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Curvature and q-strict Convexity

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Abstract. We relate q-strict convexity of compact convex sets $K \subset \mathbb{R}^d$ whose boundary ∂K is a differentiable manifold of class C^q to intrinsic curvature properties of ∂K. Furthermore we prove that the set of q-strictly convex sets is F_σ of first Baire category.

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1. Introduction

Let $\mathcal{C}$ be the set of nonempty compact convex subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric and the induced topology. By $\mathcal{C}^k$ we denote the subset of $\mathcal{C}$ of those convex sets whose boundary is a hypersurface of class C^k. Furthermore let $\mathcal{S} \subset \mathcal{C}$ be the set of strictly convex subsets of $\mathbb{R}^d$, i.e. of those $K \subset \mathbb{R}^d$ whose boundary ∂K does not contain a line segment. It is proved in [3, 5], see also [2], that $\mathcal{C} \setminus (\mathcal{C}^1 \cap \mathcal{S})$ is a F_σ subset of first category and that $\mathcal{C}^2$ is of first category in $\mathcal{C}$. This was strengthened in [10], showing that $\mathcal{C} \setminus (\mathcal{C}^1 \cap \mathcal{S})$ is σ-porous.

We are concerned with analogous questions within the spaces $\mathcal{C}^k$, $k \geq 2$. For arbitrary convex sets it was shown in [12], see also [11], that the lower and upper principal curvatures of the boundary of an arbitrary convex set are almost all 0 and ∞, respectively. Therefore, in order to have a meaningful notion of curvature,

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we impose a differentiability assumption. In place of strict convexity we have in this setting the stronger versions given by the order of contact with the tangent plane of the boundary: We say that $K \in \mathcal{C}^q$ is q-strictly convex if at each point $p \in \partial K$ the tangent hyperplane $T_p\partial K$ has contact of order at most $q-1$ with ∂K.

We relate q-strict convexity of a set $K \in \mathcal{C}^q$ to intrinsic curvature properties of its boundary ∂K proving that an estimate from below on the sectional curvature of ∂K implies q-strict convexity. In contrast to the results in [3, 5, 10] for $\mathcal{C}$ we obtain here that analytic strict convexity is rather exceptional, i.e. that the set $\mathcal{S}_q \subset \mathcal{C}^q$ of q-strictly convex sets is a F_σ-set of first category. Finally we show that the Hausdorff topology on the space of convex sets corresponds to the compact open topology on the set of defining functions.

2. Preliminaries

A convex set $K \subset \mathbb{R}^d$, $K \in \mathcal{C}^k$, can always be described by a convex function $\rho\colon \mathbb{R}^d \to \mathbb{R}$ of class C^k with $K = \rho^{-1}((-\infty, 0])$ and $\partial K = \rho^{-1}(0)$. Such a function ρ is called a defining function for K. A set K is said to be strictly convex if its boundary ∂K does not contain a line segment. As in [1] we say that $K \in \mathcal{C}^q$ is q-strictly convex if the boundary ∂K touches its tangent hyperplanes at most with order $q-1$. In terms of defining functions we may rephrase this as follows.

Definition 2.1. *Let $K = \rho^{-1}((-\infty, 0])$ with $\rho \in C^q(\mathbb{R}^d)$ and $d_x\rho \neq 0$ for each $x \in \partial K = M$. Then K is q-strictly convex if for each $x \in M$ and each $u \in T_xM$ there is $l \leq q$ such that $d_x^l\rho(u) > 0$.*

Here we have written $d_x^l\rho(u) = d_x^l\rho(u, \dots, u)$ for the lth derivative of ρ. Note that $d_x^l\rho$ is a symmetric l-form on $\mathbb{R}^d$ and thus, by polarization, all information is contained in its value on the diagonal. We will denote by $\mathcal{S}_q$ the subspace of $\mathcal{C}^q$ consisting of q-strictly convex sets. We have inclusions

$$\mathcal{C}^{q+1} \cap \mathcal{S}_q \subset \mathcal{S}_{q+1} \ .$$

Thus the present terminology slightly differs from that in [1] where the $\mathcal{S}_q$ were defined to be mutually exclusive.

Proposition 2.2. *Let $K \in \mathcal{C}^q$ and for $x \in \partial K =: M$ denote by n_x the interior normal vector. Then $K \in \mathcal{S}_q$ if and only if for each $x \in M$ there are $\epsilon, c > 0$ and a function $f\colon T_xM \to \mathbb{R}$ with $f(v) \geq c\|v\|^q$, for $v \in T_xM$ with $\|v\| \leq \epsilon$ such that*

$$M \cap B_\epsilon(x) = \{x + v + f(v)n_x \in B_\epsilon(x) \mid v \in T_xM\} \ . \tag{2.3}$$

Thus ∂K locally looks like the graph of a function $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ with $f(0) = 0$ and $f(x) \geq c\|x\|^q$.

Proof. By the implicit function theorem we have a smooth function $f\colon T_xM \to \mathbb{R}$ such that

$$\rho(x + v + f(v)n_x) = 0 \ . \tag{2.4}$$

Inductively we assume that the first $(k-1)$ derivatives of f and ρ in the v-direction vanish. Then

$$0 = \frac{d^k}{dt^k}\bigg|_{t=0} \rho(x + tv + f(tv)n_x) = d^k\rho(x)(v) + d\rho(x)(n_x)d^k f(0)(v) \ . \qquad (2.5)$$

Thus the first novanishing derivatives of ρ and f in the v-direction have the same order. Since ρ is negative on the interior of K we also get from (2.5) that $d^k f(0)(v)$ is positive.

To prove the proposition first assume that $f(v) \geq c\|v\|^q$ for all $v \in T_xM$ with $\|v\|$ sufficiently small. If, for fixed $v_0 \in T_xM$, $\|v_0\| = 1$, k is the order of the first non-vanishing derivative of f in the v_0-direction, then by Taylor's theorem we have

$$f(tv_0) = c't^k + o(t^{k+1})$$

where

$$h(t) = o(t^{k+1}) \ \text{ if } \ \lim_{t\to 0} h(t)/t^{k+1} = 0 \ .$$

If $k > q$, then

$$f(tv_0) = c't^k + o(t^{k+1}) \leq ct^q$$

for sufficiently small t, but this contradicts the initial assumption on f. Therefore $k \leq q$ is the order of the first non-vanishing derivative of f in the v_0-direction, and the same holds for ρ by the preceding remark.

Conversely, assume that K is q-strictly convex at x. Let f be defined by (2.4). For each $v \in T_xM$, $\|v\| = 1$, we have that $d^k f(0)(v) > 0$ and $d^k\rho(x)(v) > 0$ for the same $k \leq q$ by the remark above. Again by Taylor's theorem we find $c(v) > 0$ depending continuously on v such that

$$f(tv) = c'(v)t^k + o(t^{k+1}) \geq c(v)t^q \ .$$

Hence $c := \min_{v\in T_xM, \|v\|=1} \min_t \frac{f(tv)}{t^q} > 0$ and $f(w) \geq ct^q$ for all $w = tv \in T_xM$. $\square$

3. Curvature and strict convexity

For $y \in \mathbb{R}^d$, $n \in \mathbb{R}^d \setminus \{0\}$, $q \in \mathbb{N}_0$ let

$$y_n = \langle y \mid n\rangle \in \mathbb{R} \quad \text{and} \quad y_{n^\perp} = y - \frac{y_n}{\|n\|^2} n \in \mathbb{R}^d$$

denote the projections. The "q-cone" at $x \in \mathbb{R}^d$ in direction of n is then defined as

$$C_q(x,n) := \{y \in \mathbb{R}^d \mid (y-x)_n \geq \|(y-x)_{n^\perp}\|^q\} \ . \qquad (3.1)$$

This set is congruent to the cone at $x = 0$, $n = (0,\dots,0,\lambda)$, $\lambda > 0$, i.e

$$C_q(0,n) := \{(y_1, y_2, \dots, y_{d-1}, y_d) \in \mathbb{R}^d \mid y_d \geq \frac{1}{\lambda}\|(y_1, y_2, \dots, y_{d-1})\|^q\} \ .$$

For $K \in \mathcal{C}$ and $x \in M = \partial K$ we define the "q-curvature" of M at x by

$$\kappa^q(x) = \sup\{\|n\|^{-1} \mid K \cap B_\epsilon(x) \subset C_q(x,n) \text{ for some } \epsilon > 0\} \ .$$

In the case $q = 2$, $\kappa^2(x)$ is the minimal principal curvature of M at x. If $\kappa^p(x) > 0$ at some $x \in M$ then $\kappa^q(x) = \infty$ for all $q > p$.

Theorem 3.2. *A set $K \in \mathcal{C}^q$ is q-strictly convex if and only if the q-curvature of ∂K is positive, i.e. for each $x \in \partial K = M$ there are $n_x \in T_xM^\perp$, $n_x \neq 0$, such that $K \subset C_q(x, n_x)$.*

Proof. It follows from Proposition 2.2 that the assertion holds locally, i.e. K is q-strictly convex if and only if for each point $x \in M$ we find a cone $C_q(x, n_x)$ and $\epsilon_x > 0$ such that $K \cap B_{\epsilon_x}(x) \subset C_q(x, n_x)$. (Then automatically n_x is a normal vector to M pointing in the inward direction.) By compactness, possibly replacing n_x by a larger normal vector λn_x , we get a q-cone containing all of K: By strict convexity K is contained in the half-space $E_x = x + T_xM + \mathbb{R}_0^+ n_x$ of the hyperplane $x + T_xM$ and $x + T_xM \cap K = \{x\}$. Since

$$\bigcup_{\lambda \in \mathbb{R}^+} \operatorname{int} C_q(x, \lambda n_x) = \operatorname{int} E_x \supset K \setminus B_{\epsilon_x}(x)$$

and $K \setminus B_{\epsilon_x}(x)$ is compact, this latter set is contained in $\operatorname{int} C_q(x, \lambda n_x)$ for some $\lambda > 0$. Thus $K \subset C_q(x, \max\{1, \lambda\} n_x)$. □

In the case $q = 2$ we could have replaced the q-cones $C_q(x, n)$ above by balls $\bar{B}_{\|n\|}(x + n)$. Thus the above proof has the immediate

Corollary 3.3. *$K \in \mathcal{C}^2$ is 2-strictly convex if and only if there is $r > 0$ such that for each point $x \in \partial K$ there is $y \in \mathbb{R}^d$, $\|y - x\| = r$, such that $K \subset B_r(y)$.*

We finish this section considering the relation between the sectional curvature of M and q-strict convexity. The minimal sectional curvature of M at $x \in M$ is defined as

$$K(x) := \min\{K(\sigma) \mid \sigma \subset T_xM, \ \dim \sigma = 2\}$$

where $K(\sigma)$ denotes the sectional curvature of the plane σ. If σ is spanned by $u, v \in T_xM$ then $K(\sigma)$ is computed by

$$\begin{aligned} K(\sigma) &= \frac{K(u,v)}{\|u \wedge v\|^2} \quad \text{where} \\ K(u,v) &= \langle R(u,v)v, u\rangle = d_x^2\rho(u,u) d_x^2\rho(v,v) - (d_x^2\rho(u,v))^2 \quad \text{and} \\ \|u \wedge v\|^2 &= u^2v^2 - \langle u \mid v\rangle^2 \ . \end{aligned} \tag{3.4}$$

Proposition 3.5. *Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, $\rho^{-1}(0) = M$ and $d_x\rho \neq 0$ for each $x \in M$. The sectional curvature of M is positive iff ρ or $-\rho$ is the defining function of a 2-strictly convex set.*

Proof. Let $x \in M$ and let ρ or $-\rho$ be the defining function of a 2-strictly convex set. Then $d_x^2\rho(y,y) > 0$ or $d_x^2\rho(y,y) < 0$ for every $y \in T_xM$, i.e. $d_x^2\rho(y,y)$ is a positive or negative definite, symmetric bilinear form. Let E be a 2-dimensional subspace of T_xM and (u,v) an orthonormal basis of E. Then $d_x^2\rho|_E(y_1,y_2) = \langle y_1, Ay_2\rangle$, where

$$A = \begin{pmatrix} d_x^2\rho(u,u) & d_x^2\rho(u,v) \\ d_x^2\rho(v,u) & d_x^2\rho(v,v) \end{pmatrix} .$$

Because A is positive or negative definite

$$\det A = d_x^2\rho(u,u)d_x^2\rho(v,v) - (d_x^2\rho(u,v))^2 > 0 .$$

So from (3.4) we have $K(u,v) > 0$.

We now assume that M has positive sectional curvature. Let $(u_1,\ldots,u_{n-1})$ be an orthonormal basis of eigenvectors of $d_x^2\rho$ in T_xM. Then $d_x^2\rho(u_i,u_j) = \lambda_j\delta_{ij} = \langle u_i, Au_j\rangle$. Because $K(u_i,u_j) > 0$ we get that $K(u_i,u_j) = \lambda_i\lambda_j > 0$. Thus all eigenvalues have the same sign. Therefore $d_x^2\rho$ is negative or positive definite. □

Theorem 3.6. *Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $d_x\rho \neq 0$ for all $x \in M := \rho^{-1}(0)$. Assume that each $x \in M$ has a neighbourhood $U \subset M$ such that on U the sectional curvature K of M satisfies $K(x') \geq Cd(x',x)^m$ with some constant $C = C(U) > 0$ independent of x'. Then for each component M_0 of M one of the two components of $\mathbb{R}^d \setminus M_0$ is strictly $(m+2)$-convex.*

Proof. By a theorem of Sacksteder (see [7], or [4]), M_0 is convex. Assume that M is not strictly $(m+2)$-convex. Then there is a point $x \in M$ and a unit vector $u \in T_xM \subset \mathbb{R}^d$ such that

$$d_x^l\rho(u) = 0 \text{ for all } l \leq m+2 . \tag{3.7}$$

We fix x and u from now on and choose a vector field w on M such that $w(x)$ is a unit vector perpendicular to u. As in Proposition 2.2 we choose $f\colon T_xM \to \mathbb{R}$ satisfying (2.3) with $n_x := -\mathrm{grad}_x\rho/\|\mathrm{grad}_x\rho\|$ and let $\alpha(t) := -f(tu)/\|\mathrm{grad}_x\rho\|$. Thus we have $\alpha\colon(-\epsilon,\epsilon) \to \mathbb{R}$ such that

$$\rho(x + tu - \alpha(t)\mathrm{grad}_x\rho) = 0 .$$

It follows from (2.5) that

$$\left.\frac{d^l}{dt^l}\right|_{t=0} \alpha(t) = 0 \text{ for } l \leq m+2 , \quad \alpha(t) = o(t^{m+2}) . \tag{3.8}$$

Let γ be the curve in M given by

$$\gamma(t) := x + tu - \alpha(t)\mathrm{grad}_x\rho .$$

We claim that

$$d_{\gamma(t)}^2\rho(\dot\gamma(t)) = o(t^m) . \tag{3.9}$$

To see this, we note that

$$0 = \frac{d^2}{dt^2}\rho(\gamma(t)) = d^2_{\gamma(t)}\rho(\dot\gamma(t)) + d_{\gamma(t)}\rho(\gamma''(t)) \ ,$$

hence

$$d^2_{\gamma(t)}\rho(\dot\gamma(t)) = -d_{\gamma(t)}\rho(\gamma''(t)) = \alpha''(t) d_{\gamma(t)}\rho(\mathrm{grad}_x\rho) = o(t^m)$$

because of (3.8).

We now consider the minimal sectional curvature K along the curve γ. From (3.4) we estimate

$$\begin{aligned} K(\gamma(t)) &\leq \frac{K(\dot\gamma(t), w(\gamma(t)))}{\|\dot\gamma(t)\wedge w(\gamma(t))\|^2} \\ &\leq \frac{d^2_{\gamma(t)}\rho(\dot\gamma(t))\ d^2_{\gamma(t)}\rho(w(\gamma(t)))}{\|\dot\gamma(t)\wedge w(\gamma(t))\|^2} \\ &= o(t^m) \ , \end{aligned} \tag{3.10}$$

since

$$\begin{aligned} \lim_{t\to 0} \frac{d^2_{\gamma(t)}\rho(\dot\gamma(t))\ d^2_{\gamma(t)}\rho(w(\gamma(t)))}{t^m\ \|\dot\gamma(t)\wedge w(\gamma(t)\|^2} &= \lim_{t\to 0}\frac{d^2_{\gamma(t)}\rho(\dot\gamma(t))}{t^m}\ \lim_{t\to 0}\frac{d^2_{\gamma(t)}\rho(w(\gamma(t)))}{\|\dot\gamma(t)\wedge w(\gamma(t)\|^2} \\ &= 0\cdot \frac{d^2_x\rho(w(x))}{1} = 0 \end{aligned}$$

because of (3.9). Since the interior distance d^M in M dominates the Euclidean distance in $\mathbb{R}^d$ we have

$$d^M(x,\gamma(t)) \geq |tu - \alpha(t)\mathrm{grad}_x\rho| \geq t \ . \tag{3.11}$$

From (3.11) and (3.10) we have

$$\frac{K(\gamma(t))}{d^M(x,\gamma(t))^m} \leq \frac{K(\gamma(t))}{t^m} \overset{t\to 0}{\longrightarrow} 0.$$

Therefore there can not hold an estimate $K(\gamma(t)) \geq C(d^M(x,\gamma(t))^m) \geq Ct^m$ with a positive constant C as in the assumption of the theorem. □

The following example shows that there is no characterization of q-strict convexity, $q > 2$, by an isotropic growth condition for the sectional curvature as in the assumption of the theorem. To see this look at the function $\rho\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\rho(x,y,z) = x^{2k} + y^{2l} + z$$

for $k \geq l > 2$. Near $(0,0,0)$ this function describes a k-strictly convex set contained in the half space $\{z \leq 0\}$ in $\mathbb{R}^3$. Gradient and Hessian of ρ are

$$\begin{aligned} d\rho(x,y,z) &= (2kx^{2k-1}, 2ly^{2l-1}, 1) \\ d^2\rho(x,y,z) &= \begin{pmatrix} 2k(2k-1)x^{2k-2} & 0 & 0 \\ 0 & 2l(2l-1)y^{2l-2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \end{aligned}$$

We use the tangent vectors $u = (1, 0, -2kx^{2k-1})$ and $v = (0, 1, -2ly^{2l-1})$. Then $\|u\wedge v\|^2 = 1+\mathcal{o}(\|(x, y)\|)$ and the sectional curvature at $K(x, y, z) = K(T_{(x,y,z)}M)$ is computed from (3.4) as

$$\begin{aligned} K(x, y, z) &= 2k(2k-1)x^{2k-2}2l(2l-1)y^{2l-2}\|u \wedge v\|^2 \\ &= (2k(2k-1)2l(2l-1)(1+\mathcal{o}(\|(x, y)\|))x^{2k-2}y^{2l-2} \ . \end{aligned}$$

This vanishes on the lines $\{x = 0\}$ and $\{y = 0\}$. In particular, there is no estimate $K(x, y, z) \geq Cd((x, y, z), 0)^m$ with $C > 0$.

4. Approximation by *q*-strictly convex sets

Among the $\mathcal{S}_q$, $\mathcal{C}^q$ we have for $q \geq 2$ inclusions

$$\mathcal{S}_2 \subset \mathcal{S}_q \subset \mathcal{C}^q \subset \mathcal{C}^2 \ . \tag{4.1}$$

It is shown in [1] that $\mathcal{S}_2 \subset \mathcal{C}^2$ is dense. Hence all the inclusions in (4.1) are dense as well. We proceed to show that $\mathcal{S}_q \subset \mathcal{C}^q$ is F_σ of first category.

Lemma 4.2. *For $K \in \mathcal{S}_q$ and $x \in \partial K$ let n_x denote the inward unit normal vector of ∂K at x. Then the global q-curvature*

$$\kappa^q(K) := \sup\{\lambda^{-1} \mid K \subset C_q(x, \lambda n_x) \text{ for all } x\ \} \tag{4.3}$$

is positive.

Proof. Since n_x depends continuously on x the function

$$\Phi\colon K \times \partial K \to \mathbb{R} \quad \Phi(y, x) := \frac{\|(y-x)_{n^\perp}\|^q}{\langle y - x \mid n_x\rangle} \tag{4.4}$$

is continuous. In particular its maximum $\max\Phi$ is finite since $K\times\partial K$ is compact. From the definition (3.1) we have $K \subset C_q(x, \lambda n_x)$ if and only if $\lambda \geq \Phi(y, x)$ for all $y \in K$. Thus $\kappa^q(K) = \frac{1}{\max\Phi} > 0$. □

Theorem 4.5. *$\mathcal{S}_q \subset \mathcal{C}^q$ is a F_σ-set of first category.*

Proof. We filter $\mathcal{S}_q$ by the global q-curvature κ^q defined in (4.3). Let

$$F_n := \{K \subset \mathcal{C}^q \mid \kappa^q(K) \geq 1/n\} \ .$$

Form Lemma 4.2 we have $\mathcal{S}_q = \bigcup_n F_n$. It remains to show that the F_n are closed in $\mathcal{C}^q$ and nowhere dense.

To that end let $K_\nu \in \mathcal{S}_q$ be a sequence, $K_\nu \overset{\nu\to\infty}{\longrightarrow} K \in \mathcal{C}^q$ with respect to the Hausdorff distance. In order to show that $K \in \mathcal{S}_q$, let $x \in \partial K$ be arbitrary and let $x_\nu \in \partial K_\nu$ converge to x. We also have

$$K_\nu \subset C_q(x_\nu, \frac{1}{n} n_{x_\nu})$$

where n_{x_ν} denotes as before the inward unit normal vector.

Passing to a subsequence if necessary we may assume that n_{x_ν} converges to some vector (which must then coincide with the unit normal vector n_x). We will show that $K \subset C_q(x, \frac{1}{n}n_x)$: Let $y \in K$ and $y_\nu \in K_\nu$ be a convergent sequence, $y = \lim_{\nu\to\infty} y_\nu$. From (3.1) we infer that

$$(y_\nu - x_\nu)_{\frac{1}{n}n_{x_\nu}} \geq \|(y_\nu - x_\nu)_{n_{x_\nu}\perp}\|^q$$

for each ν. By continuity we get

$$(y - x)_{\frac{1}{n}n_x} \geq \|(y - x)_{n_x\perp}\|^q \tag{4.6}$$

and therefore $y \in C_q(x, \frac{1}{n}n_x)$.

Finally, to see that F_n is nowhere dense in $\mathcal{C}^q$, we show that for each $K \in F_n$ we find $K' \in \mathcal{C}^q \setminus F_n$ with arbitrarily small Hausdorff distance $d(K, K')$. To that end let ρ be a defining function for K, i.e. $K = \rho^{-1}((-\infty, 0])$, and pick $x \in \partial K$. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a smooth convex function with $\chi(t) = 0$ for $t \leq 0$ and $\chi(t) > 0$ for $t > 0$. For $\epsilon, \lambda \geq 0$, $v \in T_x\partial K$ and $t \in \mathbb{R}$ define $\rho_{\epsilon,\lambda} \in C^q(\mathbb{R}^n)$ by

$$\rho_{\epsilon,\lambda}(x + v + tn_x) = \rho(x + v + tn_x) + \lambda\chi(\epsilon - t)$$

and let $K_{\epsilon,\lambda} = \rho_{\epsilon,\lambda}^{-1}((-\infty, 0])$ be the convex set defined by $\rho_{\epsilon,\lambda}$. We also set

$$K_{\epsilon,\infty} := K \cap (x + T_x\partial K + [\epsilon, \infty)n_x) = \bigcap_{\lambda\geq 0} K_{\epsilon,\lambda}\ . \tag{4.7}$$

This is the intersection of K with a half space. (In view of the results of the next Section 5, the set $K_{\epsilon,\infty}$ is just the Hausdorff limit $\lambda \to \infty$ of the sets $K_{\epsilon,\lambda}$.)

We have $K_{\epsilon,\lambda} \in \mathcal{C}^q$, $K_{0,\lambda} = K_{\epsilon,0} = K$ and inclusions

$$K_{\epsilon,\infty} \subset K_{\epsilon,\lambda} \subset K\ .$$

It is immediate from (4.7) that $d(K, K_{\epsilon,\infty}) = \epsilon$, hence, for all λ,

$$d(K, K_{\epsilon,\lambda}) \leq \epsilon\ .$$

On the other hand, the $K_{\epsilon,\lambda}$ can not be in F_n for all λ: Therefore let $x_{\epsilon,\lambda} \in \partial K_{\epsilon,\lambda}$ be a sequence converging to $x + \epsilon n_x \in \partial K_{\epsilon,\infty}$ and let $n_{\epsilon,\lambda}$ denote the inward unit normal vector of $\partial K_{\epsilon,\lambda}$ at $x_{\epsilon,\lambda}$. (For instance, choose $t_{\epsilon,\lambda} \in [0, \epsilon]$ such that $\rho(x + t_{\epsilon,\lambda}n_x) + \lambda\chi(\epsilon - t_{\epsilon,\lambda}) = 0$ and set $x_{\epsilon,\lambda} = x + t_{\epsilon,\lambda}n_x$.) If we had $K_{\epsilon,\lambda} \subset C_q(x_{\epsilon,\lambda}, \frac{1}{n}n_{\epsilon,\lambda})$ for all λ then, by the same continuity argument as in the proof of (4.6), we would have $K_{\epsilon,\infty} \subset C_q(x+\epsilon n_x, \frac{1}{n}n_{\epsilon,\infty})$ for some accumulation point $n_{\epsilon,\infty}$ of the $n_{\epsilon,\lambda}$. But this is not possible since $K_{\epsilon,\infty}$ contains line segments through $x + \epsilon n_x$ in its boundary. □

5. Hausdorff convergence versus uniform convergence on compacta of defining functions

Theorem 5.1. *Let* $K_\nu, K \in \mathcal{C}$ *with* $\operatorname{int} K \neq \emptyset$, *and* $\rho_\nu, \rho \in C(\mathbb{R}^d)$ *defining functions of them. Assume that* $\rho_\nu \overset{\nu\to\infty}{\longrightarrow} \rho$ *uniformly on compact subsets of* $\mathbb{R}^d$. *Then* $K_\nu \overset{\nu\to\infty}{\longrightarrow} K$ *with respect to the Hausdorff distance.*

Proof. There is the following criterion for convergence in the Hausdorff topology, (see [8]). A sequence K_ν of compact convex sets in $\mathbb{R}^d$ converges to a set K if and only if

$$K = \{x \in \mathbb{R}^d \mid \text{ there are } x_\nu \in K_\nu, x_\nu \overset{\nu\to\infty}{\longrightarrow} x\} \tag{5.2}$$

and whenever $x_{k_\nu} \overset{\nu\to\infty}{\longrightarrow} x$, $x_{k_\nu} \in K_{k_\nu}$, then $x \in K$.

Let $x_0 \in \operatorname{int} K$. Then $\rho(x_0) < 0$. As $\rho_\nu(x_0) \overset{\nu\to\infty}{\longrightarrow} \rho(x_0)$ we may assume that $x_0 \in \operatorname{int} K_\nu$ for any $\nu \in \mathbb{N}$.

For arbitrary $x \in K$ we may select $y_\nu \in K_\nu$ such that $y_\nu \overset{\nu\to\infty}{\longrightarrow} x$ as follows: In case that $x \in \operatorname{int} K$ taking $y_\nu = x$ we have the result. In case that $x \in \partial K$ we define $y_\nu = x$ if $x \in K_\nu$ and $y_\nu \in \partial K_\nu \cap (x_0, x)$ if x not in K_ν. Let now a convergent subsequence $(y_{k_\nu})_{\nu\in\mathbb{N}}$ of it with $y_{k_\nu} \overset{\nu\to\infty}{\longrightarrow} y_0 \in [x_0, x]$. As $\rho_{k_\nu}(y_{k_\nu}) \overset{\nu\to\infty}{\longrightarrow} \rho(y_0)$ and $\rho_{k_\nu}(y_{k_\nu}) \leq 0$ we deduce that $\rho(y_0) \leq 0$. If $\rho(y_0) < 0$ then $y_{k_\nu} \in \operatorname{int} K_{k_\nu}$ so $y_{k_\nu} = x$ for sufficiently large ν. Then $y_0 = x \in \partial K$ contradicts the fact $\rho(y_0) < 0$. So $\rho(y_0) = 0$ which means that $y_0 \in [x_0, x] \cap \partial K = \{x\}$. Hence any convergent subsequence of the bounded sequence (y_ν) converges to x and the same is true for (y_ν). We deduce that $K_\nu \overset{\nu\to\infty}{\longrightarrow} K$. □

As a converse, for a Hausdorff convergent sequence in $\mathcal{C}$ we find a sequence of defining functions converging uniformly on compacta. For a compact convex set $A \subset \mathbb{R}^d$ with $0 \in \operatorname{int} A$ the Minkowski function is

$$\lambda_A(x) := \inf\{t > 0 \mid x \in tA\}$$

for $x \in \mathbb{R}^d$. Then $\lambda_A - 1$ is a defining function of A.

Lemma 5.3. *Let* $K_\nu \overset{\nu\to\infty}{\longrightarrow} K$ *be a Hausdorff convergent sequence of compact convex sets with* $0 \in \operatorname{int} K$. *Then* $\lambda_{K_\nu} \overset{\nu\to\infty}{\longrightarrow} \lambda_K$ *uniformly on compact sets.*

Proof. Let $D \subset \mathbb{R}^d$ an arbitrary compact set and $B = B_1(0) \subset \mathbb{R}^d$ be the unit ball. Choose $R, \rho > 0$ such that $D \subset RB$ and $\rho B \subset \operatorname{int} K$. Let $\epsilon > 0$ and $\lambda > 1$ such that $(\lambda - 1)R/\rho < \epsilon$. If $0 < \alpha \leq (\lambda - 1)\rho$ and $Q \in \mathcal{C}$ with $\mathcal{H}(K, Q) \leq \alpha$ we easily get that $\rho B \subset Q$. Thus omitting the first elements of the sequence K_ν we may assume $\mathcal{H}(K, K_\nu) \leq \alpha \leq (\lambda - 1)\rho$ for all ν. Then $\rho B \subset K_\nu$. So we obtain

$$K \subset K_\nu + (\lambda - 1)\rho B \subset K_\nu + (\lambda - 1)K_\nu = \lambda K_\nu$$
$$K_\nu \subset K + (\lambda - 1)\rho B \subset K + (\lambda - 1)K = \lambda K.$$

Hence $K \subset \lambda K_\nu$ and $K_\nu \subset \lambda K$ and therefore

$$\lambda_K(x) \geq \frac{\lambda_{K_\nu}(x)}{\lambda}$$
$$\lambda_{K_\nu}(x) \geq \frac{\lambda_K(x)}{\lambda}.$$

Thus

$$\begin{aligned} \lambda_{K_\nu}(x) - \lambda_K(x) &\le (\lambda - 1)\lambda_K(x) \\ \lambda_K(x) - \lambda_{K_\nu}(x) &\le (\lambda - 1)\lambda_{K_\nu}(x). \end{aligned} \tag{5.4}$$

For $x \in D \subset RB$ we have $\lambda_{RB}(x) \le 1$ and

$$\lambda_{RB}(x) = \lambda_{\frac{R}{\rho}\rho B}(x) = \frac{\rho}{R}\lambda_{\rho B}(x) \ge \frac{\rho}{R}\lambda_{K_\nu}(x), \frac{\rho}{R}\lambda_K(x) \ .$$

Hence

$$\lambda_K(x), \lambda_{K_\nu}(x) \le \frac{R}{\rho} \ .$$

By (5.4) this gives

$$|\lambda_{K_\nu}(x) - \lambda_K(x)| \le (\lambda - 1)\frac{R}{\rho} < \epsilon$$

for $x \in D$. □

Theorem 5.5. *Let $K_\nu \overset{\nu\to\infty}{\longrightarrow} K$ be a Hausdorff convergent sequence of compact convex sets with $0 \in \operatorname{int} K$ and let ρ be a defining function for K. Then there are defining functions ρ_ν for the K_ν converging uniformly to ρ on a suitable compact neighbourhood of ∂K.*

Proof. The defining function ρ for K can be divided by $\lambda_K - 1$,

$$\rho = h(\lambda_K - 1)$$

with some positive continuous function h in a compact neighbourhood V of ∂K (see [6]). Then, the $\rho_\nu := h(\lambda_{K_\nu} - 1)$ are defining functions for the K_ν. By Lemma 5.3 $\lambda_{K_\nu} \overset{\nu\to\infty}{\longrightarrow} \lambda_K$ uniformly on any compact subset of $\mathbb{R}^d$ and since h is bounded away from 0 on any compact set, we deduce that $\rho_\nu \overset{\nu\to\infty}{\longrightarrow} \rho$ uniformly on V. □

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