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Spherical Quadrangles

Catarina P. Avelino* **A. M. d'Azevedo Breda†** **Altino F. Santos**

Department of Mathematics, UTAD
5001 - 801 Vila Real, Portugal
e-mail: cavelino@utad.pt

Department of Mathematics, University of Aveiro
3810-193 Aveiro, Portugal
e-mail: ambreda@ua.pt

Department of Mathematics, UTAD
5001 - 801 Vila Real, Portugal
e-mail: afolgado@utad.pt

Abstract. We introduce the notion of quasi-well-centered spherical quadrangle, or QWCSQ for short, describing a geometrical method to construct any QWCSQ. It is shown that any spherical quadrangle is congruent to a QWCSQ. We classify such quadrangles taking into account the relative position of the spherical moons containing their sides. This allows us to conclude that the class of all QWCSQ is a differentiable manifold of dimension five.
Keywords: spherical geometry, applications of spherical trigonometry

1. Introduction

Let S^2 be the unit 2-sphere. The notion of well-centered spherical moon was introduced in [1] (a spherical moon whose vertices belong to the great circle $x = 0$, and whose bisecting semi-great circle contains the point $C = (1, 0, 0)$; in Figure 1

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a well centered spherical moon, L_1 is presented). Also in [1] it was established that any spherical (geodesic) quadrangle with congruent opposite angles is congruent to the intersection of two well-centered spherical moons, i.e., a well-centered spherical quadrangle.

In this paper we generalize these results to the class of all spherical quadrangles, by introducing the notion of quasi-well-centered spherical quadrangle. Some of the obtained results are based in spherical trigonometry formulas. The cosine rules state that the angles α_1, α_2 and α_3 of a spherical triangle satisfy

$$\cos\alpha_1 = \frac{\cos a - \cos b\,\cos c}{\sin b\,\sin c} \quad \text{and} \quad \cos a = \frac{\cos\alpha_1 + \cos\alpha_2\,\cos\alpha_3}{\sin\alpha_2\,\sin\alpha_3}, \tag{1.1}$$

where a, b and c are the lengths of the edges opposite to α_1, α_2 and α_3, respectively. For a detailed discussion on spherical trigonometry see [2].

2. Spherical Quadrangles

By a *quasi-well-centered spherical quadrangle* (QWCSQ) we mean a spherical quadrangle Q which is the intersection of a well-centered spherical moon L_1 with vertices N and $-N$ (where $N = (0,0,1)$) and a spherical moon L_2 with one of its vertices, say v, in the first octant $(x, y, z \geq 0)$, see Figure 1.

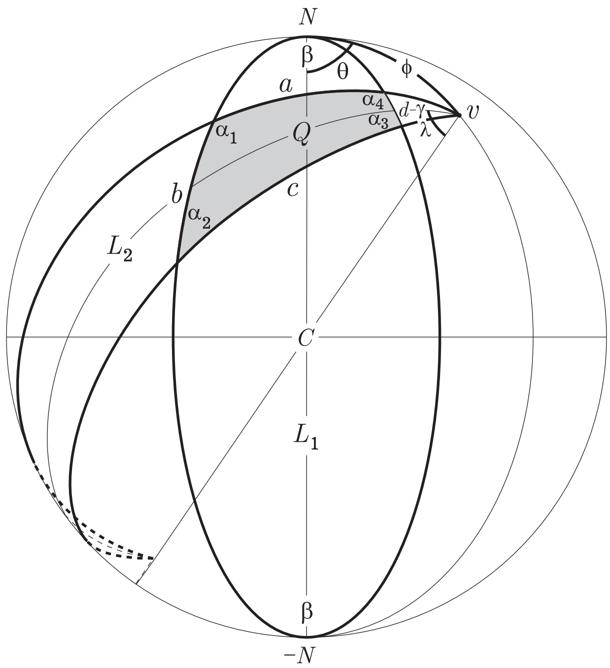

Figure 1. A quasi-well-centered spherical quadrangle Q

We have used the following notation:

- β and γ are the angles measure of the spherical moons L_1 and L_2, respectively; β, $\gamma \in (0, \pi)$;

- α_1, α_2, α_3, α_4 and a, b, c, d are, respectively, the internal angles and the edge lengths of $Q = L_1 \cap L_2$;
- θ and ϕ are the spherical coordinates of the vertex v of L_2; in other words, considering $v = (x, y, z)$, then $x = \cos\theta \sin\phi$, $y = \sin\theta \sin\phi$ and $z = \cos\phi$. Geometrically, θ is the oriented angle between the bisector of L_1 and the meridian through N that contains v; $\theta \in \left(\frac{\beta}{2}, \frac{\pi}{2}\right]$. On the other hand, ϕ is the oriented angle between N and the vertex v; $\phi \in \left(0, \frac{\pi}{2}\right]$;
- λ is the oriented angle between the line connecting v and $C = (1, 0, 0)$, and the bisector of L_2; $\lambda \in \left(-\frac{\pi-\gamma}{2}, \frac{\pi-\gamma}{2}\right)$.

Remark 1. With the above notation we have the following properties.

1. If $\theta = \frac{\pi}{2}$ and $\lambda = 0$, then $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$, i.e., $Q = L_1 \cap L_2$ is a spherical parallelogram. In addition, if $\phi = \frac{\pi}{2}$, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ and Q is a spherical rectangle.
2. If $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$, then $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$, and so $Q = L_1 \cap L_2$ is an isosceles trapezoid.
3. If $\phi = \frac{\pi}{2}$ and $\lambda = 0$, then $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$, and so $Q = L_1 \cap L_2$ is also an isosceles trapezoid.

Proposition 2.1. *Let Q be a spherical quadrangle with internal angles α_1, α_2, α_3 and α_4, and edge lengths a, b, c and d. Then any three of these parameters are completely determined by the remaining five.*

Proof. Let Q be a spherical quadrangle as described above. We shall show how to determine α_4, c and d as functions of α_1, α_2, α_3, a and b. Other cases are treated in a similar way.

Let l be the diagonal of Q through α_2 and α_4 as illustrated in Figure 2.

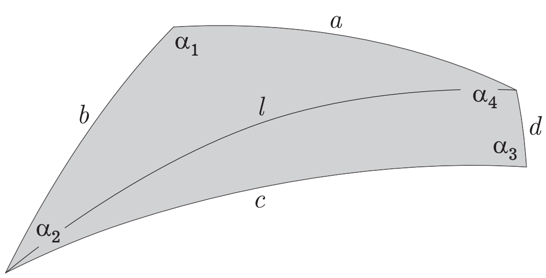

Figure 2. A spherical quadrangle

By (1.1), we have

$$\cos\alpha_1 = \frac{\cos l - \cos a \cos b}{\sin a \sin b} \qquad \text{and} \qquad \cos\alpha_3 = \frac{\cos l - \cos c \cos d}{\sin c \sin d},$$

and so

$$\cos a \cos b + \sin a \sin b \cos\alpha_1 = \cos c \cos d + \sin c \sin d \cos\alpha_3. \tag{2.1}$$

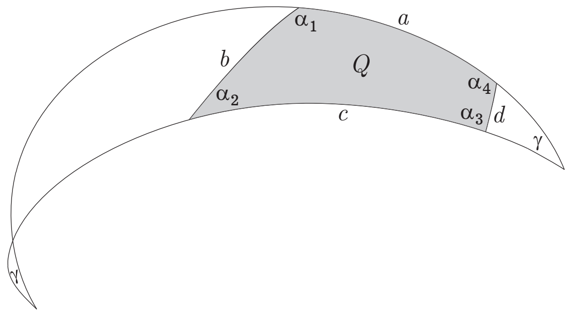

Figure 3. A spherical moon obtained by extending a pair of opposite sides of Q

Now, extending the sides a and c of Q one gets a spherical moon as shown in Figure 3. Let γ be its angle measure.
Using again (1.1), one gets

$$\cos b = \frac{\cos\gamma + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2} \qquad \text{and} \qquad \cos d = \frac{\cos\gamma + \cos\alpha_3\cos\alpha_4}{\sin\alpha_3\sin\alpha_4},$$

and so

$$-\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2\cos b = -\cos\alpha_3\cos\alpha_4 + \sin\alpha_3\sin\alpha_4\cos d. \quad (2.2)$$

Similarly, extending the other pair of opposite sides of Q we get the formula

$$-\cos\alpha_1\cos\alpha_4 + \sin\alpha_1\sin\alpha_4\cos a = -\cos\alpha_2\cos\alpha_3 + \sin\alpha_2\sin\alpha_3\cos c. \quad (2.3)$$

From equations (2.2) and (2.3) we may obtain d and c as functions of α_4, respectively. Replacing c and d in (2.1) by the obtained expressions, we get α_4 as function of α_1, α_3, a and b. The expressions for c and d follow immediately. Therefore, α_4, c and d are completely determined when α_1, α_2, α_3, a and b are fixed values. □

Proposition 2.2. *Any spherical quadrangle is congruent to a QWCSQ. Besides, its sides and angles are completely determined by the five parameters* β, γ, θ, ϕ *and* λ *defined in Figure* 1.

Proof. Suppose that Q is a spherical quadrangle with internal angles, α_1, α_2, α_3 and α_4, and edge lengths a, b, c and d. The extension of the edges of Q give rise to two spherical moons L_1 and L_2, such that $Q = L_1 \cap L_2$. Now it follows that there is a spherical isometry σ such that $\sigma(L_1)$ is a well centered spherical moon with vertices N and $-N$ ($N = (0, 0, 1)$) and $\sigma(L_2)$ has one of these vertices in the first octant. And so Q is congruent to a QWCSQ. By Proposition 2.1 the knowledge of α_1, α_2, α_3, a and b determines α_4, c and d.

Using the labelling of Figure 1 one gets the following system of equations in the five variables β, γ, θ, ϕ and λ.

$$\begin{cases} \cos\alpha_1 &= \cos\frac{\beta+2\theta}{2}\sin\frac{\gamma+2\lambda}{2} - \sin\frac{\beta+2\theta}{2}\cos\frac{\gamma+2\lambda}{2}\cos\phi \\ \cos\alpha_2 &= \cos\frac{\beta+2\theta}{2}\sin\frac{\gamma-2\lambda}{2} + \sin\frac{\beta+2\theta}{2}\cos\frac{\gamma-2\lambda}{2}\cos\phi \\ \cos\alpha_3 &= -\cos\frac{\beta-2\theta}{2}\sin\frac{\gamma-2\lambda}{2} + \sin\frac{\beta-2\theta}{2}\cos\frac{\gamma-2\lambda}{2}\cos\phi \\ \cos a &= \frac{\cos\beta+\cos\alpha_1\cos\alpha_4}{\sin\alpha_1\sin\alpha_4} \\ \cos b &= \frac{\cos\gamma+\cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2} \end{cases}$$

We obtain the expressions of β and γ from the two last equations. Replacing these expressions in the first three equations and solving now the 3×3 system of equations, we also get θ, ϕ and λ. □

Let $\mathcal{Q}$ be the set of all QWCSQ.

Corollary 2.1. *The degree of freedom given by the five parameters β, γ, θ, ϕ and λ allows us to conclude that $\mathcal{Q}$ is a differentiable manifold of dimension five; $\beta,\gamma\in(0,\pi)$, $\theta\in\left(\frac{\beta}{2},\frac{\pi}{2}\right]$, $\phi\in\left(0,\frac{\pi}{2}\right]$, $\lambda\in\left(-\frac{\pi-\gamma}{2},\frac{\pi-\gamma}{2}\right)$.*
The set of all isosceles trapezoids contains a manifold of dimension three. The submanifold contained in the border of $\mathcal{Q}$ defined by the equations $\theta=\frac{\pi}{2}$ and $\phi=\frac{\pi}{2}$ has dimension three.

References

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