delta}_{g_1}\bar{H}_1 - 2m\bar{H}_1 - m(2 - |\bar{H}_1|^2)g_1, \tag{3.5}$$
$$BIH_2 = \bar{\Delta}_{g_2}\bar{H}_2 - 2n\bar{H}_2 - n(2 - |\bar{H}_2|^2)g_2, \tag{3.6}$$
$$BIH = \bar{\Delta}_{g_1\otimes g_2}\bar{H} - 2(m+n)\bar{H} - (m+n)(2 - |\bar{H}|^2)g_1 \otimes g_2. \tag{3.7}$$

Then, the vanishing of BIH_1, BIH_2 and BIH is equivalent to the biharmonicity of g_1, g_2 and $g_1 \otimes g_2$ respectively (see [4]). We have

$$\bar{H} = \tfrac{1}{m+n}\Big(m\bar{H}_1 \otimes g_2 + ng_1 \otimes \bar{H}_2\Big), \tag{3.8}$$

$$\bar{\Delta}_{g_1\otimes g_2}\bar{H} = \tfrac{1}{m+n}\Big(m\bar{\Delta}_{g_1}\bar{H}_1 \otimes g_2 - 2mn\bar{H}_1 \otimes \bar{H}_2 + ng_1 \otimes \bar{\Delta}_{g_2}\bar{H}_2\Big). \tag{3.9}$$

By (3.5)-(3.9) we get

$$BIH = \frac{m}{m+n} BIH_1 \otimes g_2 + \frac{n}{m+n} g_1 \otimes BIH_2 - \frac{2mn}{m+n} H_1 \otimes H_2, \tag{3.10}$$

where H_1 (resp. H_2) is the mean curvature field of M^m in $S^{p-1}(1)$ (resp. $S^{q-1}(1)$). It follows from (3.10) that $BIH = 0$ if and only if one of g_1 and g_2 is proper biharmonic and the other is minimal. □

From Proposition 5, we can construct infinity proper biharmonic submanifolds in the unit sphere.

Inoguchi studied biharmonic Hopf cylinders in Sasakian 3-space forms (see, Corollary 3.2 in [9]). We remark that Hopf cylinders are anti-invariant surfaces. By the similar way as in [9], we can prove the following:

Proposition 6. *Let M^2 be a proper biharmonic anti-invariant surface in Sasakian space forms $N^3(c)$. Then $c > 1$ and M^2 is a surface of constant mean curvature $\frac{\sqrt{c-1}}{2}$.*

Proof. We choose an orthonormal frame $\{e_1, e_2\}$ such that $e_2 = \xi$. Then from (2.4) we have $\langle \nabla_{e_i} e_j, e_k \rangle = 0$ for $i, j = 1, 2$. Moreover $h(e_1, e_1) = 2\kappa\phi e_1$, $h(e_2, e_2) = 0$ and $h(e_1, e_2) = -\phi e_1$ for some function κ. We may assume that κ is positive. The equation of Coddazi (2.9) gives $e_2\kappa = 0$. Thus, by the similar computations due to [9], κ^2 is constant and equal to $\frac{c-1}{4}$. This proves the proposition. □

Corollary 7. *There exists no proper biharmonic Legendre curve in Sasakian space forms $N^3(c)$ with $c \leq 1$.*

By applying Theorem 1 and 2, we see that a surface in Proposition 6 exists uniquely. From Corollary 7, we state that there exist no proper biharmonic anti-invarint surfaces in $S^3(1)$. To the contrary, by using proper biharmonic Legendre immersion (3.4), we can construct proper biharmonic anti-invariant 3-submanifolds in $S^5(1) \subset C^3$ as follows:

$$f(x, y, z) = \frac{1}{\sqrt{2}}(e^{ix}, ie^{-ix}\sin\sqrt{2}y, ie^{-ix}\cos\sqrt{2}y)e^{iz}. \tag{3.11}$$

Theorems 1, 2, Proposition 6 and (3.11) motivate us to consider the following problem:

In the case of $n > 1$, classify proper biharmonic anti-invariant $(n+1)$-submanifolds in Sasakian $(2n+1)$-space forms.

In the next section, in the case of $n = 2$ we obtain the existence and uniqueness theorem of such submanifolds.

4. Biharmonic anti-invariant 3-submanifolds

Let M^3 be a proper anti-invariant 3-submanifold in Sasakian space forms $N^5(c)$ and $\{e_i\}$ orthonormal frame fields along M^3 such that $e_3 = \xi$. We may assume that $H = \alpha\phi e_1$, where $\alpha \in C^\infty(M)$ and $\alpha > 0$. Then using (2.3), (2.4) and (2.6), we see that the second fundamental forms take the following forms:

$$\begin{aligned} h(e_1, e_1) &= \lambda\phi e_1 + \mu\phi e_2, \\ h(e_2, e_2) &= (3\alpha - \lambda)\phi e_1 - \mu\phi e_2, \\ h(e_3, e_3) &= 0, \\ h(e_1, e_2) &= \mu\phi e_1 + (3\alpha - \lambda)\phi e_2, \\ h(e_1, e_3) &= -\phi e_1, \\ h(e_2, e_3) &= -\phi e_2, \end{aligned} \tag{4.1}$$

for some functions λ and μ.

We put $\omega_i^j(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle$. Using (2.4) we obtain

$$\omega_i^3 = 0, \quad (i = 1, 2, 3). \tag{4.2}$$

From the Codazzi equation (2.9), we have the following Lemma.

Lemma 8.

$$\begin{aligned} e_1(3\alpha - \lambda) + 3\mu\omega_1^2(e_1) &= e_2\mu + 3(\lambda - 2\alpha)\omega_1^2(e_2), & (4.3)\\ -e_1\mu + 3(3\alpha - \lambda)\omega_1^2(e_1) &= e_2(3\alpha - \lambda) + 3\mu\omega_1^2(e_2), & (4.4)\\ e_1\mu + 3(\lambda - 2\alpha)\omega_1^2(e_1) &= e_2\lambda - 3\mu\omega_1^2(e_2), & (4.5)\\ \omega_1^2(e_3) &= 0, & (4.6)\\ e_3(\lambda) &= e_3(\mu) = e_3(\alpha) = 0. & (4.7) \end{aligned}$$

Proof. Since M^3 is an anti-invariant submanifold in Sasakian space forms, we get

$$(\bar{R}(X, Y)Z)^\perp = 0$$

by (2.5). From $(\bar{\nabla}_{e_1} h)(e_2, e_2) = (\bar{\nabla}_{e_2} h)(e_1, e_2)$ and $(\bar{\nabla}_{e_1} h)(e_1, e_2) = (\bar{\nabla}_{e_2} h)(e_1, e_1)$ by (2.9), we have (4.3), (4.4) and (4.5). Putting $X = e_1$, $Y = e_3$ and $Z = e_3$ in (2.9), the relation (4.6) is obtained. Similarly, by using $(\bar{\nabla}_{e_1} h)(e_1, e_3) = (\bar{\nabla}_{e_3} h)(e_1, e_1)$ and $(\bar{\nabla}_{e_2} h)(e_2, e_3) = (\bar{\nabla}_{e_3} h)(e_2, e_2)$, we get (4.7). □

Assume that $f : M^3 \to N^5(c)$ is biharmonic, namely M^3 satisfies $\mathcal{J}_f H = 0$. We shall compute $\mathcal{J}_f H$ by using ω_i^j, λ, α and μ. Due to Chen [5],

$$\bar{\Delta}_f H = \mathrm{tr}(\bar{\nabla} A_H) + \Delta^D H + (\mathrm{tr} A^2_{\phi e_1})H + a(H), \tag{4.8}$$

where $a(H) = \mathrm{trace}(A_H A_{\phi e_2})\phi e_2$ and $\mathrm{tr}(\bar{\nabla} A_H) = \sum_{i=1}^3 (A_{D_{e_i} H} e_i + (\nabla_{e_i} A_H) e_i)$.

Using (2.5), (4.1), (4.2), (4.6) and (4.7), we get the following lemma by straight-forward computations.

Lemma 9.

(i) $\mathrm{tr}(\bar{\nabla} A_H) = [2\{(e_1\alpha)\lambda + (e_2\alpha)\mu\} + \alpha\{(e_1\lambda) + (e_2\mu) + \mu\omega_2^1(e_1) + \lambda\omega_1^2(e_2)\}]e_1$
$+[2\{(e_1\alpha)\mu + (e_2\alpha)(3\alpha-\lambda)\} + \alpha\{e_1\mu + e_2(3\alpha-\lambda) + \lambda\omega_1^2(e_1) + \mu\omega_1^2(e_2)\}]e_2$
$-2\{e_1\alpha + \alpha\omega_1^2(e_2)\}e_3,$ (4.9)

(ii) $\Delta^D H = [-e_1e_1\alpha - e_2e_2\alpha + \alpha\{\omega_1^2(e_1) + \omega_1^2(e_2)\}]\phi e_1$
$-[2\{(e_1\alpha)\omega_1^2(e_1) + (e_2\alpha)\omega_1^2(e_2)\} + \alpha\{e_1(\omega_1^2(e_1)) + e_2(\omega_1^2(e_2))\}]\phi e_2,$ (4.10)

(iii) $\mathrm{tr}(A_{\phi e_1}^2) = \lambda^2 + 2\mu^2 + (3\alpha-\lambda)^2 + 2,$ (4.11)

(iv) $a(H) = 3\alpha^2\mu\phi e_2,$ (4.12)

(v) $\mathcal{R}_f(H) = (2c+1)H.$ (4.13)

From the biharmonicity, we have

$$\bar{\Delta}_f H = (2c+1)H.$$

Remark 10. In [14]–[17], the fourth author studied surfaces satisfying $\bar{\Delta}_f H = \beta H$ for a constant β in Sasakian space forms and complex space forms.

Hence, using Lemma 9 we obtain the following system of partial differential equations.

Lemma 11.

(i) $2\{(e_1\alpha)\lambda + (e_2\alpha)\mu\} + \alpha\{(e_1\lambda) + (e_2\mu) + \mu\omega_2^1(e_1) + \lambda\omega_1^2(e_2)\} = 0,$ (4.14)

(ii) $2\{(e_1\alpha)\mu + (e_2\alpha)(3\alpha-\lambda)\}$
$+\alpha\{e_1\mu + e_2(3\alpha-\lambda) + \lambda\omega_1^2(e_1) + \mu\omega_1^2(e_2)\} = 0,$ (4.15)

(iii) $e_1\alpha + \alpha\omega_1^2(e_2) = 0,$ (4.16)

(iv) $-e_1e_1\alpha - e_2e_2\alpha + \alpha\{\omega_1^2(e_1) + \omega_1^2(e_2)\}$
$+\alpha\{\lambda^2 + 2\mu^2 + (3\alpha-\lambda)^2 + 2\} - \alpha(2c+1) = 0,$ (4.17)

(v) $-2\{(e_1\alpha)\omega_1^2(e_1) + (e_2\alpha)\omega_1^2(e_2)\}$
$+\alpha\{e_1(\omega_1^2(e_1)) + e_2(\omega_1^2(e_2))\} + 3\alpha^2\mu = 0.$ (4.18)

Combining (4.4) and (4.5) yields

$$e_2\alpha = \alpha\omega_1^2(e_1). \tag{4.19}$$

It follows from (4.2), (4.6), (4.7), (4.16) and (4.19) that

$$\left[\frac{1}{\alpha}e_1, \frac{1}{\alpha}e_2\right] = 0, \tag{4.20}$$

$$\left[\frac{1}{\alpha}e_1, e_3\right] = 0, \tag{4.21}$$

$$\left[\frac{1}{\alpha}e_2, e_3\right] = 0. \tag{4.22}$$

Hence there exists a local coordinate system $\{x, y, z\}$ such that

$$\frac{1}{\alpha}e_1 = \frac{\partial}{\partial x}, \quad \frac{1}{\alpha}e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}. \tag{4.23}$$

From (4.7) we obtain that α, λ and μ are functions of x and y. Also by (4.16) and (4.19) we have

$$\omega_1^2(e_1) = \alpha_y, \quad \omega_1^2(e_2) = -\alpha_x, \tag{4.24}$$

where $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ for a function f. Substituting (4.23) and (4.24) into (4.18), we obtain that $\mu = 0$, i.e., M^3 is a Chen submanifold. Replacing (4.3), (4.4), (4.14) and (4.15) by derivatives with respect to x and y, we get

$$\alpha(3\alpha - \lambda)_x = -3(\lambda - 2\alpha)\alpha_x, \tag{4.25}$$
$$3(3\alpha - \lambda)\alpha_y = \alpha(3\alpha - \lambda)_y, \tag{4.26}$$
$$(\lambda\alpha)_x = 0, \tag{4.27}$$
$$9\alpha\alpha_y = (\lambda\alpha)_y. \tag{4.28}$$

By solving this system, we obtain α, λ are constant and hence $\omega_i^j = 0$ from (4.2), (4.6) and (4.24). Therefore by (4.17) we have

$$\lambda^2 + (3\alpha - \lambda)^2 + 1 - 2c = 0. \tag{4.29}$$

Also, by using the Gauss equation (2.8) we obtain

$$\frac{c+3}{4} + \lambda(3\alpha - \lambda) - (3\alpha - \lambda)^2 = 0. \tag{4.30}$$

Since α and λ are real numbers, c must satisfy $c \geq \frac{1+14\sqrt{2}}{23}$ from (4.29) and (4.30). Further $\alpha = \frac{\sqrt{11c-9\pm\sqrt{23c^2-2c-17}}}{6}$ and $\lambda = \frac{7(c-1)}{12\alpha}$.

By using a coordinate change $\alpha\tilde{x} = x$, $\alpha\tilde{y} = y$, we can rewrite the metric tensor as $g = d\tilde{x}^2 + d\tilde{y}^2 + dz^2$. Then $e_1 = \frac{\partial}{\partial \tilde{x}}$, $e_2 = \frac{\partial}{\partial \tilde{y}}$. Consequently, by applying Theorem 1 and 2 we can state the following:

Theorem 12. *Let M^3 be a proper biharmonic anti-invariant submanifold in Sasakian space forms $N^5(c)$. Then $c \geq \frac{1+14\sqrt{2}}{23}$ and at each point $p \in M^3$ there exists a suitable local coordinate system $\{x, y, z\}$ on a neighborhood of p such that the metric tensor g and the second fundamental form h take the following forms:*

(I) $$g = dx^2 + dy^2 + dt^2,$$

(II)
$$\begin{aligned}
h(\partial_x, \partial_x) &= \frac{7(c-1)}{12\alpha}\phi\partial_x,\\
h(\partial_y, \partial_y) &= \left(3\alpha - \frac{7(c-1)}{12\alpha}\right)\phi\partial_x,\\
h(\partial_z, \partial_z) &= 0,\\
h(\partial_x, \partial_y) &= \left(3\alpha - \frac{7(c-1)}{12\alpha}\right)\phi\partial_y,\\
h(\partial_x, \partial_z) &= -\phi\partial_x,\\
h(\partial_y, \partial_z) &= -\phi\partial_y,
\end{aligned}$$

where $\partial_x = \frac{\partial}{\partial_x}$, $\partial_y = \frac{\partial}{\partial_y}$, $\partial_z = \frac{\partial}{\partial_z}$, *and* $\alpha = \frac{\sqrt{11c-9\pm\sqrt{23c^2-2c-17}}}{6} (\neq 0)$.

Conversely, suppose that c *is a constant satisfying* $c \geq \frac{1+14\sqrt{2}}{23}$ *and let* g *be the metric tensor on a simply-connected domain* $V \subset \mathbf{R}^3$ *defined by* (I). *Then, up to rigid motions of* $N^5(c)$, *there exists a unique anti-invariant immersion of* (V, g) *into* $N^5(c)$ *whose second fundamental form is given by* (II). *Moreover such an immersion is proper biharmonic.*

Corollary 13. *Let* $A(c)$ *be the number of proper biharmonic anti-invariant* 3-*submanifold in Sasakian space forms* $N^5(c)$. *Then we have:*

(i) *if* $c < \frac{1+14\sqrt{2}}{23}$, $A(c) = 0$;

(ii) *if* $c = 1$ *or* $\frac{1+14\sqrt{2}}{23}$, $A(c) = 1$;

(iii) *if* $c > \frac{1+14\sqrt{2}}{23}$ *and* $c \neq 1$, $A(c) = 2$.

Corollary 14. (3.11) *is the only proper biharmonic anti-invariant* 3-*submanifolds in* $S^5(1)$.

Proof. We can easily check that the metric tensor and the second fundamental form of (3.11) take the form (I) and (II) in Theorem 12. □

In the case of $n > 2$, the classification has not been completed yet.

5. Stability of biharmonic anti-invariant submanifolds

In [11] Jiang obtained the second variation formula for the bienergy E_2. But in case that the ambient space is not locally symmetric, it is difficult to compute the formula. We remark that Sasakian space forms are not locally symmetric in general. In this section, we shall compute the second variation formula for a biharmonic anti-invariant immersion into Sasakian space forms by the similar way as in [12].

Let $f : M^{n+1} \to N^{2n+1}(c)$ be a biharmonic anti-invariant immersion from a compact n-dimensional manifold into a $(2n+1)$-dimensional Sasakian space form. Let $F : \mathbf{R} \times M^{n+1} :\to N^{2n+1}(c)$ be a smooth variation of f such that $F(0,p) = f(p)$ for any $p \in M$. Let $\left(\frac{\partial}{\partial t}\right)_{(t,p)}$ and $X_{(t,p)}$ be the vector fields which are the extension of $\frac{\partial}{\partial t}$ on $\mathbf{R}$ and X on M^{n+1} to $\mathbf{R} \times M^n$, respectively. We put $f_t(p) = F(t,p)$. The corresponding variational vector field V is given by

$$V(p) = \frac{d}{dt}\Big|_{t=0} f_t(p) = dF\Big(\frac{\partial}{\partial t}\Big)_{(0,p)}.$$

We recall the following from [12].

$$\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_2(f_t) = \int_{M^n} \langle I(V), V \rangle dv_g, \tag{5.1}$$

where

$$I(V) = \tilde{\nabla}_{\frac{\partial}{\partial t}} \{-\bar{\Delta}_{f_t}\tau_t - \mathrm{trace} R^N(df_t\cdot, \tau_t) df_t\cdot\}\big|_{t=0}, \tag{5.2}$$

$\tilde{\nabla} = \nabla^F$ and $\tau_t = \tau(f_t)$.

If (5.1) is non-negative for any vector field V, then f or M^{n+1} is said to be *stable*. Otherwise it is said to be *unstable*.

We shall calculate (5.2) more precisely.

$$\begin{aligned}
&-\tilde{\nabla}_{\frac{\partial}{\partial t}}\bar{\Delta}_{f_t}\tau_t \sum\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\tau_t - \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\nabla_{e_i}e_i}\tau_t\Big)\\
&= \sum\Big\{R^N\Big(df_t\Big(\frac{\partial}{\partial t}\Big), df_t(e_i)\Big)(\tilde{\nabla}_{e_i}\tau_t) + \tilde{\nabla}_{e_i}\tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{e_i}\tau_t + \tilde{\nabla}_{[\frac{\partial}{\partial t},e_i]}\tilde{\nabla}_{e_i}\tau_t\Big\}\\
&\quad -\sum\Big\{R^N\Big(df_t\Big(\frac{\partial}{\partial t}\Big), df_t(\nabla_{e_i}e_i)\Big)\tau_t + \tilde{\nabla}_{\nabla_{e_i}e_i}\tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t + \tilde{\nabla}_{[\frac{\partial}{\partial t},\nabla_{e_i}e_i]}\tau_t\Big\}. \qquad (5.3)
\end{aligned}$$

As in [12], we have

$$\tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t\Big|_{t=0} = -\bar{\Delta}_f V - \mathrm{trace} R^N(df\cdot, V)df\cdot = -\mathcal{J}_f(V). \qquad (5.4)$$

Let $\{e_i\}$ be a geodesic frame field around an arbitrary point $p \in M^{n+1}$. Then from (5.3) and (5.4), when $t = 0$, at p we get

Lemma 15.

$$-\tilde{\nabla}_{\frac{\partial}{\partial t}}\bar{\Delta}_{f_t}\tau_t\Big|_{t=0} = \sum\Big\{R^N(V,e_i)(\bar{\nabla}_{e_i}\tau) + \bar{\nabla}_{e_i}(R^N(V,e_i)\tau)\Big\} + \bar{\Delta}\mathcal{J}_f V, \qquad (5.5)$$

where $\bar{\nabla} = \nabla^f$, $\tau = \tau_0$.

We need the following lemma in order to compute (5.5) more precisely.

Lemma 16.

$$\begin{aligned}
\text{(i)}\quad R^N(V,e_i)(\bar{\nabla}_{e_i}\tau) &= \frac{c+3}{4}\Big(\langle e_i, \bar{\nabla}_{e_i}\tau\rangle V - \langle\bar{\nabla}_{e_i}\tau, V\rangle e_i\Big)\\
&+ \frac{c-1}{4}\Big\{\eta(V)\eta(\bar{\nabla}_{e_i}\tau)e_i - \langle e_i, \bar{\nabla}_{e_i}\tau\rangle\eta(V)\xi \qquad (5.6)\\
&+ \langle\bar{\nabla}_{e_i}\tau, \phi e_i\rangle\phi V - \langle\bar{\nabla}_{e_i}\tau, \phi V\rangle\phi e_i + 2\langle V, \phi e_i\rangle\phi(\bar{\nabla}_{e_i}\tau)\Big\},\\
\text{(ii)}\quad \bar{\nabla}_{e_i}(R^N(V,e_i)\tau) &= -\frac{\epsilon+3}{4}\bar{\nabla}_{e_i}(\langle\tau, V\rangle e_i) + \frac{\epsilon-1}{4}\Big\{\bar{\nabla}_{e_i}\Big(\langle\tau, \phi e_i\rangle\phi V\\
&\quad -\langle\tau, \phi V\rangle\phi e_i + 2\langle V, \phi e_i\rangle\phi\tau\Big)\Big\}. \qquad (5.7)
\end{aligned}$$

Proof. By using the fact that τ is normal to M^{n+1} and ξ, we can easily obtain (5.6) and (5.7) from (2.5). □

We continue to calculate (5.2). Using (2.2)–(2.5), we have

$$
\begin{aligned}
&-\tilde{\nabla}_{\frac{\partial}{\partial t}} \mathrm{trace} R^N(df_t\cdot,\tau_t)df_t\cdot \\
&= -\frac{c+3}{4}\sum \tilde{\nabla}_{\frac{\partial}{\partial t}}\Big\{\big\langle \tau_t, dF(e_i)\big\rangle dF(e_i) - \big\langle dF(e_i), dF(e_i)\big\rangle \tau_t\Big\} \\
&-\frac{c-1}{4}\sum \tilde{\nabla}_{\frac{\partial}{\partial t}}\Big\{\eta(dF(e_i))\eta(dF(e_i))\tau_t - \eta(\tau_t)\eta(dF(e_i))dF(e_i) \\
&+\big\langle dF(e_i), dF(e_i)\big\rangle \eta(\tau_t)\xi - \big\langle \tau_t, dF(x_i)\big\rangle \eta(dF(e_i))\xi \\
&+3\big\langle dF(e_i), \phi\tau_t\big\rangle \phi(dF(e_i)) - \big\langle dF(e_i), \phi(dF(e_i))\big\rangle \phi\tau_t\Big\}, \\
&= -\frac{c+3}{4}\sum\Big\{\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t, dF(e_i)\big\rangle dF(e_i) + \big\langle \tau_t, \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i)\big\rangle dF(e_i) \\
&+\big\langle \tau_t, dF(e_i)\big\rangle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i) - 2\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i), dF(e_i)\big\rangle \tau_t - \big\langle dF(e_i), dF(e_i)\big\rangle \tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t\Big\} \\
&-\frac{c-1}{4}\sum\Big[2\big\langle dF(e_i),\xi\big\rangle\Big\{\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i),\xi\big\rangle - \big\langle dF(e_i), \phi(dF\Big(\frac{\partial}{\partial t}\Big))\big\rangle\Big\}\tau_t \\
&+\eta(dF(e_i))^2\tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t - \Big\{\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t,\xi\big\rangle - \big\langle \tau_t, \phi(dF\Big(\frac{\partial}{\partial t}\Big))\big\rangle\Big\}\eta(dF(e_i))dF(e_i) \\
&-\eta(\tau_t)\Big\{\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i),\xi\big\rangle - \big\langle dF(e_i), \phi(dF\Big(\frac{\partial}{\partial t}\Big))\big\rangle\Big\}dF(e_i) \\
&-\eta(\tau_t)\eta(dF(e_i))\tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i) + 2\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i), dF(e_i)\big\rangle \eta(\tau_t)\xi \\
&+\big\langle dF(e_i), dF(e_i)\big\rangle\Big\{\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t,\xi\big\rangle - \big\langle \tau_t, \phi(dF\Big(\frac{\partial}{\partial t}\Big))\big\rangle\Big\}\xi \\
&-\big\langle dF(e_i), dF(e_i)\big\rangle \eta(\tau_t)\phi(dF\Big(\frac{\partial}{\partial t}\Big)) - \big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t, dF(e_i)\big\rangle \eta(dF(e_i))\xi \\
&-\big\langle \tau_t, \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i)\big\rangle \eta(dF(e_i))\xi \\
&-\big\langle \tau_t, dF(e_i)\big\rangle\Big\{\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i),\xi\big\rangle - \big\langle dF(e_i), \phi(dF\Big(\frac{\partial}{\partial t}\Big))\big\rangle\Big\}\xi \\
&+\big\langle \tau_t, dF(e_i)\big\rangle \eta(dF(e_i))\phi(dF\Big(\frac{\partial}{\partial t}\Big)) + 3\Big\{\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i), \phi\tau_t\big\rangle \phi(dF(e_i)) \\
&+\big\langle dF(e_i), \big\langle dF\Big(\frac{\partial}{\partial t}\Big), \tau_t\big\rangle \xi - \eta(\tau_t)dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t)\big\rangle \phi(dF(e_i)) \\
&+\big\langle dF(e_i), \phi\tau_t\big\rangle\Big(\big\langle dF\Big(\frac{\partial}{\partial t}\Big), dF(e_i)\big\rangle \xi - \eta(dF(e_i))dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i))\Big)\Big\} \\
&-\big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i), \phi(dF(e_i))\big\rangle \phi\tau_t \\
&-\big\langle dF(e_i), \big\langle dF\Big(\frac{\partial}{\partial t}\Big), dF(e_i)\big\rangle \xi - \eta(dF(e_i))dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i))\big\rangle \phi\tau_t \\
&-\big\langle dF(e_i), \phi(dF(e_i))\big\rangle\Big\{\big\langle dF\Big(\frac{\partial}{\partial t}\Big), \tau_t\big\rangle \xi - \eta(\tau_t)dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t)\Big\}\Big]. \qquad (5.8)
\end{aligned}
$$

We need the following lemma.

Lemma 17. $\tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i)\Big|_{t=0} = \tilde{\nabla}_{e_i} dF(\frac{\partial}{\partial t})\Big|_{t=0} = \bar{\nabla}_{e_i} V.$

From (5.8) and Lemma 17 we deduce the following:

Lemma 18.

$$\begin{aligned}
&-\tilde{\nabla}_{\frac{\partial}{\partial t}} \mathrm{trace} R^N(df_t\cdot, \tau_t) df_t \cdot |_{t=0} \\
&= -\frac{c+3}{4}\Big\{ -(\mathcal{J}_f V)^\top + \sum\Big(\big\langle \tau, \bar{\nabla}_{e_i} V\big\rangle e_i - 2\big\langle \bar{\nabla}_{e_i} V, e_i\big\rangle \tau\Big) + (n+1)\mathcal{J}_f V\Big\} \\
&-\frac{c-1}{4}\Big\{2\big\langle \bar{\nabla}_\xi V, \xi\big\rangle\tau + (1-n)\Big(\big\langle \mathcal{J}_f V, \xi\big\rangle + \big\langle \tau, \phi V\big\rangle\Big)\xi - \big\langle \tau, \bar{\nabla}_\xi V\big\rangle \xi + 3(\mathcal{J}_f V)^\perp \\
&+3\sum\Big(\big\langle \bar{\nabla}_{e_i} V, \phi\tau\big\rangle \phi e_i + \big\langle e_i, \phi\tau\big\rangle\Big(\big\langle V, e_i\big\rangle\xi + \phi(\bar{\nabla}_{e_i} V)\Big)\Big)\Big\},
\end{aligned} \tag{5.9}$$

where $(\mathcal{J}_f V)^\top$ (resp. $(\mathcal{J}_f V)^\perp$) denotes the tangent (resp. normal) part of $\mathcal{J}_f V$.

Consequently, we obtain the second variation formula as follows:

Theorem 19. *Let f be a biharmonic anti-invariant immersion from a compact $(n+1)$-dimensional manifold M^{n+1} into a Sasakian space form $N^{2n+1}(c)$. Let $\{f_t\}$ be a smooth variation of f such that $f_0 = f$ and V be the corresponding variational vector field. Then we have*

$$\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_2(f_t) = \int_{M^{n+1}} \big\langle I(V), V\big\rangle dv_g,$$

where

$$\begin{aligned}
I(V) \;=\; & -\frac{c+3}{4}\Big\{|\tau|^2 V + 2\mathrm{trace}\big\langle \bar{\nabla}.\tau, V\big\rangle \cdot + 2\mathrm{trace}\big\langle \tau, \bar{\nabla}.V\big\rangle \cdot + \big\langle \tau, V\big\rangle \tau \\
& -2\mathrm{trace}\big\langle \bar{\nabla}.V, \cdot\big\rangle \tau - (\mathcal{J}_f V)^\top + (n+1)\mathcal{J}_f V\Big\} \\
& +\frac{c-1}{4}\Big\{ -2\big\langle \bar{\nabla}_\xi V, \xi\big\rangle\tau + \big\langle \tau, \bar{\nabla}_\xi V\big\rangle\xi + \eta(V)\mathrm{trace}(\eta(\bar{\nabla}.\tau)\cdot) + |\tau|^2\eta(V)\xi \\
& +2\mathrm{trace}\big\langle \bar{\nabla}.\tau, \phi\cdot\big\rangle\phi V - 2\mathrm{trace}\big\langle \bar{\nabla}.\tau, \phi V\big\rangle \phi\cdot - 4\phi(\bar{\nabla}_{(\phi V)^\top} V) - \big\langle V, \phi\tau\big\rangle\xi \\
& +\eta(V)\phi\tau - 4\phi(\bar{\nabla}_{\phi\tau} V) + 2\mathrm{trace}\big\langle \tau, \phi(\bar{\nabla}.V)\big\rangle\phi\cdot - 3\big\langle \tau, \phi V\big\rangle\phi\tau \\
& +2\mathrm{trace}\big\langle \bar{\nabla}.V, \phi\cdot\big\rangle\phi\tau + 2n\eta(V)\phi\tau + 2\eta(V)(\phi V)^\top \\
& +(n-1)\eta(\mathcal{J}_f V)\xi - 3(\mathcal{J}_f V)^\perp\Big\} + \bar{\Delta}\mathcal{J}_f V.
\end{aligned} \tag{5.10}$$

Proof. When we compute (5.7), we use the following:

$$\begin{aligned}
\bar{\nabla}_{e_i}(\phi V) &= \big\langle e_i, V\big\rangle\xi - \eta(V)e_i + \phi(\bar{\nabla}_{e_i} V), \\
\bar{\nabla}_{e_i}(\phi e_i) &= \xi + \phi h(e_i, e_i).
\end{aligned}$$

Combining (5.2), Lemma 15, 16 and 18 we get (5.10). □

We put

$$F(X) := \big\langle h(X,X), \phi X \big\rangle$$

for a vector field X of M^{n+1}. $F(\phi\tau)$ is globally defined on M^{n+1}. In the case of $n=1$, then $F(\phi\tau)$ coincides with $-||\tau||^2$. However it is not true in general. In terms of $||\tau||$ and $F(\phi\tau)$, we give the sufficient conditions for proper biharmonic anti-invariant submanifolds to be unstable.

Theorem 20. *Let M^{n+1} be a compact proper biharmonic anti-invariant submanifold in a Sasakian space form $N^{2n+1}(c)$. If*

$$\int_{M^{n+1}} \Big\{(c+3)||\tau||^4 - 3(c-1)F(\phi\tau)\Big\} dv_g > 0, \tag{5.11}$$

then M^{n+1} is unstable.

Proof. We take τ as the variational vector field V. By Theorem 19, (2.6) and (2.7) we have

$$\big\langle I(\tau), \tau \big\rangle = -(c+3)||\tau||^4 - 3(c-1)\big\langle h(\phi\tau, \phi\tau), \tau \big\rangle. \tag{5.12}$$

This completes the proof. □

It follows from Proposition 6 and (II) in Theorem 12 that $(c+3)||\tau||^4 - 3(c-1)F(\phi\tau) > 0$ if $n=1$ or 2. Therefore applying Theorem 20 we state the following:

Corollary 21. *Let M^{n+1} be a compact proper biharmonic anti-invariant submanifold in Sasakian space form $N^{2n+1}(c)$. If $n \le 2$, then M^{n+1} is unstable.*

There is a special vector field along submanifolds in contact manifolds, i.e., Reeb vector field ξ. Thus, it is natural and interesting to consider variations $V \in \mathrm{Span}\{\xi\} := \{a\xi | a \in C^\infty(M^{n+1})\}$. We call such variations *R-variations*. If the second variation (5.1) under any R-variation is non-negative, f or M^{n+1} is said to be *R-stable*. Otherwise it is said to be *R-unstable*.

Theorem 22. *Let M^{n+1} be a compact proper biharmonic anti-invariant submanifold in Sasakian space forms $N^{2n+1}(c)$. Then M^{n+1} is R-stable if and only if $\lambda_1 \ge \frac{5c-17}{4}$, where λ_1 is the first eigenvalue of the Laplacian acting on $C^\infty(M^{n+1})$.*

Proof. Let f be an isometric proper biharmonic anti-invariant immersion from M^{n+1} into $N^{2n+1}(\epsilon)$. We take $a\xi$ as the variational vector field, where $a \in C^\infty(M^{n+1})$. We can easily see the following:

$$\bar{\Delta}_f(a\xi) = (\Delta_M a + na)\xi + 2\phi \mathrm{grad} a + a\phi\tau, \tag{5.13}$$

$$\mathcal{R}_f(a\xi) = an\xi. \tag{5.14}$$

By using Theorem 19, (5.13), (5.14) and Stokes' theorem, we obtain

$$\begin{aligned}
&\int_{M^{n+1}} \langle I(a\xi), a\xi\rangle dv \\
&= \int_{M^{n+1}} \Big\{ -a^2|\tau|^2 + \langle \bar{\Delta}_f(\mathcal{J}_f(a\xi)), a\xi\rangle + \frac{1-5\epsilon-4n}{4}\langle \mathcal{J}_f(a\xi), a\xi\rangle \Big\} dv_g \\
&= \int_{M^{n+1}} \Big\{ -a^2|\tau|^2 + \langle \mathcal{J}_f(a\xi), \bar{\Delta}_f(a\xi)\rangle + \frac{1-5c-4n}{4}(\Delta_M a)a \Big\} dv_g \\
&= \int_{M^{n+1}} \Big\{ (\Delta_M a)^2 + n(\Delta_M a)a + 4||\mathrm{grad} a||^2 + \frac{1-5c-4n}{4}(\Delta_M a)a \Big\} dv_g \\
&= \int_{M^{n+1}} \Big\{ (\Delta_M a)^2 + \frac{17-5c}{4}(\Delta_M a)a \Big\} dv_g. \qquad (5.15)
\end{aligned}$$

This completes the proof. □

Corollary 23. *Compact biharmonic anti-invariant $(n+1)$-submanifolds of $N^{2n+1}(c)$ with $c \leq \frac{17}{5}$ are R-stable.*

Theorem 22 implies that the spectral geometry of compact proper biharmonic anti-invariant submanifolds of maximum dimension in Sasakian space forms is important.

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