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Ideal Structure of Hurwitz Series Rings

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Abstract. We study the ideals, in particular, the maximal spectrum and the set of idempotent elements, in rings of Hurwitz series.
Let A be a commutative ring with identity. The elements of the ring HA of Hurwitz series over A are formal expressions of the type $f = \sum_{i=0}^{\infty} a_i X^i$ where $a_i \in A$ for all i. Addition is defined termwise. The product of f by $g = \sum_{i=0}^{\infty} b_i X^i$ is defined by $f * g = \sum_{n=0}^{\infty} c_n X^n$ where $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ and $\binom{n}{k}$ is a binomial coefficient. Recently, many authors turned to this ring and discovered interesting applications in it. See for example [1] and [2]. The natural homomorphism $\epsilon : HA \longrightarrow A$, is defined by $\epsilon(f) = a_0$.

1. Generalities

1.1. Proposition. *HA is an integral domain if and only if A is an integral domain with zero characteristic.*

Proof. $\Longleftarrow$ See [1, Corollary 2.8].
$\Longrightarrow$ Since $A \subset HA$, then A is a domain. Suppose that A has a positive characteristic m. Then $X * X^{m-1} = \binom{m-1+1}{1} X^m = mX^m = 0$.

1.2. Proposition. *Let I be an ideal of A. Then $HA/\epsilon^{-1}(I) \simeq A/I$ and $HA/HI \simeq H(A/I)$. In particular*

a) *$\epsilon^{-1}(I)$ is a radical ideal of $HA \Longleftrightarrow I$ is a radical ideal of A.*

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b) $\epsilon^{-1}(I) \in Spec(HA) \Longleftrightarrow I \in Spec(A)$.

c) $\epsilon^{-1}(I) \in Max(HA) \Longleftrightarrow I \in Max(A)$.

d) $HI \in Spec(HA) \Longleftrightarrow I \in Spec(A)$ *and* A/I *has zero characteristic.*

Proof. The map $\psi : HA \longrightarrow A/I$, defined by $\psi = \tau \circ \epsilon$ where τ is the canonical surjection of A onto A/I, is a surjective homomorphism with $ker\ \psi = \epsilon^{-1}(I)$, so $HA/\epsilon^{-1}(I) \simeq A/I$.

The map $\phi : HA \longrightarrow H(A/I)$, defined for $f = \sum_{i=0}^{\infty} a_i X^i$ by $\phi(f) = \sum_{i=0}^{\infty} \bar{a}_i X^i$, is a surjective homomorphism, with $ker\ \phi = HI$, so $HA/HI \simeq H(A/I)$.

Now (a), (b) and (c) follow from the first isomorphism.

(d) $HI \in Spec(HA) \Longleftrightarrow HA/HI$ an integral domain $\Longleftrightarrow H(A/I)$ an integral domain $\Longleftrightarrow A/I$ an integral domain with zero characteristic $\Longleftrightarrow I \in Spec(A)$ and A/I has zero characteristic.

The inverse implication in (d) of the proposition was proved in [1, Prop. 2.7].

Example. Let $A = \mathbb{F}_q$ be the finite field of q elements. Since $X * X^{q-1} = qX^q = 0$, then $H0 = 0$ is not prime in $H\mathbb{F}_q$.

1.3. Corollary. *The set of maximal ideals of* HA *is* $Max(HA) = \{\epsilon^{-1}(M) : M \in Max(A)\}$. *In particular, the Jacobson radical* $Rad(HA) = \epsilon^{-1}(Rad(A))$. *The ring* HA *is local (resp. quasi local) if and only if* A *is local (resp. quasi local).*

Proof. By the part (c) of the preceding proposition, we have only to prove that for any $\mathcal{M} \in Max(HA)$ there is $M \in Max(A)$ such that $\mathcal{M} = \epsilon^{-1}(M)$. The set $M = \epsilon(\mathcal{M})$ is an ideal of A and $M \neq A$ since in the contrary case, by [1, Proposition 2.5], $\mathcal{M}$ contains a unit of HA. Therefore $\mathcal{M} \subseteq \epsilon^{-1}(M) \subset HA$ and by the maximality of $\mathcal{M}$, $\mathcal{M} = \epsilon^{-1}(M)$. By Proposition 1.2 (c), $M \in Max(A)$.

Examples. 1) $Max(H\mathbb{Z}) = \{\epsilon^{-1}(p\mathbb{Z}) :\ p \text{ prime integer}\}$.

2) For any field K, HK is local with maximal ideal $\epsilon^{-1}(0)$.

3) Contrary to the case of the ring of usual formal power series over a field, the element X does not generate the maximal ideal $\epsilon^{-1}(0)$ of $H\mathbb{F}_2$. Indeed, for any $f = \sum_{n=0}^{\infty} a_n X^n \in H\mathbb{F}_2$, $X * f = \sum_{n=0}^{\infty} \binom{n+1}{1} a_n X^{n+1} = \sum_{n=0}^{\infty} (n+1) a_n X^{n+1} = \sum_{k=0}^{\infty} a_{2k} X^{2k+1}$.

1.4. Proposition. *If* $P \subset Q$ *are consecutive prime ideals in* A, *then* $\epsilon^{-1}(P) \subset \epsilon^{-1}(Q)$ *are consecutive prime ideals in* HA.

Proof. Let $R \in Spec(HA)$ such that $\epsilon^{-1}(P) \subset R \subseteq \epsilon^{-1}(Q)$. There is an $f = a_0 + a_1 X + \cdots \in R \setminus \epsilon^{-1}(P)$. Then $a_0 \notin P$ and $a_0 = f - (a_1 X + \cdots) \in R$ since $a_1 X + \cdots \in \epsilon^{-1}(P) \subset R$. Therefore $a_0 \in R \cap A$ and $P = \epsilon^{-1}(P) \cap A \subset R \cap A \subseteq \epsilon^{-1}(Q) \cap A = Q$. Since $P \subset Q$ are consecutive, then $R \cap A = Q$. For any element $g = b_0 + b_1 X + \cdots \in \epsilon^{-1}(Q)$, $b_0 \in Q \subset R$ and $b_1 X + \cdots \in \epsilon^{-1}(P) \subseteq R$, so $g \in R$ and $\epsilon^{-1}(Q) = R$.

2. Idempotent elements in Hurwitz series ring

For $f \in HA$, the ideal $c(f)$ generated by the coefficients of f in A is called the content of f.

2.1. Proposition. *Suppose that for any $P \in Spec(A)$, A/P has zero characteristic. If f and $g \in HA$ are such that $f * g = 0$, then $c(f)c(g) \subseteq Nil(A)$. Moreover, if A is reduced, then each coefficient of f annihilates g.*

Proof. By Proposition 1.2, for any $P \in Spec(A)$, $HP \in Spec(HA)$. Since $f * g = 0 \in HP$, then f or $g \in HP$. If a is a coefficient of f and b a coefficient of g, then $ab \in P$. So $ab \in \bigcap\{P : P \in Spec(A)\} = Nil(A)$ and $c(f)c(g) \subseteq Nil(A)$.

Example. The result is not true in general. Suppose for example that A has positive characteristic n. Then $X * X^{n-1} = \binom{n-1+1}{1} X^n = nX^n = 0$, with $c(X) = c(X^{n-1}) = A$, so $c(X)c(X^{n-1}) = A \not\subseteq Nil(A)$.

As usual, $Bool(A)$ will mean the set of idempotent elements in the ring A.

2.2. Corollary. *Suppose A is reduced and A/P has zero characteristic, for every $P \in Spec(A)$. Then $Bool(HA) = Bool(A)$.*

Proof. Let $f = \sum_{i=0}^{\infty} a_i X^i \in HA$, with $f * f = f$. Then $f - 1 = (a_0 - 1) + \sum_{i=1}^{\infty} a_i X^i$ and $f * (f-1) = 0$. By Proposition 2.1, for $i \geq 1$, $a_i^2 = 0$, so $a_i = 0$ and $f = a_0 \in A$.

More generally, we have the following result.

2.3. Proposition. *For any ring A, $Bool(HA) = Bool(A)$.*

Proof. Let $f = \sum_{i=0}^{\infty} a_i X^i \in HA$ be such that $f * f = f$. Then $a_0^2 = a_0$ and $2a_0 a_1 = a_1 \implies 2a_0^2 a_1 = a_0 a_1 \implies 2a_0 a_1 = a_0 a_1 \implies a_0 a_1 = 0$. Suppose by induction that $a_0 a_i = 0$, for $1 \leq i < n$. The coefficient of X^n in $f * f = f$ is $\sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} = a_n \implies a_0 \left(\sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} \right) = a_0 a_n \implies a_0 \left(\binom{n}{0} a_0 a_n + \binom{n}{n} a_n a_0 \right) = a_0 a_n \implies 2a_0^2 a_n = a_0 a_n \implies 2a_0 a_n = a_0 a_n \implies a_0 a_n = 0$. So for each $i \geq 1$, $a_0 a_i = 0$. Suppose that $f \notin A$ and let $k = inf\{i \in \mathbb{N}^* : a_i \neq 0\}$, $g = \sum_{i=k}^{\infty} a_i X^i$, then $k \geq 1$, $a_k \neq 0$, $f = a_0 + g$, $a_0 * g = \sum_{i=k}^{\infty} a_0 a_i X^i = 0$. Since $f * f = f$, then $(a_0 + g) * (a_0 + g) = a_0 + g \implies a_0^2 + g * g = a_0 + g \implies g * g = g \implies \binom{2k}{k} a_k^2 X^{2k} + \cdots = a_k X^k + \cdots \implies a_k = 0$, which is impossible. So $f = a_0 \in A$.

A ring A is called PS if the socle $Soc({}_A A)$ is projective. By [3, Theorem 2.4], a ring A is PS if and only if for every maximal ideal M of A there is an idempotent e of A such that $(0 : M) = eA$. In [2, Theorem 3.2], Zhongkui Liu proved the following result:

"If A has zero characteristic and if A is a PS-ring, then HA is a PS-ring".

His proof is not correct, it uses in many places the wrong fact:

"If A has zero characteristic, $n \in \mathbb{N}^*$ and $x \in A$, then $nx = 0$ implies $x = 0$".

But this is not true. Take for example: $A = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, $n \geq 2$ an integer and $x = (0, \bar{1})$. When I wrote to Liu, he proposed to replace the condition "A has zero characteristic" by "A is $\mathbb{Z}$-torsion free". With this change the proof becomes correct.

In the next proposition, I avoid the hypothesis "A is a PS-ring" in the theorem of Liu and I give a short and simple proof.

2.4. Proposition. *If A is torsion free as a $\mathbb{Z}$-module, then HA is a* PS-*ring.*

Proof. If $\mathcal{M} \in Max(HA)$, there is $M \in Max(A)$ such that $\mathcal{M} = \epsilon^{-1}(M)$ by Corollary 1.3, so $X \in \mathcal{M}$. Let $f = \sum_{i=0}^{\infty} a_i X^i \in (0 : \mathcal{M})$, then $0 = X * f = \sum_{i=0}^{\infty} \binom{i+1}{1} a_i X^{i+1} = \sum_{i=0}^{\infty} (i+1) a_i X^{i+1}$. For each $i \in \mathbb{N}$, $(i+1)a_i = 0$, but A is $\mathbb{Z}$-torsion free, then $a_i = 0$ and $f = 0$.

2.5. Lemma. *Suppose that A is reduced and A/P has zero characteristic for any $P \in Spec(A)$. For $f \in HA$, let $I_f = (0 : c(f))$. Then:*

a) *For every $f \in HA$, $(0 : f) = HI_f$.*

b) *If J is an ideal of HA and $L = \sum_{f \in J} c(f)$, then $(0 : J) = H(0 : L)$.*

Proof. a) Put $f = \sum_{i=0}^{\infty} a_i X^i$. By Proposition 2.1, $g = \sum_{i=0}^{\infty} b_i X^i \in (0 : f) \iff f * g = 0 \iff \forall i, j \in \mathbb{N},\ a_i b_j = 0 \iff \forall j \in \mathbb{N},\ b_j \in (0 : c(f)) = I_f \iff g \in HI_f$.
b) By part a), $(0 : J) = \bigcap_{f \in J} (0 : f) = \bigcap_{f \in J} HI_f = H(\bigcap_{f \in J} I_f)$. But $\bigcap_{f \in J} I_f = \bigcap_{f \in J} (0 : c(f)) = (0 : \sum_{f \in J} c(f)) = (0 : L)$. So $(0 : J) = H(0 : L)$.

2.6. Proposition. *If A is reduced with A/P has zero characteristic for every $P \in Spec(A)$, then HA is a PS-ring.*

Proof. Let $\mathcal{M} \in Max(HA)$. By Corollary 1.3, $X \in \mathcal{M}$, then $\sum\limits_{f \in \mathcal{M}} c(f) = A$. By the preceding lemma, $(0 : \mathcal{M}) = H(0 : A) = H0 = (0)$.

Conjecture. In [4, Proposition 4], Xue showed that the ring $A[[X]]$ is always PS, for any ring A. In the light of this theorem and the preceding results I conjecture that the ring HA is also PS.

2.7. Definition. *A quasi-Baer ring is a ring A such that for any ideal I of A there is an idempotent e of A with $(0 : I) = eA$.*

The following lemma is well known. We include its proof for the sake of the reader.

2.8. Lemma. *Any quasi-Baer ring is reduced.*

Proof. Let a be a nilpotent element of the quasi-Baer ring A and $n \geq 1$ the smallest integer such that $a^n = 0$. Let $(0 : aA) = eA$, with $e \in A$ and $e^2 = e$. If $n \geq 2$, then $a^{n-1} \in eA$, put $a^{n-1} = eb$, with $b \in A$. Since $ae = 0$, then $0 = a^{n-1}e = be^2 = be = a^{n-1}$, which is impossible.

2.9. Proposition. *If A is a quasi-Baer ring with A/P has zero characteristic for every $P \in Spec(A)$, then HA is a quasi-Baer ring.*

Proof. Let J be an ideal of HA and $L = \sum_{f \in J} c(f)$. There is $e \in Bool(A)$ such that $(0 : L) = eA$. By Lemma 2.5, $(0 : J) = H(0 : L) = H(eA) = e * HA$.

3. Hurwitz series over a noetherian ring

3.1. Lemma. *Let I be an ideal of A. Then $HI = I * HA$ if and only if for any countable subset S of I there is a finitely generated ideal F of A such that $S \subseteq F \subseteq I$.*

Proof. $\Longrightarrow$ A countable subset of I is a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of I. Let $f = \sum_{i=0}^{\infty} a_i X^i \in HI = I * HA$. There are $b_1, \ldots, b_n \in I$ and $g_1, \ldots, g_n \in HA$ such that $f = b_1 * g_1 + \cdots + b_n * g_n$. If $F = b_1 A + \cdots + b_n A$, then $\{a_i : i \in \mathbb{N}\} \subseteq F$.

$\Longleftarrow$ Since $I \subset HI$, then $I * HA \subseteq HI$. Now, let $f = \sum_{i=0}^{\infty} a_i X^i \in HI$. There is a finitely generated ideal $F = b_1 A + \cdots + b_n A$ of A such that $\{a_i : i \in \mathbb{N}\} \subseteq F \subseteq I$. For each $i \in \mathbb{N}$, $a_i = \sum_{j=1}^{n} a_{ij} b_j$, with $a_{ij} \in A$. So $f = \sum_{i=0}^{\infty} (\sum_{j=1}^{n} a_{ij} b_j) X^i = \sum_{j=1}^{n} b_j * (\sum_{i=0}^{\infty} a_{ij} X^i) \in I * HA$.

Example. Let (A, M) be a non-discrete valuation domain of rank one, defined by a valuation v with group G. We can suppose that G is a dense subgroup of $\mathbb{R}$. Let $(\alpha_i)_{i \in \mathbb{N}}$ be a strictly decreasing sequence of elements of G converging to zero. For each $i \in \mathbb{N}$, there is $a_i \in M$, with $v(a_i) = \alpha_i$. Let $f = \sum_{i=0}^{\infty} a_i X^i \in HM$. Suppose that $f \in M * HA$, there is $b \in M$ and $g = \sum_{i=0}^{\infty} c_i X^i \in HA$ such that $f = b * g$. For each $i \in \mathbb{N}$, $a_i = b c_i$, so $\alpha_i = v(a_i) = v(b) + v(c_i) \geq v(b)$, which is impossible.

3.2. Corollary. *If I is a finitely generated ideal, then $HI = I * HA$.*

3.3. Proposition. *The ring A is noetherian if and only if for each ideal I of A, $HI = I * HA$.*

Proof. Suppose that A is not noetherian and let $(I_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of ideals of A and put $I = \bigcup_{i=0}^{\infty} I_i$. For each $i \in \mathbb{N}^*$, there is $a_i \in I_i \setminus I_{i-1}$. Since $HI = I * HA$, there is a finitely generated ideal $F = b_1 A + \cdots + b_n A$ of A such that $\{a_i : i \in \mathbb{N}^*\} \subseteq F \subseteq I$. Since the sequence $(I_i)_{i \in \mathbb{N}}$ is increasing, there is $k \in \mathbb{N}$ such that $b_1, \ldots, b_n \in I_k$ so $F \subseteq I_k$ and $\{a_i : i \in \mathbb{N}^*\} \subseteq I_k$, which is impossible.

Example. Let K be a commutative field and $\{Y_i : i \in \mathbb{N}\}$ a sequence of indeterminates. The ring $A = K[Y_i : i \in \mathbb{N}]$ is not noetherian because its ideal $I = (Y_i : i \in \mathbb{N})$ is not finitely generated. Suppose that $HI = I * HA$, by Lemma 3.1, there is a finitely generated ideal F of A such that $\{Y_i : i \in \mathbb{N}\} \subseteq F \subseteq I$, so $I = F$, which is impossible.

3.4. Proposition. *Let I and J be ideals of the ring A, with $HJ = J * HA$ and $J \subseteq \sqrt{I}$. Then there is $n \in \mathbb{N}^*$ such that $J^n \subseteq I$.*

Proof. Suppose that for each $m \in \mathbb{N}^*$, $J^m \not\subseteq I$, there are $b_{m1}, \ldots, b_{mm} \in J$ such that the product $b_{m1} \cdots b_{mm} \notin I$. Let C be the ideal of A generated by the countably subset $\{b_{mi} : m \in \mathbb{N}^*, 1 \leq i \leq m\}$, then $C \subseteq J$ and $C^m \not\subseteq I$ for every $m \in \mathbb{N}^*$. Since $HJ = J * HA$, by Lemma 3.1, there is a finitely generated ideal F of A such that $C \subseteq F \subseteq J \subseteq \sqrt{I}$, so $F \subseteq \sqrt{I}$. But F is finitely generated, there is $n \in \mathbb{N}^*$ such that $F^n \subseteq I$, so $C^n \subseteq I$, which is impossible.

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