Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 1, 147-151.

Tangent Segments in Minkowski Planes

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Abstract. A Minkowski plane is Euclidean iff the two tangent segments from any exterior point to the unit circle have the same length.

MSC 2000: 52A10, 46B20, 46C99, 51M04
Keywords: Birkhoff orthogonality, Busemann angular bisector, Glogovskij angular bisector, James orthogonality, Minkowski plane, normed plane, Radon plane

1. Introduction

In Euclidean planar geometry, there is a number of interesting theorems referring to tangent segments and secant segments of circles. E.g., the circle is the only closed convex curve in the Euclidean plane with the property that its two tangent segments from any exterior point have equal lengths (see Theorem 2.1). This is an easy consequence of the fact that only the circle has an axis of symmetry in any direction, which is possibly due to Hermann Brunn (see [1, (3.5')]). A similar result for higher dimensions was recently derived in [4]. It is interesting to ask whether such results have analogues in normed linear spaces. For dimension two we will prove a related characterization of the Euclidean plane.

We denote by X an arbitrary normed (or Minkowski) plane with norm $\|\cdot\|$ and origin o, by $S(X)$ and $B(X)$ the *unit circle* and the *unit disc* of X, respectively. Basic references to the geometry of Minkowski planes and spaces are [8] and the monograph [9]. For $x \neq y$, a set l is called a *line* through x and y iff $l = \{tx + (1-$

[*]This work is partially supported by NSFC (China) grant 10671048

$t)y: t \in \mathbb{R}\}$; the *segment* between x and y is the set $\{tx + (1-t)y: t \in [0,1]\}$ and denoted by $[x, y]$; and the set $\{(1-t)x + ty: t \geq 0\}$ is called the ray with starting point x passing through y and denoted by $[x, y\rangle$. A point y is said to be an *exterior point* with respect to a closed convex curve C if y is an exterior point of the convex region bounded by C. Similarly, a point y is said to be an *interior point* with respect to a closed convex curve C if y is an interior point of the convex region bounded by C. A chord $[a, b]$ of a closed convex curve C is called an *affine diameter* of C if there exist two parallel supporting lines $l_1 \neq l_2$ of C such that $a \in l_1$ and $b \in l_2$.

Let $x, y \in X$. The point x is said to be *isosceles orthogonal to* y if $\|x+y\| = \|x-y\|$, and in this case we write $x \perp_I y$. On the other hand, x is said to be *Birkhoff orthogonal to* y if $\|x+ty\| \geq \|x\|$ holds for all $t \in \mathbb{R}$, and this situation is denoted by $x \perp_B y$. A Minkowski plane is said to be a *Radon plane* iff the Birkhoff orthogonality of that plane is symmetric. We refer to [5] and [6] for basic properties of isosceles orthogonality and Birkhoff orthogonality, respectively, and to [1] for a detailed study of the relations between them.

We continue by defining tangent segments and secant segments in Minkowski planes.

Definition 1.1. *Let C be a closed convex curve in X, x be an exterior point with respect to C, l_1 be a supporting line of C through x, and l_2 be a line that intersects (but does not support) C. Then the segment $[x, y]$ is called the tangent segment (from x to C) if*

$$y \in l_1 \cap C \text{ and } \|x-y\| = \inf\{\|x-w\|: w \in l_1 \cap C\},$$

and the segments $[x, z]$, $[x, z']$ are called the secant segment and external secant segment (from x to C along l_2), respectively, if

$$z \in l_2 \cap C \text{ and } \|x-z\| = \sup\{\|x-w\|: w \in l_2 \cap C\},$$

$$z' \in l_2 \cap C \text{ and } \|x-z'\| = \inf\{\|x-w\|: w \in l_2 \cap C\}.$$

The following lemmas will be necessary for the discussion following them.

Lemma 1.2. (cf. [1] (4.1)) *If the implication*

$$x \perp_I y \Rightarrow x \perp_B y$$

holds for any $x, y \in X$, then the Minkowski plane X is Euclidean.

For non-collinear rays $[o, x\rangle$ and $[o, y\rangle$ with unit vectors x and y the ray $[o, x+y\rangle$ is called the *Busemann angular bisector* of the rays $[o, x\rangle$ and $[o, y\rangle$. And the set of all points z lying in the convex cone $\{\alpha x + \beta y: \alpha, \beta \geq 0\}$ and having the same distance to both lines $\langle o, x\rangle$ and $\langle o, y\rangle$ is called the *Glogovskij angular bisector*.

Lemma 1.3. (cf. [3]) *A Minkowski plane is a Radon plane iff Busemann's and Glogovskij's definitions of angular bisectors coincide.*

cf. [7], [8], and [3] for more information about Radon curves and angular bisectors

2. Main results

We begin with a characterization of Euclidean circles in the Euclidean plane. We are sure that this statement was proved already in ancient times, but we could not locate a proof of it.

Theorem 2.1. *The circle is the only closed convex curve in the Euclidean plane with the property that its two tangent segments from any exterior point have equal lengths.*

Proof. First we show that C has to be strictly convex. Suppose the contrary. Then there is a nontrivial maximal segment $[a,b] \subset C$, and we can suppose, without loss of generality, that o is an interior point with respect to C. Therefore for any $t > 1$ the point $p_t = t(\frac{1}{3}a + \frac{2}{3}b)$ is an exterior point of the convex region bounded by C, and as $t \to 1$, tangent segments from p_t to C cannot have equal lengths.

Second we show that the curve C has to be smooth. Suppose to the contrary, i.e., there is a point p on C such that there are two different supporting lines l_1 and l_2 supporting C at p. Let l_3 be a line supporting C at another point q and intersecting l_1 and l_2 at m and n (we can require that $m \neq n$), respectively. Thus, by the assumption of the theorem we have $\|m-p\| = \|m-q\|$ and $\|n-p\| = \|n-q\|$. Therefore $(m-\frac{1}{2}(p+q)) \perp (p-q)$ and $(n-\frac{1}{2}(p+q)) \perp (p-q)$, which is impossible.

For any direction u, let l_1 and l_1' be two parallel supporting lines of C perpendicular to u and supporting C at p and p', respectively. For any chord $[a,b]$ parallel to l_1 of C between p and the affine diameter $[a_0,b_0]$ parallel to l_1, let l_2 and l_3 be the lines supporting C at a and b, respectively. Then l_2 and l_3 will intersect at some point q and meet l_1 at e and f, respectively. By the assumption of the theorem we have the following equalities:

$$\|q-a\| = \|q-b\|\,, \quad \|e-a\| = \|e-p\|\,, \|f-b\| = \|f-p\|\,,$$

and therefore

$$\|e-p\| = \|e-a\| = \|f-b\| = \|f-p\|\,.$$

Thus, the midpoint of $[a,b]$ will lie in the line through p and perpendicular to l_1. Since $[a,b]$ is an arbitrary chord parallel to l_1 of C between p and $[a_0,b_0]$, the midpoint of $[a_0,b_0]$ will lie in the line through p and perpendicular to l_1, which implies that the midpoint of any chord of C parallel to l_1 will lie in the line through p and perpendicular to l_1. Therefore C has an axis of symmetry in the direction of u. Since the direction u is arbitrary, C has an axis of symmetry in any direction, which completes the proof. □

We have shown that among all closed convex curves in Euclidean planes only the circle has the property that the two tangent segments from any exterior point have equal lengths. Now we will show that among all Minkowski planes only the Euclidean plane has the property that the two tangent segments from any exterior point of the unit (Minkowski) circle have equal (Minkowski) lengths.

Theorem 2.2. *A Minkowski plane X is Euclidean iff for any exterior point x of $S(X)$ the lengths of the two corresponding tangent segments from x to $S(X)$ are equal.*

Proof. It is obvious that we only have to prove sufficiency. First we show that by the assumption of equal lengths of the tangent segments the plane is strictly convex.

Suppose that X is not strictly convex. Then there will be a nontrivial maximal segment $[a, b]$ on $S(X)$. Let $x = t(\frac{1}{3}a + \frac{2}{3}b)$ for $t > 1$. Then x is an exterior point of $S(X)$. As $t \to 1$, tangent segments from x to $S(X)$ cannot have equal lengths. Second we show that the plane has to be Radon. (For the geometry of Radon planes we refer to [8] and [7].)

For any $x \in S(X)$ let $y \in S(X)$ be a point with $x \perp_B y$. Then $[x + y, x]$ will be a tangent segment from $x + y$ to $S(X)$. Since X is strictly convex by the statement above, any unit vector of norm < 2 is the sum of two unit vectors in a unique way, which will imply, together with the fact that the length of the segment $[x+y, y]$ is 1, that $[x+y, y]$ is also a tangent segment from $x+y$ to $S(X)$. Hence $y \perp_B x$. Since x is arbitrary, the Birkhoff orthogonality is symmetric.

For any $x, y \in X\backslash\{o\}$ with the property that $x \perp_I y$, the triangle $(-x)xy$ is an isosceles one. Let o' be the center of the circle inscribed to the triangle $(-x)xy$, and let $a \in [x, y]$, $b \in [-x, y]$, $c \in [-x, x]$ be the points such that $o' - a \perp_B y - x$, $o' - b \perp_B y + x$, $o' - c \perp_B x$. By strict convexity this means that $[y, a]$, $[y, b]$, $[-x, b]$, $[-x, c]$, $[x, a]$, $[x, c]$ are tangent segments from $x, y, -x$ to the inscribed circle, respectively. Thus

$$\|y - a\| = \|y - b\| , \|x - a\| = \|x - c\| , \|-x - b\| = \|-x - c\| .$$

These equalities together with the fact that

$$\|y - a\| + \|x - a\| = \|x - y\| = \|x + y\| = \|y - b\| + \|-x - b\|$$

imply $\|-x - c\| = \|x - c\|$, which means that $[y, c]$ is the median (i.e., the segment joining one vertex and the midpoint of the opposite side of a triangle) of $[-x, x]$.

Now, since Busemann's and Glogovskij's definitions of angular bisectors coincide in Radon planes (by Lemma 1.3), o' must lie on the segment $[c, y]$ or, equivalently, on $[o, y]$, and hence $y \perp_B x$ as well as $x \perp_B y$.

Now we have proved that the implication $x \perp_I y \Rightarrow x \perp_B y$ holds for all $x, y \in X\backslash\{o\}$. This implication is trivial when either $x = o$ or $y = o$. Therefore $x \perp_I y \Rightarrow x \perp_B y$ holds for all $x, y \in X$. By Lemma 1.2, X is Euclidean and the proof is complete. □

Corollary 2.3. *Let X be a Minkowski plane. If for any exterior point x of $S(X)$ the squared length of the tangent segment from x to $S(X)$ equals the product of the lengths of the secant segment and the external secant segment, then X is Euclidean.*

The following corollary is an immediate consequence of the well known result that a d-dimensional Minkowski space ($d \geq 2$) is Euclidean iff all its 2-dimensional subspaces are Euclidean.

Theorem 2.4. *Let X be a d-dimensional Minkowski space ($d \geq 2$). The space X is Euclidean iff for any exterior point x of $S(X)$ and any 2-dimensional subspace X_0 through x and o the lengths of the two corresponding tangent segments from x to $S(X_0)$ are equal.*

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Received October 1, 2006