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Remarks on Reflexive Modules, Covers, and Envelopes

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Abstract. We present results on reflexive modules over Gorenstein rings which generalize results of Serre and Samuel on reflexive modules over regular local rings. We characterize Gorenstein rings of dimension at most two by the property that the dual module $\mathrm{Hom}_R(M, R)$ has G-dimension zero for every finitely generated R-module M. In the second section we introduce the notions of a reflexive cover and a reflexive envelope of a module. We show that every finitely generated R-module has a reflexive cover if R is a Gorenstein local ring of dimension at most two. Finally we show that every finitely generated R-module has a reflexive envelope if R is quasi-normal or if R is locally an integral domain.

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Introduction

It is well known that if R is a regular local ring of dimension at most 2, then every reflexive module is free. This result was proved in the late 1950s by J. P. Serre [15]. In Section 1 we generalize this result by observing that if R is a Gorenstein local ring of dimension at most 2, then every reflexive module is G-projective (i.e. has G-dimension zero). This is our Corollary 1.2. We also generalize the following result of P. Samuel:

For a regular local ring R of dimension 3, an R-module M is reflexive if and only if $\operatorname{pd}_R M \leq 1$ and the localizations $M_{\mathfrak{p}}$ are free over $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ distinct from the maximal ideal.

We prove (Proposition 1.7) that for a Gorenstein local ring R of dimension 3, an R-module M is reflexive if and only if $\operatorname{G-dim}_R M \leq 1$ and the localizations $M_{\mathfrak{p}}$ have G-dimension zero over $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ distinct from the maximal ideal. We also prove (Theorem 1.9) that R is Gorenstein of dimension at most two if and only if M^* is G-projective for every finitely generated R-module M.

In Section 2 we define reflexive covers and reflexive envelopes. We show that if R is a Gorenstein local ring of dimension at most two, then every finitely generated R-module has a reflexive cover. We also prove that if R is a quasi-normal ring or if R is locally an integral domain then every finitely generated R-module has a reflexive envelope.

All rings in this paper will be assumed to be commutative Noetherian rings with identity. As usual, M^* denotes $\operatorname{Hom}_R(M, R)$, where R is any ring and M any R-module. We call M^* the *algebraic dual* of M. There is a natural evaluation map $\phi_M : M \to M^{**}$ defined by $\phi_M(m)(f) = f(m)$, for $m \in M$, $f \in M^*$, and we say M is a *reflexive module* if this natural map is an isomorphism.

The Gorenstein dimension, or G-dimension, of a module was introduced by Auslander [1] and Auslander-Bridger [2] in the mid 1960s.

Definition. *A finitely generated R-module M is said to have G-dimension zero (notation:* $\operatorname{G-dim}_R M = 0$*) if and only M satisfies the following three properties:*
1. *M is reflexive,*
2. $\operatorname{Ext}^i_R(M, R) = 0$ *for each* $i \geq 1$*,*
3. $\operatorname{Ext}^i_R(M^*, R) = 0$ *for each* $i \geq 1$.

A module of G-dimension zero is also called *G-projective* or *Gorenstein projective.* The term *totally reflexive* is also used by some authors. For simplicity we will sometimes write simply G-dim instead of $\operatorname{G-dim}_R$ if the ring R is understood. The same convention applies to projective dimension (denoted pd or pd_R) and injective dimension (denoted id or id_R). G-dimension is a refinement of projective dimension in the sense that there is always an inequality $\operatorname{G-dim} M \leq \operatorname{pd} M$. Moreover equality holds if $\operatorname{pd} M < \infty$. Like projective dimension, the G-dimension of a module may range from 0 to ∞. If R is a Gorenstein ring, then every R-module has finite G-dimension. G-dimension enjoys many of the nice properties of projective dimension. For example, if R is local and M is a finitely generated R-module with $\operatorname{G-dim} M < \infty$, then $\operatorname{G-dim} M + \operatorname{depth} M = \operatorname{depth} R$, an equation which is now known as the Auslander-Bridger formula. In fact, G-dimension is to Gorenstein local rings what projective dimension is to regular local rings (cf. the regularity theorem and Gorenstein theorems in the synopsis of [8]). Modules of G-dimension zero over Gorenstein local rings are simply maximal Cohen-Macaulay modules. For these and other standard facts about G-dimension, see [8].

1. Gorenstein rings and reflexive modules

In this section we present generalizations of results of Serre and Samuel, in the context of Gorenstein rings. We also prove that Gorenstein rings R of dimension at most two are characterized by the property that $\mathrm{Hom}_R(M, R)$ has G-dimension zero for all finitely generated R-modules M.

We begin by demonstrating the first result stated in the introduction.

Proposition 1.1. *Let R be a local ring with $\operatorname{depth} R \leq 2$ and M an R-module with $\text{G-dim}\, M < \infty$. If M is a reflexive R-module, then $\text{G-dim}\, M = 0$.*

Proof. By [7, Exercise 1.4.19] (or [3, Proposition 4.7]) we have

$$\operatorname{depth} M = \operatorname{depth} \mathrm{Hom}_R(M^*, R) \geq \min\{2, \operatorname{depth} R\} = \operatorname{depth} R.$$

Since $\text{G-dim}\, M < \infty$ we may apply the Auslander-Bridger formula

$$\text{G-dim}\, M + \operatorname{depth} M = \operatorname{depth} R$$

to conclude that $\text{G-dim}\, M = 0$. □

Corollary 1.2. *Let R be a local Gorenstein ring with $\dim R \leq 2$. If M is a reflexive R-module, then $\text{G-dim}\, M = 0$.*

Remarks. This generalizes Serre's result [15] that reflexive modules are free if R is regular local and $\dim R \leq 2$. Using other terminology, the proposition says reflexive implies totally reflexive if R is Gorenstein local and $\dim R \leq 2$.

Our next goal is to generalize the following result of Samuel ([14, Proposition 3]):

Proposition 1.3. *Let R be a regular local ring of dimension 3. For an R-module M to be reflexive, it is necessary and sufficient that $\operatorname{pd} M \leq 1$, and that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ distinct from the maximal ideal $\mathfrak{m}$.*

We will use the following nice characterization of reflexive modules over Gorenstein rings due to Vasconcelos [17, (1.4)]: "A necessary and sufficient condition for M to be reflexive is that every R-sequence of two or less elements be also an M-sequence." We will refer to this as Vasconcelos' theorem. First we quote the following result from Samuel, where $h(\mathfrak{p})$ denotes the height of the prime ideal $\mathfrak{p}$.

Proposition 1.4. *Let R be Cohen-Macaulay ring, M an R-module and $q \geq 1$ an integer. The following are equivalent:*

$(\mathrm{i})_q$ $\operatorname{depth}(M_{\mathfrak{p}}) \geq \inf(q, h(\mathfrak{p}))$ *for all prime ideals $\mathfrak{p}$ of R;*

$(\mathrm{ii})_q$ *every R-sequence of length $\leq q$ is an M-sequence.*

This result is stated and proved in [14], Proposition 6, page 246.

Corollary 1.5. *Let R be a Gorenstein local ring, M an R-module, and $q \geq 1$ an integer. Then $(\mathrm{i})_q$ and $(\mathrm{ii})_q$ are equivalent to*

$(\mathrm{iii})_q$ $\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \sup(h(\mathfrak{p}) - q, 0)$ *for all prime ideals* $\mathfrak{p}$ *of* R.

Proof. Modify the proof of [14, Corollary 2, p. 247], using the Auslander-Bridger formula. □

Corollary 1.6. *Let* R *be a Gorenstein local ring. An* R*-module* M *is reflexive if and only if*

$$\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \sup(h(\mathfrak{p}) - 2, 0)$$

for all prime ideals $\mathfrak{p}$ *of* R.

Proof. Suppose M is reflexive. For any prime ideal $\mathfrak{p}$ of R, we have $M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module. If $\dim R_\mathfrak{p} \leq 2$ then $\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by Corollary 1.2 and the result is true. Assume therefore that $\dim R_\mathfrak{p} > 2$. By [17, Theorem (1.4)], $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq 2$, and then by the Auslander-Bridger formula, $\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq h(\mathfrak{p}) - 2$.

Conversely, if the condition is true, then by the equivalence of $(\mathrm{iii})_2$ and $(\mathrm{ii})_2$, M is a reflexive R-module by Vasconcelos' theorem. □

Now we can generalize the result of Samuel [14, Proposition 3, p. 239] mentioned above.

Proposition 1.7. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring of dimension* 3*. An* R*-module* M *is reflexive if and only if* $\text{G-dim}_R(M) \leq 1$ *and* $\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ *for all prime ideals* $\mathfrak{p}$ *distinct from* $\mathfrak{m}$*.*

Proof. Assume M is reflexive. By Vasconcelos' theorem, $\operatorname{depth} M \geq 2$ and then by Auslander-Bridger $\text{G-dim}_R(M) \leq 1$. For any prime $\mathfrak{p} \neq \mathfrak{m}$, $M_\mathfrak{p}$ is a reflexive module over the Gorenstein local ring R_p and we have $\dim R_p \leq 2$; therefore $\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by Corollary 1.2.

Conversely, we show that M is reflexive by showing that M satisfies the condition of Corollary 1.6. If $\mathfrak{p} = \mathfrak{m}$, then because $\text{G-dim}\, M \leq 1$ and $h(\mathfrak{m}) = 3$ the condition is satisfied. If $\mathfrak{p} \neq \mathfrak{m}$ then $h(\mathfrak{p}) \leq 2$ and the condition is also satisfied in this case. □

Our third goal of this section (Theorem 1.9) is a characterization of Gorenstein rings of dimension at most two.

Reminders about pure submodules. Recall that a submodule A of a module B is called a *pure submodule* if the induced sequence

$$0 \to \operatorname{Hom}(N, A) \to \operatorname{Hom}(N, B) \to \operatorname{Hom}(N, B/A) \to 0$$

is exact for every finitely presented R-module N, or equivalently the sequence

$$0 \to M \otimes A \to M \otimes B \to M \otimes B/A \to 0$$

is exact for every R-module M (see e.g. [12, Theorem 1.27] for a proof). It is not hard to see that if A is a pure submodule of an injective module, then A is itself an injective module.

Lemma 1.8. *Let N be an R-module. If $\operatorname{Ext}^1_R(M^*, N) = 0$ for every finitely generated R-module M, then $\operatorname{id}_R N \leq 2$.*

Proof. Let $0 \to N \to E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} E^2 \xrightarrow{d^3} \cdots$ be an injective resolution of N. The complex

$$E^\bullet : 0 \longrightarrow E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} E^2 \xrightarrow{d^3} \cdots$$

is a complex of injective modules, and the Hom evaluation morphism

$$\theta_{MRE^\bullet} : M \otimes_R \operatorname{Hom}_R(R, E^\bullet) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), E^\bullet)$$

is an isomorphism of complexes (see (0.3)(b) of [9]). By hypothesis, the first cohomology module of $M \otimes_R E^\bullet$ is zero. It follows that $0 \to M \otimes_R \operatorname{Im}(d^2) \to M \otimes_R E^2$ is exact. This means that $\operatorname{Im}(d^2)$ is a pure submodule of the injective module E^2. Therefore $\operatorname{Im}(d^2)$ is injective, and $\operatorname{id}_R N \leq 2$. $\square$

Theorem 1.9. *For a commutative Noetherian ring R, the following are equivalent.*
1. *R is a Gorenstein ring of dimension at most two.*
2. G-dim $M^* = 0$ *for every finitely generated R-module M.*

Proof. (1) $\implies$ (2). Assume (1) and let M be a finitely generated R-module. By [17, Corollary 1.5], M^* is a reflexive module. For any prime ideal $\mathfrak{p}$ of R, we have that $(M^*)_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module. Since $\dim R_\mathfrak{p} \leq 2$, by Corollary 1.2 $\text{G-dim}_{R_\mathfrak{p}} (M^*)_\mathfrak{p} = 0$. By the localization property of G-dimension, $\text{G-dim}_R M^* = 0$.
(2) $\implies$ (1). Let M be a finitely generated R-module. By hypothesis and the definition of G-dimension, we have $\operatorname{Ext}^1_R(M^*, R) = 0$. By Lemma 1.8 with $N = R$, we conclude that $\operatorname{id}_R R \leq 2$ and R is a Gorenstein ring. $\square$

2. Reflexive covers and envelopes

In this section we study the covering and enveloping properties of the class of finitely generated reflexive modules. The reflexive covers (when they exist) lie between the projective covers and the torsion-free covers, as finitely generated projectives are reflexive, and reflexive modules are torsion-free.

Let $\mathcal{X}$ be a class of R-modules, and let M be any R-module. An *$\mathcal{X}$-precover* of M is defined to be an R-homomorphism $\phi : C \to M$ from some $C \in \mathcal{X}$ to M with the property:

(i) for any R-homomorphism $f : D \to M$ from a module $D \in \mathcal{X}$ to M, there is a homomorphism $g : D \to C$ such that $\phi g = f$.

An $\mathcal{X}$-precover $\phi : C \to M$ is called an *$\mathcal{X}$-cover* if it satisfies property

(ii) whenever $g : C \to C$ is such that $\phi g = \phi$, then g is an automorphism of C.

For example, a projective precover of a module M is just a surjection $\phi : P \to M$ where P is a projective module. Projective covers were originally studied by Bass in [5]. If $\mathcal{X}$ is the class of torsion-free modules over an integral domain, we get the *torsion-free covers*, which were shown to exist by Enochs in [10]. Flat covers exist for all modules over any ring; see [6] or [11, Theorem 7.4.4].

We note that an $\mathcal{X}$-cover, when it exists, is unique up to isomorphism.

Definition. *Let $\mathcal{X}$ be the class of finitely generated reflexive R-modules. An $\mathcal{X}$-cover of a finitely generated R-module M will be called a reflexive cover.*

A Gorenstein projective module is reflexive according to the definition. Corollary 1.2 states the converse is true for a Gorenstein local ring R with $\dim R \leq 2$. Thus we immediately get the following.

Proposition 2.1. *Let R be local Gorenstein with dimension at most 2. A finitely generated R-module M is Gorenstein projective if and only if M is a reflexive R-module.*

Theorem 2.2. *Let R be a local Gorenstein ring of dimension at most 2, and let M be a finitely generated R-module. Then M has a finitely generated reflexive cover $C \to M$.*

Proof. By [11, Theorem 11.6.9], M has a Gorenstein projective cover $C \to M$ and C is finitely generated. It follows from the previous proposition that $C \to M$ is the reflexive cover of M. ☐

Example of a reflexive cover. We now give a nontrivial example of a reflexive cover. Let k be a field and consider the one-dimensional local Gorenstein domain $R = k[[t^2, t^3]]$. Since R is a one-dimensional local Gorenstein domain, a finitely generated module is reflexive if and only if it is torsion-free [Kaplansky, Theorem 222]. In particular the maximal ideal $\mathfrak{m}$ of R is reflexive. Let I be any principal ideal of R. We first show that the natural map $\phi : \mathfrak{m} \to \mathfrak{m}/I$ is a reflexive precover. Let P be any finitely generated reflexive R-module. Since $\text{G-dim}_R\, P = 0$, and I is free on one generator, we have $\operatorname{Ext}^1_R(P, I) = 0$. This shows that $\mathfrak{m} \to \mathfrak{m}/I$ is a reflexive precover. We know that $\mathfrak{m}/I$ has a reflexive cover, and any reflexive cover is a direct summand of any precover. That is, $\mathfrak{m} = M_1 \oplus K$ for submodules M_1 and K such that the restriction $\phi|_{M_1} : M_1 \to \mathfrak{m}/I$ is a reflexive cover of $\mathfrak{m}/I$ ([Xu, Theorem 1.2.7], for example). But $\mathfrak{m}$ is indecomposable, being of rank 1, and so $K = 0$. Thus $\mathfrak{m} = M_1$ and $\mathfrak{m} \to \mathfrak{m}/I$ is the reflexive cover of $\mathfrak{m}/I$.

Remarks. (1) Other than Gorenstein local rings of dimension at most two, we would like to know which other rings R (if any) have the property that all finitely generated R-modules have reflexive covers.

(2) If R is a ring with the property that there are only finitely many isomorphism classes of indecomposable reflexive R-modules, then every R-module has a reflexive precover. The argument uses the idea in the proof of [4, Proposition 4.2]. See also [16, page 409].

We now turn to envelopes, which are dual to covers. For a class $\mathcal{X}$ of modules, an $\mathcal{X}$*-preenvelope* of an R-module M is an R-homomorphism $\phi : M \to X$ with $X \in \mathcal{X}$ such that

(i) for any R-homomorphism $f : M \to Y$ where $Y \in \mathcal{X}$, there is a homomorphism $g : X \to Y$ such that $g\phi = f$.

An $\mathcal{X}$-preenvelope $\phi : M \to X$ is called an $\mathcal{X}$*-envelope* if it satisfies

(ii) whenever $g : X \to X$ is such that $g\phi = \phi$, then g is an automorphism of X.

Definition. *Let $\mathcal{X}$ be the class of finitely generated reflexive R-modules. An $\mathcal{X}$-envelope of a finitely generated R-module M will be called a reflexive envelope of M.*

Recall that when R is an integral domain we have the notion of the rank of a module M, which can be defined to be the maximal number of R-linearly independent elements contained in M. If K is the field of fractions of the domain R, then $\operatorname{rank} M$ is equivalently, the dimension of $M \otimes_R K$ as a vector space over K, or the dimension of the localization $S^{-1}M$, where $S = R \setminus \{0\}$. We denote the torsion submodule of an R-module M by $t(M)$. That is, $t(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}$. We say M is a torsion module if $M = t(M)$, and M is torsion-free if $t(M) = 0$. The proof of the next lemma is omitted, as the results are either well-known or are easily proved.

Lemma 2.3. *Let R be an integral domain, and let M and N be finitely generated R-modules.*

1. *The map $\phi_M : M \to M^{**}$ is injective if and only if M is torsion-free.*
2. *The module M is a torsion module if and only if $S^{-1}M = 0$.*
3. $\operatorname{rank} M = \operatorname{rank} M^*$.
4. *If M is a torsion and N is a torsion-free R-module, then* $\operatorname{Hom}_R(M, N) = 0$.
5. *The modules M and $M/t(M)$ have the same duals and biduals.*
6. *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of R-modules, then* $\operatorname{rank} M = \operatorname{rank} M' + \operatorname{rank} M''$.

The next lemma is almost certainly known but will be needed later; we sketch a proof for lack of a convenient reference.

Lemma 2.4. *Let R be a integral domain. If a finitely generated R-module N is of the form $N = M^*$ for some R-module M, then N is reflexive.*

Proof. The exact sequence $0 \to N \xrightarrow{\phi_N} N^{**} \to C \to 0$ is split exact, where $C = \operatorname{Coker}(\phi_N)$ (by [8, Proposition 1.1.9(a)] for example). Since N and N^{**} have the same rank, $\operatorname{rank} C = 0$ and C is a torsion module. But C is isomorphic to a direct summand of the torsion-free module M^{**} and hence $C = 0$. $\square$

The previous result is true for rings which are locally domains. To prepare for the proof of this we quote [13, Corollary IV.1.6, p. 94]. In the following $\operatorname{Max}(R)$ denotes the set of all maximal ideals of R.

Lemma 2.5. *A linear mapping $\alpha : M \to N$ of finitely generated R-modules (R any ring) is bijective if and only if for all $\mathfrak{m} \in \mathrm{Max}(R)$ the mapping $\alpha_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is bijective.*

We note that for a ring R, a multiplicatively closed subset S, and a finitely generated R-module M we have

$$S^{-1}(M^*) = S^{-1}\,\mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}R) = (S^{-1}M)^*.$$

Of course we are using $(-)^*$ here in two different ways: the first is the dual with respect to R; the second the dual with respect to $S^{-1}R$. In particular, $(M^*)_{\mathfrak{p}} = (M_{\mathfrak{p}})^*$ for any prime ideal $\mathfrak{p}$ of R and this module will be denoted simply $M^*_{\mathfrak{p}}$.

Proposition 2.6. *Let R be any ring. A finitely generated R-module M is reflexive if and only if it is locally reflexive for all $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. Since localization is exact, it is clear that if the natural map $\phi : M \to M^{**}$ is bijective R-linear, then $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to M^{**}_{\mathfrak{m}}$ is bijective $R_{\mathfrak{m}}$-linear for all $\mathfrak{m} \in \mathrm{Max}(R)$. Conversely, if $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to M^{**}_{\mathfrak{m}}$ is bijective $R_{\mathfrak{m}}$-linear for all $\mathfrak{m} \in \mathrm{Max}(R)$, then $\phi : M \to M^{**}$ is a bijective R-linear map by Lemma 2.5. □

Proposition 2.7. *If R is a ring with the property that $R_{\mathfrak{m}}$ is a domain for all $\mathfrak{m} \in \mathrm{Max}(R)$, then every dual module is reflexive; that is, if $N = M^*$ for some finitely generated R-module M, then N is reflexive.*

Proof. For all $\mathfrak{m} \in \mathrm{Max}(R)$ the $R_{\mathfrak{m}}$-module $N_{\mathfrak{m}} = M^*_{\mathfrak{m}}$ is reflexive over $R_{\mathfrak{m}}$ by Lemma 2.4. Now since $N_{\mathfrak{m}} \to N^{**}_{\mathfrak{m}}$ is bijective for all $\mathfrak{m} \in \mathrm{Max}(R)$, N is reflexive by Lemma 2.5. □

Proposition 2.8. *Let M be a finitely generated R-module and suppose that M^* is reflexive. Then the natural map $\phi_M : M \longrightarrow M^{**}$ is a reflexive envelope of M.*

Proof. Since M^* is reflexive, M^{**} is reflexive. Let Y be a reflexive R-module and $f : M \to Y$ a homomorphism. Then the homomorphism $\phi_Y^{-1} \circ f^{**} : M^{**} \to Y$ satisfies $\phi_Y^{-1} f^{**} \phi_M = f$. This shows that $\phi_M : M \to M^{**}$ is a reflexive preenvelope. Now suppose $g : M^{**} \to M^{**}$ is a homomorphism and $g\phi_M = \phi_M$. Then $\phi_M^* g^* = \phi_M^*$ and since ϕ_M is invertible $g^* = 1$. Taking duals once again gives $g^{**} = 1$. Then $\phi_{M^{**}} g = g^{**} \phi_{M^{**}}$ and since $\phi_{M^{**}}$ is invertible we have $g = 1$. This completes the proof. □

Note. The proof shows that $\phi : M \to M^{**}$ actually enjoys a property stronger than that of an envelope: any map $g : M^{**} \to M^{**}$ such that $g\phi = \phi$ is not just an automorphism of M^{**} but is in fact the identity map on M^{**}.

Definition. (cf. [17]) *A ring R is said to be quasi-normal if for any prime ideal $\mathfrak{p}$ of R the following hold:* (1) *if $ht(\mathfrak{p}) \geq 2$, then depth $R_{\mathfrak{p}} \geq 2$; and* (2) *if $ht(\mathfrak{p}) \leq 1$, then $R_{\mathfrak{p}}$ is Gorenstein.*

Theorem 2.9. *If R is either a quasi-normal ring or locally an integral domain, then for every finitely generated R-module M, the natural map $\phi_M : M \to M^{**}$ is a reflexive envelope of M.*

Proof. Let M be an R-module. If R is a quasi-normal ring, then M^* is reflexive by [17, Corollary 1.5]; then by 2.8 the natural map $M \to M^{**}$ is a reflexive envelope. If R is locally a domain, then by 2.7 and 2.8 the natural map $\phi_M : M \to M^{**}$ is a reflexive envelope. □

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