Real and Complex Fundamental Solutions — A Way for Unifying Mathematical Analysis.*

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Goal of the mini-course

The Fundamental Theorem of Calculus says that a differentiable function h defined in an interval $a \leq x \leq b$ can be recovered from its derivative h' and its boundary values:

$$h(x) = h(a) + \int_a^x h'(\xi)d\xi.$$

The mini-course will show that an analogous result is true for partial differential operators:

Suppose $\mathcal{L}$ is a differential operator of order k. Moreover, let u be a function defined and k times continuously differentiable in the closure of a domain Ω of $\mathbf{R^n}$. Provided the adjoint differential operator possesses a fundamental solution, we shall see that u can be recovered from $\mathcal{L}u$ and the boundary values of u. Strictly speaking, we shall get an integral representation of u in form of the sum of two integrals. One of them is a boundary integral, the other is a domain integral whose integrand is the product of $\mathcal{L}u$ and the fundamental solution of the adjoint operator. Such integral representations can be used for solving boundary value problems.

Since for getting this result we need basic concepts of distribution theory, the mini-course will also include an elementary approach to distribution theory as far as it will be essential for our goals.

While the first part of the mini-course (Section 1) will prove general statements, the second part (Section 2) will consider the case of the complex plane more in detail. This concerns, especially, boundary value problems for non-linear systems in the plane.

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The third part (Section 3), finally, deals with initial value problems of type

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{F}\left(t, x, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) \\ u(0,x) &= \varphi(x). \end{aligned}$$

We shall see that such initial value problems can be solved using interior estimates for solutions of elliptic differential equations. Since interior estimates can be obtained from integral representations by fundamental solutions, the above mentioned initial value problems can also be solved within the framework of the theory of fundamental solutions.

At the end of the mini-course (Section 4) we shall discuss some further generalizations and open problems.

1 Integral representations using fundamental solutions

1.1 Differential operators of divergence type and their Green's Formulae

Let Ω be a bounded domain in $\mathbf{R^n}$ with sufficiently smooth boundary. A differential operator $\mathcal{L}$ of order k is called a differential operator of *divergence type* if there exist another operator $\mathcal{L}^*$ of order k and n differential operators P_i of order $k-1$ such that

$$v\mathcal{L}u + (-1)^{k+1}u\mathcal{L}^*v = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}[u,v],$$

u and v being k times continuously differentiable. The operator $\mathcal{L}^*$ is called *adjoint* to $\mathcal{L}$. In case $\mathcal{L}^* = \mathcal{L}$, the operator $\mathcal{L}$ is called *self-adjoint*.

Example 1 The Laplace operator $\mathcal{L} = \Delta$ is a self-adjoint differential operator of divergence type because

$$P_i = v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}$$

leads to

$$\sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}[u,v] = v\Delta u - u\Delta v.$$

Example 2

$$\mathcal{L}u = \sum_{i,j} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\Big)\frac{\partial u}{\partial x_j} + \sum_k b_k(x)\frac{\partial u}{\partial x_k} + c(x)u$$

is a differential operator of divergence type. Here we have

$$P_i = v\sum_j a_{ij}\frac{\partial u}{\partial x_j} - u\sum_j a_{ji}\frac{\partial v}{\partial x_j} + b_i uv,$$

and the adjoint diferential operator is

$$\mathcal{L}^* v = \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ji}(x)\frac{\partial v}{\partial x_j}\right) - \sum_i \frac{\partial}{\partial x_i}\left(b_i(x)v\right) + c(x)v.$$

Applying the Gauss Integral Formula, one gets the following *Green Integral Formula* for differential operators of divergence type

$$\int_\Omega \Big(v\mathcal{L}u + (-1)^{k+1}u\mathcal{L}^* v\Big)dx = \int_{\partial\Omega}\sum_{i=1}^n P_i[u,v]N_i d\mu \tag{1}$$

where $(N_1, ..., N_n) = N$ is the outer unit normal and $d\mu$ is the measure element of $\partial\Omega$.

1.2 The concept of distributional solutions

Using the Green Integral Formula for differential operators of divergence type, one gets a characterization of solutions by integral relations. For this purpose introduce so-called *test functions*. A test function for a differential equation of order k is a k times continuously differentiable function vanishing identically in a neighbourhood of the boundary. Consequently, replacing v by a test function, the boundary integral in the Green Integral Formula (1) is equal to zero and thus we have

$$\int_\Omega \Big(\varphi\mathcal{L}u + (-1)^{k+1}u\mathcal{L}^*\varphi\Big)dx = 0 \tag{2}$$

for each choice of the test function φ.

Now assume that u is a classical solution of the differential equation $\mathcal{L}u = 0$, i.e., u is k times continuously differentiable and the differential equation is pointwise satisfied everywhere in Ω. Then (2) implies that

$$\int_\Omega u\mathcal{L}^*\varphi dx = 0 \tag{3}$$

for each choice of the test function φ. Conversely, if the relation (3) is satisfied for any φ, then one has also

$$\int_\Omega \varphi\mathcal{L}u dx = 0$$

for each φ in view of (2). Taking into account the Fundamental Lemma of Variational Calculus, the last relation implies $\mathcal{L}u = 0$ everywhere in Ω. To sum up, the following statement has been proved:

A k times continuously differentiable function u is a classical solution of $\mathcal{L}u = 0$ *if and only if relation (3) is true for each φ.*

On the other hand, it may happen that relation (3) is satisfied for each φ if u is only an integrable function. Then u is called a *distributional solution* of $\mathcal{L}u = 0$.

Similarly, if u is a (classical) solution of the inhomogeneous equation $\mathcal{L}u = h$ where the right-hand side $h = h(x)$ is a given function in Ω, then instead of (3) the relation

$$\int\limits_{\Omega} \Big(\varphi h + (-1)^{k+1} u \mathcal{L}^* \varphi\Big) dx = 0 \tag{4}$$

is satisfied for each φ. Therefore, a distributional solution of the inhomogeneous equation $\mathcal{L}u = h$ is an integrable function u satisfying (4) for each φ.

1.3 The concept of fundamental solutions

In order to apply Green's Integral Formula to functions having an isolated singularity at an interior point ξ of Ω, one has to omit a neighbourhood of ξ. Introduce the domain $\Omega_\varepsilon = \Omega \setminus \overline{U_\varepsilon}$ where U_ε means the ε-neighbourhood of ξ. Notice that the boundary of Ω_ε consists of two parts, the boundary $\partial\Omega$ of the given domain Ω and of the ε-sphere centred at ξ.

Now let u be any (k times continuously differentiable) function, while $v = E^*(x, \xi)$ is supposed to be a solution of the adjoint equation $\mathcal{L}^* v = 0$ having an isolated singularity at ξ. Then the Green Integral Formula applied to u and $v = E^*(x, \xi)$ yields the relation

$$\int\limits_{\Omega_\varepsilon} E^*(x, \xi) \mathcal{L}u dx = \tag{5}$$

$$\int\limits_{\partial\Omega} \sum_{i=1}^{n} P_i[u, E^*(x, \xi)] N_i d\mu + \int\limits_{|x-\xi|=\varepsilon} \sum_{i=1}^{n} P_i[u, E^*(x, \xi)] N_i d\mu.$$

This relation leads to the concept of a fundamental solution (see [23]):

Definition The function $v = E^*(x, \xi)$ is said to be a *fundamental solution* of the equation $\mathcal{L}^* v = 0$ with the singularity at ξ if the following three conditions are satisfied:

1. $E^*(x,\xi)$ is a solution of $\mathcal{L}^*v = 0$ for $x \neq \xi$.

2. The boundary integral over the ε-sphere in (5) tends to $(-1)^k u(\xi)$ as ε tends to zero, i.e., if one has

$$\lim_{\varepsilon \to 0} \int\limits_{|x-\xi|=\varepsilon} \sum_{i=1}^{n} P_i[u, E^*(x,\xi)]N_i d\mu = (-1)^k u(\xi)$$

   where u is any k times continuously differentiable function.

3. The function $E^*(x,\xi)$ is weakly singular at ξ, i.e., it can be estimated by

$$|E^*(x,\xi)| \leq \frac{\text{const}}{|x-\xi|^\alpha}$$

   where $\alpha < n$.

Example If ω_n means the surface measure of the unit sphere in $\mathbf{R^n}$, then

$$-\frac{1}{(n-2)\omega_n |x-\xi|^{n-2}}$$

is a fundamental solution of the Laplace equation in $\mathbf{R^n}$, $n \geq 3$. Indeed, Example 1 of Section 1.1 implies that

$$\sum_i P_i[u,v]N_i = v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}.$$

On the ε-sphere centered at ξ one has

$$\frac{\partial}{\partial N} = -\frac{\partial}{\partial r}$$

where $r = |x-\xi|$. Hence for $v = c/r^{n-2}$ where c is a constant it follows

$$\sum_i P_i[u,v]N_i = -\frac{c}{\varepsilon^{n-2}} \cdot \frac{\partial u}{\partial r} - \frac{c(n-2)}{\varepsilon^{n-1}} \cdot u$$

on the sphere $r = \varepsilon$. Moreover, $d\mu = \varepsilon^{n-1} d\mu_1$ where $d\mu_1$ is the measure element of the unit sphere. This shows that the limit of the integral over the ε-sphere equals $u(\xi)$ in case $-c(n-2)\omega_n = 1$.

1.4 Integral representations for smooth functions

In view of the third condition on fundamental solutions (see the preceding Section 1.3), a fundamental solution is integrable in Ω and thus the limiting process $\varepsilon \to 0$ in (5) leads to the integral representation formula

$$u(\xi) = (-1)^{k+1} \int\limits_{\partial\Omega} \sum_{i=1}^{n} P_i[u, E^*(x,\xi)] N_i d\mu + (-1)^k \int\limits_{\Omega} E^*(x,\xi) \mathcal{L} u dx \qquad (6)$$

where u is any k times continuously differentiable function and $E^*(x,\xi)$ is a fundamental solution of the adjoint equation $\mathcal{L}^* v = 0$. Formula (6) is called the *generalized Cauchy-Pompeiu Formula* because in the special case of the Cauchy-Riemann operator in the complex plane it passes into the Cauchy-Pompeiu Formula. Replacing the function u in (6) by a (k times continuously differentiable) test function $u = \varphi$, one gets the important relation

$$\varphi(\xi) = (-1)^k \int\limits_{\Omega} E^*(x,\xi) \mathcal{L} \varphi dx. \qquad (7)$$

showing that a test function φ can be recovered from $\mathcal{L}\varphi$ by an integration provided a fundamental solution of $\mathcal{L}^* u = 0$ is known. Interchanging $\mathcal{L}$ and $\mathcal{L}^*$, formula (7) leads to

$$\varphi(\xi) = (-1)^k \int\limits_{\Omega} E(x,\xi) \mathcal{L}^* \varphi dx$$

Taking into account this relation, and using Fubini's Theorem for weakly singular integrals, the following theorem can be proved easily:

Theorem 1 *Suppose $E(x,\xi)$ is a fundamental solution of $\mathcal{L}u = 0$ with singularity at ξ. Then the function u defined by*

$$u(x) = \int\limits_{\Omega} E(x,\xi) h(\xi) d\xi \qquad (8)$$

turns out to be a distributional solution of the inhomogeneous equation $\mathcal{L}u = h$.

Proof Denoting Ω as domain of the x- and the ξ-space by Ω_x and Ω_ξ resp., one has

$$\int\limits_{\Omega_x} u \mathcal{L}^* \varphi dx \quad = \quad \int\limits_{\Omega_x} \Big(\int\limits_{\Omega_\xi} E(x,\xi) h(\xi) d\xi \Big) \mathcal{L}^* \varphi dx$$

$$= \int_{\Omega_\xi} h(\xi)\Big(\int_{\Omega_x} E(x,\xi)\mathcal{L}^*\varphi(x)dx\Big)d\xi$$
$$= (-1)^k \int_{\Omega_\xi} h(\xi)\varphi(\xi)d\xi.$$

1.5 Integral representations for solutions

Another important special case of a generalized Cauchy-Pompeiu Formula can be obtained for solutions of (homogeneous) differential equations. Suppose u is a solution of the differential equation $\mathcal{L}u = 0$, then formula (6) passes into the boundary integral representation

$$u(\xi) = (-1)^{k+1} \int_{\partial\Omega} \sum_{i=1}^{n} P_i[u, E^*(x,\xi)]N_i d\mu. \tag{9}$$

This formula (9) shows that each solution u can be expressed in (the interior of) Ω by its values and its derivatives (up to the order $k-1$) on the boundary $\partial\Omega$ of Ω.

1.6 Reduction of boundary value problems to fixed-point problems

Next consider a non-linear equation of type

$$\mathcal{L}u = \mathcal{F}(\cdot, u) \tag{10}$$

where $\mathcal{L}$ is again a differential operator of divergence type. Suppose u is a given solution of this equation (10). Define u_0 by

$$u_0(x) = u(x) - \int_{\Omega} E(x,\xi)\mathcal{F}(\xi, u(\xi))d\xi.$$

In view of the above Theorem 1 one gets $\mathcal{L}u_0 = 0$, i.e., to a given solution u of equation (10) there exists a solution u_0 of the simplified equation $\mathcal{L}u_0 = 0$ such that u satisfies the integral relation

$$u(x) = u_0(x) + \int_{\Omega} E(x,\xi)\mathcal{F}(\xi, u(\xi))d\xi.$$

This statement leads to the following method for the construction of solutions of (10):

Let u be any function belonging to a suitably chosen function space. Define an operator by

$$U(x) = u_0(x) + \int\limits_{\Omega} E(x,\xi)\mathcal{F}(\xi, u(\xi))d\xi \tag{11}$$

where u_0 is a solution of $\mathcal{L}u_0 = 0$. Then a fixed element of this operator satisfies equation (10).

Now suppose that a certain boundary condition

$$\mathcal{B}u = g$$

has to be satisfied. Choosing u_0 as solution of the boundary value problem

$$\mathcal{B}\left(u_0 + \int\limits_{\Omega} E(x,\xi)\mathcal{F}(\xi, u(\xi))d\xi\right) = g$$

for $\mathcal{L}u_0 = 0$, one sees that all of the images U satisfy the given boundary condition. The same is true, consequently, for every possibly existing fixed element. To sum up, the following theorem has been proved:

Theorem 2 *Boundary value problems for the non-linear differential equation* $\mathcal{L}u = \mathcal{F}(\cdot, u)$ *can be constructed as fixed points of the operator (11) provided* u_0 *is a solution of the simplified equation* $\mathcal{L}u = 0$ *having suitably chosen boundary values.*

Examples for the solution of boundary value problems by fixed-point methods can be found, for instance, in Section 2.5 below where boundary value problems for non-linear elliptic first order systems in the plane are reduced to fixed-point problems using a complex normal form for the systems under consideration. In F. Rihawi's papers [17, 18] the Dirichlet boundary value problem for

$$\Delta^2 u = F(z, u)$$

is solved where Δ is the Laplace operator in the z-plane. A fixed-point argument is also applied in C. J. Vanegas paper [28] where mainly non-linearly perturbed systems of form

$$D_0 w = f\left(x, w, \frac{\partial w}{\partial x_1}, ..., \frac{\partial w}{\partial x_n}\right)$$

for a desired vector $w = (w_1, ..., w_m)$ in a domain in $\mathbf{R}^\mathbf{n}$ are considered, $m \geq n$. Here D_0 is a matrix differential operator of first order with constant coefficients. Using the adjoint operator to D_0 and the determinant of D_0, the Dirichlet boundary value problem can be reduced to a fixed-point problem.

Remark Note that to each differential operator $\mathcal{L}$ belongs his own fundamental solution, in general. We shall see, however, that the Cauchy kernel

$$\frac{1}{z-\zeta}$$

of Complex Analysis (and its square) are sufficient in order to construct the necessary integral operators provided one uses a complex rewriting of the equations under consideration. In other words, general systems in the plane can be solved using the fundamental solution of the Cauchy-Riemann system (see the next Section 2)

2 Complex versions of the method of fundamental solutions

2.1 The Cauchy kernel as fundamental solution of the Cauchy-Riemann system

In the complex plane the Gauss Integral Formula for a complex-valued f reads

$$\iint\limits_{\Omega} \frac{\partial f}{\partial x} dxdy = \int\limits_{\partial\Omega} f dy \tag{12}$$

$$\text{and} \quad \iint\limits_{\Omega} \frac{\partial f}{\partial y} dxdy = -\int\limits_{\partial\Omega} f dx. \tag{13}$$

Define the partial complex differentiations $\partial/\partial z$ and $\partial/\partial \overline{z}$ by

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \end{aligned}$$

Multiplying (13) by i and adding the multiplied equation to (12), one gets the following complex version of Gauss' Integral Formula

$$\iint\limits_{\Omega} \frac{\partial f}{\partial \overline{z}} dxdy = \frac{1}{2i}\int\limits_{\partial\Omega} f dz, \tag{14}$$

whereas subtraction gives

$$\iint\limits_{\Omega} \frac{\partial f}{\partial z} dxdy = -\frac{1}{2i}\int\limits_{\partial\Omega} f d\overline{z}.$$

Substituting $f = w_1 w_2$ into the complex version (14) of Gauss' Integral Formula, one obtains the complex Green Formula

$$\iint_{\Omega} \left(w_1 \frac{\partial w_2}{\partial \overline{z}} + w_2 \frac{\partial w_1}{\partial \overline{z}} \right) dxdy = \frac{1}{2i} \int_{\partial\Omega} w_1 w_2 dz. \tag{15}$$

This formula is the special case of (1) for the Cauchy-Riemann operator

$$\mathcal{L} = \frac{\partial}{\partial \overline{z}}.$$

It shows that the Cauchy-Riemann operator $\partial/\partial\overline{z}$ is self-adjoint.

Applying this complex Green Integral Formula with

$$w_1 = w \quad \text{and} \quad w_2 = \frac{c}{z-\zeta}$$

in $\Omega_\varepsilon = \Omega \setminus \overline{U_\varepsilon}$ (where c is a complex constant), one gets

$$\iint_{\Omega_\varepsilon} \frac{\partial w}{\partial \overline{z}} \frac{c}{z-\zeta} dxdy \tag{16}$$

$$= \frac{1}{2i} \int_{\partial\Omega} w(z) \frac{c}{z-\zeta} dz - \frac{1}{2i} \int_{|z-\zeta|=\varepsilon} w(z) \frac{c}{z-\zeta} dz$$

which is the special case of (5) for the Cauchy-Riemann operator. The second term on the right-hand side tends to

$$-\frac{1}{2i} w(\zeta) c \cdot 2\pi i$$

as ε tends to zero. Consequently,

$$E(z,\zeta) = \frac{1}{\pi} \frac{1}{z-\zeta} \tag{17}$$

turns out to be a fundamental solution of the Cauchy-Riemann system. Moreover, formula (16) leads to the Cauchy-Pompeiu Formula

$$w(\zeta) = \frac{1}{2i\pi} \int_{\partial\Omega} \frac{w(z)}{z-\zeta} dz - \frac{1}{\pi} \iint_{\Omega} \frac{\partial w}{\partial \overline{z}} \frac{1}{z-\zeta} dxdy. \tag{18}$$

Note that (18) is the special case of formula (6) in Section 1.4 for the Cauchy-Riemann operator $\mathcal{L} = \partial/\partial\overline{z}$.

2.2 Complex normal forms for linear and non-linear first order systems in the plane

Let Ω be a bounded domain in the x, y-plane with sufficiently smooth boundary. We are looking for two real-valued functions $u = u(x,y)$ and $v = v(x,y)$ satisfying a system of form

$$H_j\left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = 0, \quad j = 1, 2, \tag{19}$$

in Ω. One of the simplest special cases of this system is the Cauchy-Riemann system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

which can be written in the complex form

$$\frac{\partial w}{\partial \overline{z}} = 0$$

where $z = x+iy$ and $w = u+iv$. In order to get an analogous complex rewriting of the system (19), we use the formulae

$$\begin{aligned}
\frac{\partial w}{\partial z} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
\frac{\partial w}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).
\end{aligned}$$

Now introduce the following abbreviations:

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= p_1 \\
\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= p_2 \\
\frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) &= q_1 \\
\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) &= q_2.
\end{aligned}$$

Then one has

$$\begin{aligned}
\frac{\partial u}{\partial x} &= p_1 + q_1 \\
\frac{\partial u}{\partial y} &= -p_2 + q_2 \\
\frac{\partial v}{\partial x} &= p_2 + q_2 \\
\frac{\partial v}{\partial y} &= p_1 - q_1.
\end{aligned}$$

Substituting these expressions into the system (19), this system passes into

$$H_j\left(x, y, u, v, p_1 + q_1, -p_2 + q_2, p_2 + q_2, p_1 - q_1\right) = 0, \quad j = 1, 2.$$

Now suppose that this system can be solved for q_1 and q_2. Then one gets real-valued representations

$$q_j = F_j(x, y, u, v, p_1, p_2), \quad j = 1, 2. \tag{20}$$

Since $x + iy = z$, $u + iv = w$ and $p_1 + ip_2 = \partial w/\partial z$, the variables on the right-hand sides of these equations can be expressed by z, w and $\partial w/\partial z$ (and their conjugate complex values). Denoting $F_1 + iF_2$ by F, and taking into consideration that $q_1 + iq_2 = \partial w/\partial \overline{z}$, the two equations (20) can be combined to the one complex equation

$$\frac{\partial w}{\partial \overline{z}} = F\left(z, w, \frac{\partial w}{\partial z}\right). \tag{21}$$

This equation (21) is the desired complex rewriting of the real first order system (19).

Remark Consider instead of (19) a system of $2m$ first order equations for $2m$ desired real-valued functions $u_1, v_1, ..., u_m, v_m$. Introducing the vector $w = (w_1, ..., w_m)$ where $w_\mu = u_\mu + iv_\mu$, $\mu = 1, ..., m$, such systems can also be written in the form (21), where both the desired w and the right-hand side F are vectors having m complex-valued components.

2.3 Distributional solutions of partial complex differential equations. The $\mathbf{T}_\Omega$- and the $\mathbf{\Pi}_\Omega$-operators

The inhomogeneous Cauchy-Riemann equation is the equation

$$\frac{\partial w}{\partial \overline{z}} = h \tag{22}$$

where h is a given function in a bounded domain Ω. In accordance with Section 1.2 a distributional solution of this equation is an integrable function $w = w(z)$ such that

$$\iint\limits_{\Omega} \left(\varphi h + w\frac{\partial \varphi}{\partial \overline{z}}\right) dxdy = 0$$

for each (continuously differentiable and complex-valued) test function φ. Since $\frac{1}{\pi}\frac{1}{z - \zeta}$ is a fundamental solution of the Cauchy-Riemann system, Theorem 1 of Section 1.4 shows that the so-called T_Ω-operator

$$(T_\Omega h)[z] = \frac{1}{\pi}\iint\limits_{\Omega} \frac{h(\zeta)}{z - \zeta} d\xi d\eta = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{h(\zeta)}{\zeta - z} d\xi d\eta,$$

(where $\zeta = \xi + i\eta$) defines a (special) distributional solution of the inhomogeneous Cauchy-Riemann equation (22). This statement can be formulated as follows:

Theorem 3

$$\frac{\partial}{\partial \bar{z}} T_\Omega h = h.$$

Denote by Π_Ω the strongly singular operator

$$(\Pi_\Omega h)[z] = -\frac{1}{\pi} \iint\limits_\Omega \frac{h(\zeta)}{(\zeta - z)^2} d\xi d\eta.$$

Then similar considerations lead to the following theorem

Theorem 4

$$\frac{\partial}{\partial z} T_\Omega h = \Pi_\Omega h.$$

Remark

The strongly singular integral $\Pi_\Omega h$ is defined as Cauchy's Principal Value provided it exists. Notice that Cauchy's Principal Value of an integral

$$\iint\limits_\Omega g d\xi d\eta$$

of a function g having a strong singularity at ζ is defined as limit

$$\lim_{\varepsilon \to 0} \iint\limits_{\Omega \setminus U_\varepsilon(\zeta)} g d\xi d\eta,$$

i.e., one has to omit an ε-neighbourhood, not an arbitrary neighbourhood of the singularity. For

$$g(\zeta) = \frac{h(\zeta)}{(\zeta - z)^2}$$

one has

$$g(\zeta) = \frac{h(\zeta) - h(z)}{(\zeta - z)^2} + h(z) \cdot \frac{1}{(\zeta - z)^2}. \tag{23}$$

If h is Hölder continuous with exponent λ, $0 < \lambda \leq 1$, then one has

$$|h(\zeta) - h(z)| \leq H \cdot |\zeta - z|^\lambda.$$

Consequently, the absolute value of the first term in (23) can be estimated by

$$\frac{H}{|\zeta - z|^{2-\lambda}}$$

and is thus weakly singular at ζ. This implies that the Π_Ω-operator exists for Hölder continuous integrands. — In order to prove Theorem 4 one has to use the Fubini Theorem for Principal Values of strongly singular integrals.

In order to determine the general solution of the inhomogeneous Cauchy-Riemann equation (22), consider an arbitrary solution $w = w(z)$ of that equation and define

$$\Phi = w - T_\Omega h.$$

Obviously,

$$\frac{\partial \Phi}{\partial \overline{z}} = 0$$

in the distributional sense, i.e.,

$$\iint_\Omega \Phi \frac{\partial \varphi}{\partial \overline{z}} dxdy = 0 \tag{24}$$

for each test function. Of course, every holomorphic function in the classical sense is a solution of the latter equation. The question is whether this equation (24) can have distributional solutions which are not holomorphic functions in the classical sense. The answer to this question is no in view of the famous Weyl Lemma which will be proved in the next section.

2.4 The Weyl Lemma and its applications to elliptic first order systems in the plane

Theorem 5 *A distributional solution of the homogeneous Cauchy-Riemann equation is necessarily a holomorphic function in the classical sense, i.e., it is everywhere complex differentiable.*

This statement will be proved by approximating a given distributional solution by classical solutions. For this purpose we need the concept of a mollifier.

Take any real-valued (continuously differentiable) function $\omega = \omega(\zeta)$ defined in the whole complex plane and satisfying the following conditions:

- $\omega(\zeta) > 0$ if $|\zeta| < 1$
- $\omega(\zeta) \equiv 0$ if $|\zeta| \geq 1$
- $\iint \omega(\zeta) d\xi d\eta = 1$

where the integration is to be carried out over the whole complex plane. A special function having these properties is defined by

$$\omega(\zeta) = \begin{cases} c(1-r^2)^2, & \text{if} \quad r < 1, \\ 0, & \text{if} \quad r \geq 1. \end{cases}$$

where $r = |\zeta|$ and c is suitably chosen. For fixedly chosen z define a further function ω_δ by

$$\omega_\delta(\zeta, z) = \frac{1}{\delta^2}\omega\left(\frac{\zeta - z}{\delta}\right).$$

Then ω_δ is positive in the δ-neighbourhood of x, whereas ω_δ vanishes identically outside this δ-neighbourhood. Moreover, one has

$$\iint\limits_{\mathbb{C}} \omega_\delta(\zeta, z) d\xi d\eta = \iint\limits_{|\zeta - z| \leq \delta} \omega_\delta(\zeta, z) d\xi d\eta = 1. \tag{25}$$

The function ω_δ is called a *mollifier*.

Using the mollifier ω_δ, one defines the regularization $f_\delta = f_\delta(z)$ of an integrable function $f = f(z)$ by

$$f_\delta(z) = \iint\limits_{|\zeta - z| \leq \delta} f(\zeta)\omega_\delta(\zeta, z) d\xi d\eta,$$

i.e., the values $f_\delta(z)$ are the mean values of $f = f(z)$ with the weight ω_δ in the δ-neighbourhood of z.

In view of (25) the value $f(z)$ can be rewritten in the form

$$f(z) = \iint\limits_{|\zeta - z| \leq \delta} f(z)\omega_\delta(\zeta, z) d\xi d\eta.$$

Thus one gets

$$f_\delta(z) - f(z) = \iint\limits_{|\zeta - z| \leq \delta} (f(\zeta) - f(z))\omega_\delta(\zeta, z) d\xi d\eta. \tag{26}$$

Now suppose that $f = f(z)$ is continuous. Then the supremum

$$\sup_{|\zeta - z| \leq \delta} |f(\zeta) - f(z)|$$

is arbitrarily small in case δ is sufficiently small. Moreover, in view of (26) one has

$$\begin{aligned} |f_\delta(z) - f(z)| &\leq \sup_{|\zeta - z| \leq \delta} |f(\zeta) - f(z)| \cdot \iint\limits_{|\zeta - z| \leq \delta} \omega_\delta(\zeta, z) d\xi d\eta \\ &\leq \sup_{|\zeta - z| \leq \delta} |f(\zeta) - f(z)| \end{aligned}$$

where (25) has been applied once more. Thus the $f_\delta = f_\delta(z)$ tend uniformly to $f = f(z)$ as $\delta \to 0$ provided z runs in a compact subset of the domain of definition.

Proof of Weyl's Lemma

Using chain rule, one has

$$\frac{\partial \omega_\delta}{\partial \overline{z}} = -\frac{\partial \omega_\delta}{\partial \overline{\zeta}}$$

and, consequently,

$$\begin{aligned} \frac{\partial f_\delta}{\partial \overline{z}}(z) &= \iint\limits_{|\zeta - z| \le \delta} f(\zeta) \frac{\partial \omega_\delta}{\partial \overline{z}}(\zeta, z) d\xi d\eta \\ &= -\iint\limits_{|\zeta - z| \le \delta} f(\zeta) \frac{\partial \omega_\delta}{\partial \overline{\zeta}}(\zeta, z) d\xi d\eta = 0 \end{aligned} \tag{27}$$

because $f = f(z)$ is a distributional solution of the (homogeneous) Cauchy-Riemann system by hypothesis and $\omega_\delta(\zeta, z)$ is (for each z) a special test function.

Formula (27) shows that all of the $f_\delta = f_\delta(z)$ are solutions of the (homogeneous) Cauchy-Riemann system. On the other hand, the $f_\delta = f_\delta(z)$ are continuously differentiable because the mollifiers have this property. Thus the $f_\delta = f_\delta(z)$ are holomorphic functions in the classical sense.

Now consider any compact subset of the domain under consideration. Applying Weierstrass' Convergence Theorem, the function $f = f(z)$ turns out to be holomorphic, too, as limit of uniformly convergent holomorphic functions. Since the compact subset can be chosen arbitrarily, the function $f = f(z)$ turns out to be holomorphic everywhere in the domain under consideration. This completes the proof of Weyl's Lemma.

Consider again the non-linear first order system (19) in its complex form (21). Let $w = w(z)$ be an arbitrary solution in the (bounded) domain Ω. Define

$$\Phi = w - T_\Omega F\left(z, w, \frac{\partial w}{\partial z}\right).$$

By virtue of Weyl's Lemma, Φ turns out to be a classical holomorphic function. Consequently, each solution $w = w(z)$ of equation (21) is a fixed point of the operator

$$W = \Phi + T_\Omega F\left(z, w, \frac{\partial w}{\partial z}\right) \tag{28}$$

where Φ is a suitable chosen holomorphic function. Therefore, boundary value problems for (21) can be reduced to boundary value problems for holomorphic functions. This will be sketched in the next section.

2.5 Fixed-point methods for linear and non-linear systems in the plane

In order to construct fixed points of the operator (28), one has to choose a suitable function space in which the T_Ω- and the Π_Ω-operators are bounded. Such spaces are the Hölder spaces or the Lebesgue spaces with $p > 2$. While the T_Ω-operator is also bounded in the space of continuous functions (the T_Ω-operator is even a bounded operator mapping $L_p(\Omega)$ into $\mathcal{C}^\beta(\Omega)$ with $\beta = 1 - \frac{2}{p}$), the Π_Ω-operator is not a bounded operator in the space of continuous functions.

In the paper [13], for instance, some boundary value problems for the non-linear system (21) are solved in the following space:

> w has to belong to $\mathcal{C}^\beta(\Omega)$, while $\partial w/\partial z$ has to be an element of $L_p(\Omega)$ where p has to satisfy the inequality
>
> $$2 < p < \frac{1}{1-\alpha}. \tag{29}$$

The left-hand side of this inequality (29) implies that the T_Ω-operator maps $L_p(\Omega)$ into the Hölder space $\mathcal{C}^\beta(\Omega)$ with

$$\beta = 1 - \frac{2}{p}.$$

Indeed,

$$(T_\Omega)[\zeta_1] - (T_\Omega)[\zeta_1] = -\frac{1}{\pi}(z_1 - z_2) \iint\limits_\Omega h(\zeta) \cdot \frac{1}{(\zeta - z_1)(\zeta - z_2)} d\xi d\eta$$

and thus by virtue of Hölder's inequality

$$|(T_\Omega h)[\zeta_1] - (T_\Omega h)[\zeta_1]| \leq \frac{1}{\pi} \cdot |z_1 - z_2| \cdot \|h\|_{L_p(\Omega)} \cdot \left\| \frac{1}{|\zeta - z_1| \cdot |\zeta - z_2|} \right\|_{L_q(\Omega)} \tag{30}$$

where p and q are conjugate exponents,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Since

$$\iint\limits_\Omega \frac{1}{|\zeta - z_1|^q \cdot |\zeta - z_2|^q} \leq C_1 |z_1 - z_2|^{2-2q} + C_2 \leq C_3 |z_1 - z_2|^{2-2q}$$

provided $q > 1$, the exponent of $|z_1 - z_2|$ on the right-hand side of (30) is equal to

$$1 + \frac{2-2q}{q} = \frac{2}{q} - 1 = 1 - \frac{2}{p}.$$

Consequently, $T_\Omega h$ turns out to be Hölder continuous with exponent β if $p > 2$.

The right-hand side of inequality (29) ensures that the derivative of a holomorphic function belongs to $L_p(\Omega)$ if the boundary values of the holomorphic function are Hölder-continuous with exponent α. Further, the right-hand side of (29) is equivalent to

$$\alpha > 1 - \frac{1}{p}$$

and thus we see that $\beta < \alpha$.

Since the real part of a holomorphic function is a solution of the Laplace equation, a suitable boundary value problem for holomorphic functions and, therefore, for solutions of (21), too, is the following so-called Dirichlet boundary value problem:

> One prescribes the real part of the desired solution on the whole boundary, whereas the imaginary part can be prescribed at one point z_0 only.

In order to solve the boundary value problem for the equation (21), let Ψ be the holomorphic solution of the boundary value problem under consideration. Further, let $\Phi_{(w)}$ be a holomorphic function such that

$$\Phi_{(w)} + T_\Omega F\left(z, w, \frac{\partial w}{\partial z}\right)$$

satisfy the homogeneous boundary condition of the given (linear) boundary value problem. While Ψ depends on the prescribed data only, the holomorphic function $\Phi_{(w)}$ depends on the choice of w. Choosing

$$\Phi = \Psi + \Phi_{(w)} \tag{31}$$

in the definition (28) of the corresponding operator, we see that all images W satisfy the prescribed boundary condition. The same is true for a possibly existing fixed point. Consequently, in order to solve a boundary value problem for the partial complex differential equation (21), one has to find fixed points of the operator (28) where the holomorphic function Φ is to be chosen by (31).

The Dirichlet boundary value problem for a desired holomorphic function can always be reduced to the Dirichlet boundary value problem for the Laplace equation. However, there are also other ways for solving this auxiliary problem.

Let Ω be the unit disk $\left\{z : |z| < 1\right\}$, and let g be a real-valued continuous function defined on the boundary $|z| = 1$. Then

$$\frac{1}{2\pi} \int\limits_{|z|=1} g(z) \frac{z+\zeta}{z-\zeta} ds + i \cdot C$$

is the most general holomorphic function in Ω where C is an arbitrary real constant and ds means the arc length element of the boundary $\partial\Omega$.

Another useful method for the unit disk is connected with a modified T_Ω-operator (see B. Bojarski [6]):

Let h be defined in Ω, and suppose that h belongs to the underlying function space. Then

$$H = T_\Omega h$$

is continuous in the whole complex plane (and holomorphic outside $\overline{\Omega}$). For points z on the boundary of Ω we have $\overline{z} = 1/z$ and, therefore,

$$\overline{H(z)} = -\frac{1}{\pi} \iint\limits_{\Omega} \frac{\overline{h(\zeta)}}{\overline{\zeta} - \overline{z}} d\xi d\eta = \frac{z}{\pi} \iint\limits_{\Omega} \frac{\overline{h(\zeta)}}{1 - z\overline{\zeta}} d\xi d\eta. \tag{32}$$

On the other hand, the right-hand side of (32) is holomorphic in the unit disk Ω. To sum up, the following statement has been proved:

$\overline{H} = \overline{T_\Omega h}$ *is a holomorphic function in* Ω *having the same real part as* $T_\Omega h$ *on* $\partial\Omega$.

This statement can be used in order to estimate the auxiliary function $\Phi_{(w)}$ and its derivative $\Phi'_{(w)}$ provided Ω is the unit disk. Details and also the solution of other boundary value problems (such as Riemann-Hilbert's one) for (21) can be found, for instance, in [13].

3 Reduction of initial value problems to fixed-point problems

3.1 Related integro-differential operators

Let $u = u(t,x)$ be the desired function where t means the time and $x = (x_1, ..., x_n)$ is a spacelike variable. Consider an initial value problem of type

$$\frac{\partial u}{\partial t} = \mathcal{F}\left(t, x, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) \tag{33}$$

$$u(0,x) = \varphi(x). \tag{34}$$

Then the initial value problem (33), (34) can be rewritten in the integral form[1]

$$u(t,x) = \varphi(x) + \int_0^t \mathcal{F}\left(\tau, x, u(\tau,x), \frac{\partial u}{\partial x_1}(\tau,x), ..., \frac{\partial u}{\partial x_n}(\tau,x)\right) d\tau. \tag{35}$$

[1] M. Nagumo [14] was the first who used such an equivalent integro-differential equation for a functional-analytic proof of the classical Cauchy-Kovalewskaya Theorem.

Since the integrand in (35) contains derivatives of the desired function with respect to spacelike variables, the equation (35) is an integro-differential equations.

In order to construct the solution of the integro-differential equation (35), define the integro-differential operator

$$U(t,x) = \varphi(x) + \int_0^t \mathcal{F}\left(\tau, x, u(\tau,x), \frac{\partial u}{\partial x_1}(\tau,x), ..., \frac{\partial u}{\partial x_n}(\tau,x)\right) d\tau. \qquad (36)$$

Then a fixed-point of this operator is a solution of the integro-differential equation (35) and thus a solution of the initial value problem (33), (34).

3.2 Behaviour of derivatives at the boundary. Weighted norms

Suppose the right-hand side of the differential equation (33) does not depend on the derivatives $\partial u/\partial x_j$. Suppose, further, that the right-hand side satisfies a Lipschitz condition with respect to u. Then the operator (36) is contractive provided the time interval is short enough. Since the differentiation is not a bounded operator, this argument is not applicable if the right-hand side $\mathcal{F}$ depends also of the derivatives (even if a Lipschitz condition is satisfied with respect to the derivatives, too). However, an analogous estimate of the operator (36) will be possible if $u(t,x)$ belongs to a class of functions for which the unboundedness of the differentiation is moderate in a certain sense. The following easy example will show how such unboundedness can be overcome.

Let Ω be the unit disk $|z| < 1$. Denote by $\mathcal{H}(\Omega)$ the set of all holomorphic functions in Ω. Choosing $\frac{\pi}{2} < \arg(z-1) < \frac{3\pi}{2}$, the function

$$\Phi(z) = (z-1)\log(z-1) = (z-1)\Big(\ln|z-1| + i\cdot\arg(z-1)\Big)$$

is uniquely defined and belongs to $\mathcal{H}(\Omega)$. Defining $\Phi(1) = 0$, the function is continuous and thus bounded in the closed unit disk $|z| \leq 1$, i.e., $\Phi \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Moreover,

$$\Phi'(z) = 1 + \log(z-1) \to \infty \quad \text{as} \quad z \to 1.$$

Consequently, the complex differentiation d/dz does not map $\Phi \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ into itself and thus the latter space is not suitable for solving the integro-differential equation (35), at least not when using the ordinary supremum norm. On the other hand,

$$(1-z)\cdot\Phi'(z) = (1-z) + (1-z)\log(z-1)$$

is bounded and belongs, therefore, to $\mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Since the distance $d(z)$ of a point $z \in \Omega$ from the boundary $\partial\Omega$ satisfies the estimate

$$d(z) = \inf_{|\zeta|=1} |\zeta - z| \leq |1 - \zeta|$$

it follows that

$$\sup_{\Omega} d(z) |\Phi'(z)|$$

is finite. The last expression, however, is nothing but a weighted supremum norm with the weight $d(z)$. Of course, the weighted supremum norm of the function Φ itself is also finite. Hence the complex differentiation d/dz transforms the function Φ whose weighted supremum norm is finite in the function Φ' having also a finite weighted supremum norm.

Later on we shall see that the integral operator (36) is bounded in a suitably chosen space equipped with a weighted norm. The space consists of functions depending on the time t and a spacelike variable x or z. For fixed t the elements of the space under consideration have to satisfy a partial differential equation of elliptic type (in particular, they have to be holomorphic or generalized analytic functions).

3.3 Weighted norms for time-dependent functions

The following easy example shows that singularities of the initial functions at the boundary can come into the domain in the course of time. This may lead to a reduction of the length of the time interval in which the solution exists.

Let Ω be the positive x-axis. The initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial u}{\partial x} \\ u(0,x) &= \frac{1}{x} \end{aligned}$$

has the solution

$$u(x,t) = \frac{1}{x-t}.$$

The initial function has a singularity at the boundary point $x = 0$ of Ω. At the point $x \in \Omega$ the solution $u(x,t)$ tends to ∞ as t tends to x, i.e., at the point x the solution exists only in a time interval of length x. In other words, the shorter the distance of x from the boundary of Ω, the shorter the time interval in which the solution exists.

Now let Ω be again an arbitrary bounded domain in $\mathbf{R^n}$. In order to measure the distance of a point $x \in \Omega$ from the boundary $\partial\Omega$ of Ω, introduce an exhaustion of Ω by a family of subdomains Ω_s, $0 < s < s_0$, satisfying the following conditions:

- If $0 < s' < s'' < s_0$, then $\Omega_{s'}$ is a subdomain of $\Omega_{s''}$, and the distance of $\Omega_{s'}$ from the boundary $\partial\Omega_{s''}$ of $\Omega_{s''}$ can be estimated by

$$\text{dist}\,(\Omega_{s'}, \partial\Omega_{s''}) \geq c_1(s'' - s').$$

  where c_1 does not depend on the choice of s' and s''.

- To each point $x \neq x_0$ of Ω where $x_0 \in \Omega$ is fixedly chosen there exists a uniquely determined $s(x)$ with $0 < s(x) < s_0$ such that $x \in \partial\Omega_{s(x)}$.

Define, finally $s(x_0) = 0$. Then $s_0 - s(x)$ is a measure of the distance of a point x of Ω from the boundary $\partial\Omega_{s(x)}$.

Now consider the conical set

$$M = \Big\{(t, x) : x \in \Omega,\ 0 \leq t < \eta\big(s_0 - s(x)\big)\Big\}$$

in the t, x-space. Its height is equal to ηs_0 where η will be fixed later. The base of M is the given domain Ω, whereas its lateral surface is defined by

$$t = \eta(s_0 - s(x)). \tag{37}$$

The nearer a point x to the boundary $\partial\Omega$, the shorter the correponding time interval (37). The expression

$$d(t, x) = s_0 - s(x) - \frac{t}{\eta} \tag{38}$$

is positive in M, while it vanishes identically on the lateral surface of M. Thus (38) can be interpreted as some pseudo-distance of a point (x, t) of M from the lateral surface of M. Later on this expression will be used as a certain weight for functions $u = u(x, t)$ defined in M.

In order to construct a suitable Banach space of functions defined in the conical domain M in the t, x-space, let $\mathcal{B}_s$ be the space of all Hölder continuous functions in Ω_s equipped with the Hölder norm

$$\|u\|_s = \max\left(\sup_{\Omega_s} |u|,\ \sup_{x' \neq x''} \frac{|u(x') - u(x'')|}{|x' - x''|^\lambda}\right), \quad 0 < \lambda \leq 1.$$

For a fixed $\tilde{t} < \eta s_0$ the intersection of M with the plane $t = \tilde{t}$ in the t, x-space is given by

$$\Big\{(t, x) :\ t = \tilde{t},\ s(x) < \tilde{s}\Big\}$$

where

$$\tilde{s} = s_0 - \frac{\tilde{t}}{\eta}. \tag{39}$$

Let $\mathcal{B}_*(M)$ be the set of all (real-valued) functions $u = u(t, x)$ satisfying the following conditions:

1. $u(t,x)$ is continuous in M.

2. $u(\tilde{t},x)$ belongs to $\mathcal{B}_{s(x)}$ for fixed $\tilde{t}$ if only $s(x) < \tilde{s}$ where $\tilde{s}$ is given by (39).

3. The norm
$$\|u\|_* = \sup_{(t,x)\in M} \|u(t,\cdot)\|_{s(x)}\, d(t,x) \tag{40}$$
is finite.

The definition (40) of the norm $\|\cdot\|_*$ implies the estimate

$$\|u(t,\cdot)\|_{s(x)} \le \frac{\|u\|_*}{d(t,x)} \tag{41}$$

for any point (t,x) in M.

Proposition 1 *$\mathcal{B}_*(M)$ is a Banach space.*

Proof Note that the inequality $d(t,x) \ge \delta > 0$ defines a closed subset M_δ of the conical domain M. Each point of M is contained in such a subset M_δ provided δ is suitably chosen. For points (t,x) in M_δ, the definition (40) implies the estimate
$$\|u(t,\cdot)\|_{s(x)} \le \frac{1}{\delta}\, \|u\|_*$$
Now consider a fundamental sequence u_1, u_2, ... with respect to the norm $\|\cdot\|_*$. Then one has
$$\|u_n(t,\cdot) - u_m(t,\cdot)\|_{s(x)} \le \frac{1}{\delta} \cdot \varepsilon \tag{42}$$
for points in M_δ provided n and m are sufficiently large. This implies also

$$|u_n - u_m| \le \frac{1}{\delta} \cdot \varepsilon$$

for points in M_δ. Consequently, a fundamental sequence converges uniformly in each M_δ, i.e., the u_n have a continuous limit function $u_*(t,x)$ in M. Similarly, estimate (42) shows that for $t = \tilde{t}$ and $s(x) < \tilde{s}$ the limit function belongs to $\mathcal{B}_{s(x)}$ because of the completeness of this space. Carrying out the limiting process $m \to \infty$ in the inequality $\|u_n - u_m\|_* < \varepsilon$, it follows, finally, $\|u_n - u_*\|_* \le \varepsilon$ and, therefore, $\|u_*\|_*$ is finite.

3.4 Associated differential operators and consequences of interior estimates

Of course, the operator (36) is defined only for functions $u = u(t,x)$ for which the first order derivatives $\partial u/\partial x_j$ exist. Suppose such a function $u = u(t,x)$ belongs to $\mathcal{B}_*(M)$ (while the first order derivatives have to belong to the $\mathcal{B}_{s(x)}$). We are going to answer the question under which conditions the image $U = U(t,x)$ belongs also to $\mathcal{B}_*(M)$. Consider again the Banach spaces $\mathcal{B}(\Omega)$ introduced above.

Definition Suppose Ω' is any subdomain of Ω'' having a positive distance dist $(\Omega', \partial\Omega'')$ from the boundary of Ω''. Then a function $u \in \mathcal{B}(\Omega'')$ is called a function with a *first order interior estimate* if $\partial u/\partial x_j$ belongs to $\mathcal{B}(\Omega')$ and

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{\mathcal{B}(\Omega')} \leq \frac{c_2}{\text{dist }(\Omega', \partial\Omega'')} \, \|u\|_{\mathcal{B}(\Omega'')} \tag{43}$$

where the constant c_2 depends neither on the special choice of u nor on the choice of the pair Ω', Ω''.

Applying this estimate to the exhaustion Ω_s of Ω, $0 < s < s_0$, one gets

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{s'} \leq \frac{c_2}{c_1} \cdot \frac{1}{s'' - s'} \cdot \|u\|_{s''} \tag{44}$$

provided $s' < s''$.

Now let (t,x) be an arbitrary point of M. Then

$$d(t,x) = s_0 - s(x) - \frac{t}{\eta} > 0.$$

Define

$$\tilde{s} = s(x) + \frac{1}{2} d(t,x)$$

implying

$$\tilde{s} \leq s(x) + \frac{1}{2}\Big(s_0 - s(x)\Big) = \frac{1}{2} s(x) + \frac{1}{2} s_0 < s_0$$

and thus there exists a point $\tilde{x}$ with $s(\tilde{x}) = \tilde{s}$, i.e., $\tilde{x} \in \partial\Omega_{\tilde{s}}$. One has

$$d(t,\tilde{x}) = s_0 - s(\tilde{x}) - \frac{t}{\eta} = \frac{1}{2} d(t,x).$$

Taking into account the estimate (41), the last relation gives

$$\|u(t,\cdot)\|_{\tilde{s}} \leq \frac{\|u\|_*}{d(t,\tilde{x})} = \frac{2\,\|u\|_*}{d(t,x)}.$$

In view of (44) one gets, therefore,

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{s(x)} \leq \frac{c_2}{c_1} \cdot \frac{1}{\tilde{s} - s(x)} \cdot \|u\|_{\tilde{s}} \leq \frac{c_2}{c_1} \cdot \frac{4}{d^2(t,x)} \cdot \|u\|_* \, . \tag{45}$$

To be short denote $\mathcal{F}\left(t, x, u, \dfrac{\partial u}{\partial x_j}\right)$ by $\mathcal{F}u$, i.e., in particular one has $\mathcal{F}\Theta = \mathcal{F}\Big(t, x, 0, 0\Big)$. Next we have to estimate the norm of $\mathcal{F}u$. For this purpose we assume that the right-hand side $\mathcal{F}$ of (33) satisfies the following conditions:

1. $\mathcal{F}\Theta$ is continuous.
2. The norms $\|\mathcal{F}\Theta\|_s$ are bounded and thus $\|\mathcal{F}\Theta\|_*$ is finite.
3. $\mathcal{F}u$ satisfies the (global) Lipschitz condition

$$\|\mathcal{F}u - \mathcal{F}v\|_s \leq L_0 \|u - v\|_s + \sum_j L_j \left\| \frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j} \right\|_s . \tag{46}$$

Note that $\mathcal{F}u = \mathcal{F}\Theta + (\mathcal{F}u - \mathcal{F}\Theta)$. Using (41) and (45), the Lipschitz condition (46) implies

$$\begin{aligned} \|\mathcal{F}u\|_{s(x)} &\leq \|\mathcal{F}\Theta\|_{s(x)} + L_0 \|u\|_{s(x)} + \sum_j L_j \left\| \frac{\partial u}{\partial x_j} \right\|_{s(x)} \\ &\leq \|\mathcal{F}\Theta\|_* \frac{1}{d(t,x)} + L_0 \|u\|_* \frac{1}{d(t,x)} + \frac{4c_2}{c_1} \sum_j L_j \|u\|_* \frac{1}{d^2(t,x)} \\ &\leq \|\mathcal{F}\Theta\|_* \frac{s_0}{d^2(t,x)} + L_0 \|u\|_* \frac{s_0}{d^2(t,x)} + \frac{4c_2}{c_1} \sum_j L_j \|u\|_* \frac{1}{d^2(t,x)} \end{aligned}$$

since $d(t,x) \leq s_0$. The definition (38) of the weight function $d(t,x)$ implies

$$\int_0^t \frac{1}{d^2(\tau,x)} d\tau < \frac{\eta}{d(t,x)}$$

and thus it follows

$$\left\| \int_0^t \mathcal{F}u \cdot d\tau \right\|_{s(x)} \leq \frac{\eta}{d(t,x)} \Big(\|\mathcal{F}\Theta\|_* s_0 + c_3 \|u\|_* \Big) \tag{47}$$

where

$$c_3 = s_0 L_0 + \frac{4c_2}{c_1} \sum_j L_j .$$

The estimate (47) of the $s(x)$-norm yields

$$\left\| \int_0^t \mathcal{F}u \cdot d\tau \right\|_* \leq \eta \Big(\|\mathcal{F}\Theta\|_* \, s_0 + c_3 \, \|u\|_* \Big).$$

Suppose, finally, that the norms $\|\varphi\|_s$, $0 < s < s_0$, are bounded. Then $\|\varphi\|_*$ is finite, and the following statement for the image $U(t,x)$ defined by (36) has been proved:

Proposition 2

$$\|U\|_* \leq \|\varphi\|_* + \eta \Big(\|\mathcal{F}\Theta\|_* \, s_0 + c_3 \, \|u\|_* \Big).$$

Together with $u(t,x)$ consider a second element $v(t,x)$ of $\mathcal{B}_*(M)$ with the same properties listed above. Let $V(t,x)$ be the corresponding image defined by an equation analogous to (36). Then

$$U(t,x) - V(t,x) = \int_0^t (\mathcal{F}u - \mathcal{F}v) d\tau.$$

Again in view of (46), (41) and (45), one gets

$$\begin{aligned} \|\mathcal{F}u - \mathcal{F}v\|_{s(x)} &\leq L_0 \, \|u - v\|_{s(x)} + \sum_j L_j \left\| \frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j} \right\|_{s(x)} \\ &\leq L_0 \, \|u - v\|_* \, \frac{1}{d(t,x)} + \frac{4c_2}{c_1} \sum_j L_j \, \|u - v\|_* \, \frac{1}{d^2(t,x)} \\ &\leq L_0 \, \|u - v\|_* \, \frac{s_0}{d^2(t,x)} + \frac{4c_2}{c_1} \sum_j L_j \, \|u - v\|_* \, \frac{1}{d^2(t,x)} \end{aligned}$$

and, consequently,

$$\left\| \int_0^t (\mathcal{F}u - \mathcal{F}v) d\tau \right\|_{s(x)} \leq \frac{\eta}{d(t,x)} c_3 \, \|u - v\|_* \, .$$

Thus the following statement has been proved:

Proposition 3

$$\|U - V\|_* \leq \eta c_3 \, \|u - v\|_* \, .$$

3.5 An existence theorem

The Propositions 2 and 3 are true only under the hypothesis that the first order derivatives of $u(t,x)$ with respect to the spacelike variables x_j exist (and belong to $\mathcal{B}_{s(x)}$). In addition, $u(t,x)$ must be a function with a first order interior estimate.

This is not the case for an arbitrary element of $\mathcal{B}_*(M)$. In order to apply the above estimations, one has to find a closed subset of $\mathcal{B}_*(M)$ such that the assumptions mentioned above are true everywhere in this subset. Such a subset can be defined as kernel of an elliptic operator $\mathcal{G}$. Define

$$\mathcal{B}_*^{\mathcal{G}}(M) = \Big\{ u \in \mathcal{B}_*(M) : \mathcal{G}u(t,\cdot) = 0 \quad \text{for each fixed} \quad t \Big\}.$$

Notice that $\mathcal{G}$ has to be an elliptic operator whose coefficients do not depend on t. Condition (43) can be verified using an interior estimate for solutions of elliptic differential equations (see A. Douglis and L. Nirenberg [9] and also S. Agmon, A. Douglis and L. Nirenberg [2]), whereas $\mathcal{B}_*^{\mathcal{G}}(M)$ is closed in view of a Weierstrass convergence theorem for elliptic equations.

In order to apply the contraction mapping principle, the operator (36) has to map this subspace $\mathcal{B}_*^{\mathcal{G}}(M)$ into itself.

Definition Let $\mathcal{F}$ be a first order differential operator depending on t, x, $u = u(t,x)$ and on the spacelike first order derivatives $\frac{\partial u}{\partial x_j}$, while $\mathcal{G}$ is any differential operator with respect to the spacelike variables x_j whose coefficients do not depend on the time t. Then $\mathcal{F}$, $\mathcal{G}$ is called an *associated pair* if $\mathcal{F}$ transforms solutions of $\mathcal{G}u = 0$ into solutions of the same equation for fixedly chosen t, i.e., $\mathcal{G}u = 0$ implies $\mathcal{G}(\mathcal{F}u) = 0$.

Note that $\mathcal{G}$ needs not be of first order [11].

In view of Proposition 3, the corresponding integral operator (36) is contractive in case the height ηs_0 of the conical domain M is small enough, and thus the following statement has been proved:

Theorem 6 *Suppose that $\mathcal{F}, \mathcal{G}$ is an associated pair. Suppose, further, that the solutions of $\mathcal{G}u = 0$ satisfy an interior estimate of first order. Then the initial value problem*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{F}u \\ u(0,\cdot) &= \varphi \end{aligned}$$

is solvable provided the initial function φ satisfies the side condition $\mathcal{G}\varphi = 0$. Moreover, the solution $u = u(t,x)$ satisfies the side condition $\mathcal{G}u(t,\cdot) = 0$ for each t.

There are two possiblities for the application of Theorem 6: In case $\mathcal{F}$ is given, then one has to find an associated $\mathcal{G}$. If, however, $\mathcal{G}$ is given, then one can look for operators $\mathcal{F}$ such that initial value problems with initial functions φ satisfying $\mathcal{G}\varphi = 0$ can be solved. Moreover, it is possible that to a given $\mathcal{F}$ there exist several so-called co-associated [11] operators $\mathcal{G}$. This leads to decomposition theorems [10] for the solution of initial value problems.

3.6 Conservation laws

Theorem 6 does not only show the existence of a solution $u = u(t, x)$ of the initial value problem (33), (34), but also it says that the constructed solution satisfies the side condition $\mathcal{G}u = 0$ for each t. Thus this side condition $\mathcal{G}u = 0$ may be interpreted as a conservation law for the evolution equation (33).

3.7 The special case of Complex Analysis

Interior estimates for holomorphic functions Φ can be obtained from Cauchy's Integral Formula for the derivative Φ'. Indeed, let Φ be holomorphic in the bounded domain Ω and continuous in its closure $\overline{\Omega}$. If $\zeta \in \Omega$ has at least the distance δ from the boundary $\partial\Omega$, then the Cauchy Integral Formula applied to the disk $\{z : |z - \zeta| \leq \delta\}$ yields

$$|\Phi'(\zeta)| \leq \frac{1}{2\pi} \cdot \frac{1}{\delta^2} \cdot \sup_{\Omega} |\Phi| \cdot 2\pi\delta = \frac{1}{\delta} \cdot \|\Phi\|$$

where $\|\cdot\|$ means the supremum norm. Similar estimates are true for the Hölder norm and the L_p-norm as well (see [22]).

Interior estimates for generalized analytic functions can be obtained using the T_Ω- and the Π_Ω operators. Suppose $w = w(z)$ is a solution of the Vekua equation

$$\frac{\partial w}{\partial \overline{z}} = a(z)w + b(z)\overline{w}.$$

Then in view of Theorem 3 (Section 2.3) and Weyl's Lemma (Section 2.4, Theorem 5) the function

$$\Phi = w - T_\Omega(aw + b\overline{w})$$

turns out to be holomorphic. In view of Theorem 4 of Section 2.3, differentiation with respect to z implies

$$\frac{\partial w}{\partial z} = \Phi' + \Pi_\Omega(aw + b\overline{w}).$$

Since the T_Ω- and the Π_Ω operators are bounded (cf. Section 2.5), an interior estimate for Φ gives thus an interior estimate for generalized analytic functions. Taking into account Theorem 6, in this way initial value problems with generalized analytic functions as initial functions can be solved (see [22]).

3.8 The scale method

Initial value problems of type (33), (34) can also be solved using an *abstract Cauchy-Kovalevskaya Theorem* instead of applying the contraction-mapping principle to the Banach space $\mathcal{B}_*^{\mathcal{G}}(M)$ introduced in Section 3.5. Here one starts from a *scale of Banach spaces*, i.e., one has a family of (abstract) Banach spaces $\mathcal{B}_s$ with norms $\|\cdot\|_s$, $0 < s < s_0$, which are embedded into each other. The latter means that $\mathcal{B}_s$ is a subspace of $\mathcal{B}_{s'}$ if $0 < s' < s < s_0$ where $\|u\|_{s'} \leq \|u\|_s$. An example of such a scale is given by the Banach spaces $\mathcal{B}_s = \mathcal{B}(\Omega_s)$ introduced in Section 3.3 where the Ω_s form an exhaustion of a given domain Ω in $\mathbf{R^n}$. In this case embedding means nothing but the restriction of a function in Ω_s to a smaller domain $\Omega_{s'}$.

Suppose that for every t an operator $\mathcal{F}(t,u)$ is given mapping $\mathcal{B}_s$ into $\mathcal{B}_{s'}$ where $s' < s$. Suppose, further, that the condition

$$\|\mathcal{F}(t,u) - \mathcal{F}(t,v)\|_{s'} \leq \frac{c\,\|u-v\|_s}{s-s'}$$

is satisfied with a constant c not depending on t, u, v, s, s'. Under this condition the abstract initial value problem

$$\begin{aligned} \frac{du}{dt} &= \mathcal{F}(t,u) \\ u(0) &= u_0 \end{aligned}$$

is solvable (by successive approximations) in the scale $\mathcal{B}_s$, $0 < s < s_0$. In the linear case a proof of the abstract Cauchy-Kovalevskaya Theorem can be found in F. Treves' book [19], while T.Nishida's paper [16] and L. Nirenberg's book [15] prove non-linear versions. In these publications [19, 15, 16] there are also further references.

In the booklet [22] initial value problems with generalized analytic initial functions are solved using not only the method of weighted norms but also the scale method which is based on the above mentioned abstract Cauchy-Kovalevskaya Theorem.

4 Outlook to further generalizations and open problems

4.1 Interactions between different generalizations of classical Complex Analysis

There are various generalizations of classical Complex Analysis dealing with several complex variables, monogenic functions and generalized analytic functions.

Generalized analytic functions

A generalized analytic function is a (complex-valued) solution $w = w(z)$ of a differential equation of the form

$$\frac{\partial w}{\partial \overline{z}} = F\left(z, w, \frac{\partial w}{\partial z}\right). \tag{48}$$

Important special cases of this equation are

$$\begin{aligned} &\frac{\partial w}{\partial \overline{z}} = 0 \quad \text{(Cauchy} - \text{Riemann system)} \\ &\frac{\partial w}{\partial \overline{z}} = a(z)w + b(z)\overline{w} \quad \text{(Vekua equation)} \\ &\frac{\partial w}{\partial \overline{z}} = q(z)\frac{\partial w}{\partial z},\ |q(z)| \leq q_0 < 1 \quad \text{(Beltrami equation).} \end{aligned} \tag{49}$$

If $w = (w_1, ..., w_m)$ is a desired vector satisfying an equation (48) with a vector-valued right-hand side, then w is said to be a *generalized analytic vector*.

Definition of poly-analytic functions

A complex-valued function $w = w(z)$ is a *poly-analytic* function if it is a solution of the differential equation

$$\frac{\partial^n w}{\partial \overline{z}^n} = 0, \tag{50}$$

where n is a natural number, i.e., a poly-analytic function has the form

$$w(z) = \sum_{\nu=0}^{n-1} a_\nu(z)\overline{z}^\nu$$

where the $a_\nu(z)$ are holomorphic coefficients.

Holomorphic functions in several complex variables

A (complex-valued) function depending on $n \geq 2$ complex variables $z_1, z_2, \ldots, z_n$ is said to be a *holomorphic function in several complex variables* in case the n-dimensional Cauchy-Riemann system

$$\frac{\partial w}{\partial \overline{z}_j} = 0, \quad j = 1, ..., n,$$

is satisfied.

Definition of monogenic functions

Consider a Clifford Algebra with the basis

$$e_0 = 1, e_1, ..., e_n, e_{12}, ..., e_{12\ldots n}$$

over the $(n+1)$-dimensional Euclidian space $\mathbf{R^{n+1}}$ with the coordinates $x = (x_0, x_1, ..., x_n)$. Define the $(n+1)$-dimensional Cauchy-Riemann operator D by

$$D = \sum_{\nu=0}^{n} e_\nu \frac{\partial}{\partial x_\nu}.$$

Then a Clifford-Algebra-valued function $u = u(x)$ defined in a domain Ω of $\mathbf{R^{n+1}}$ is called *(left-)monogenic* in Ω in case the first order equation

$$Du = 0$$

is satisfied. This equation is a system of 2^n equations for the 2^n real-valued components of u.

A main feature of present trends in Complex Analysis is the *combination* of methods developed in different branches. For instance, in the theory of multi-monogenic functions one can apply ideas coming from the theory of holomorphic functions in several complex variables such as methods for proving extension theorems.

Recent trends concern the following problems:

- Characterization of the set of all zeros of a solution, including factorization theorems
- Characterization and classification of the singularities of solutions
- Partial complex differential equations with singular coefficients
- Degeneration of the ellipticity, e.g., non-uniformly elliptic equations such as $\partial w/\partial \overline{z} = (1-z)(\partial w/\partial z)$ and $\partial w/\partial \overline{z} = |z|(\partial w/\partial z)$ in the unit disk (note that the Beltrami equation $\partial w/\partial \overline{z} = q(z)(\partial w/\partial z)$ is uniformly elliptic if $|q(z)| \le q_0 < 1$)
- Solution of overdetermined systems

In order to illustrate these tendencies, we consider the following four examples:

Example 1: A factorization theorem for generalized analytic functions

A holomorphic functions has *isolated zeros* unless it vanishes identically. The same is also true for solution $w = w(z)$ of the Vekua equation (49). This statement can be proved easily using the T_Ω-operator:

Suppose the coefficients $a(z)$ and $b(z)$ are continuous. Let $w = w(z)$ be a continuous solution. Define

$$g = \begin{cases} a + b\frac{\overline{w}}{w} & \text{in case } w \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h = T_\Omega g.$$

Then $\Phi = w \cdot \exp(-h)$ turns out to be holomorphic because

$$\begin{aligned} \frac{\partial \Phi}{\partial \overline{z}} &= \frac{\partial w}{\partial \overline{z}} \cdot \exp(-h) + w \cdot \exp(-h) \cdot \left(-\frac{\partial h}{\partial \overline{z}}\right) \\ &= \exp(-h) \cdot \left(\left(aw + b\overline{w}\right) - w\left(a + b\frac{\overline{w}}{w}\right)\right) = 0. \end{aligned}$$

To sum up, we have proved the factorization

$$w = \Phi \cdot \exp h \tag{51}$$

of a generalized analytic function w by a holomorphic factor Φ and a (continuous) factor $\exp h$ which is different from zero everywhere. Consequently, concerning the distribution of the zeros a generalized analytic function behaves like a holomorphic one.

As a by-product of the factorization (51) one gets a modified maximum principle for generalized analytic functions. Note that

$$|w| = |\Phi| \cdot \exp(\text{Re } h).$$

Since h is bounded, there exist constants c_1 and c_2 such that

$$c_1 \leq \text{Re } h \leq c_2$$

and, therefore,

$$\begin{aligned} &|w| \leq |\Phi| \cdot \exp c_2 \\ &|\Phi| \leq |w| \cdot \exp(-c_1). \end{aligned}$$

Applying the maximum modulus principle for holomorphic functions, it follows

$$\sup_{\Omega} |w| \leq \sup_{\partial\Omega} |w| \cdot \exp(c_2 - c_1).$$

Example 2: The inhomogeneous Cauchy-Riemann system in several complex variables ($\overline{\partial}$-equation)

In the case of n complex variables $z_1, ..., z_n$ the inhomogeneous Cauchy-Riemann systems reads

$$\frac{\partial w}{\partial \overline{z}_j} = h_j, \quad j = 1, ..., n, \tag{52}$$

where the right-hand sides h_j are given functions. Suppose the *compatibility conditions*

$$\frac{\partial h_j}{\partial \overline{z}_k} = \frac{\partial h_k}{\partial \overline{z}_j} \tag{53}$$

are satisfied.

Suppose, additionally, that the h_j are compactly supported. Then we shall show that a solution of (52) is given by the T_Ω-operator in one of the z_j-planes where Ω is here the whole z_j-plane. Choosing $j = 1$, we have

$$w(z_1, z_2, ..., z_n) = -\frac{1}{\pi} \iint \frac{h_1(\zeta_1, z_2, ..., z_n)}{\zeta_1 - z_1} d\xi_1 d\eta_1 \tag{54}$$

where $\zeta_1 = \xi_1 + i\eta_1$. Indeed, Theorem 3 of Section 2.3 gives $\partial w/\partial \overline{z}_1 = h_1$. Further, for $k \neq 1$ we get

$$\frac{\partial w}{\partial \overline{z}_k} = -\frac{1}{\pi} \iint \frac{\partial h_1}{\partial \overline{z}_k}(\zeta_1, z_2, ..., z_n) \frac{1}{\zeta_1 - z_1} d\xi_1 d\eta_1. \tag{55}$$

In view of the compatibility conditions, $\partial h_1/\partial \overline{z}_k$ can be replaced by $\partial h_k/\partial \overline{z}_1$. Since h_k has a compact support, then formula (7) of Section 1.4 is applicable for the Cauchy-Riemann operator in the z_1-plane. Consequently, the right-hand side of (55) equals h_k. This proves formula (54).

The above formula (54) is true if the h_j are given in the whole of $\mathbf{C^n}$ and are identically equal to zero outside a bounded set of $\mathbf{C^n}$. Next assume that the h_k are given only in a (bounded) poly-cylinder $\Omega_1 \times \cdots \times \Omega_n$ of $\mathbf{C^n}$. Suppose, however, that the compatibility conditions (53) are satisfied and, moreover, that the derivatives of the h_j with respect to different variables $\overline{z}_k$ exist up to the order $n - 1$. Then a special solution of the inhomogeneous Cauchy-Riemann system (52) is given by iterated T_{Ω_j}-operators:

$$w = \sum_{\lambda=1}^{n} (-1)^{\lambda+1} \sum_{j_1,\dots,j_\lambda}^{*} T_{\Omega_{j_1}} \cdots T_{\Omega_{j_\lambda}} h_{j_1 \dots j_\lambda} \tag{56}$$

where the $h_{j_1 \dots j_\lambda}$ are defined recursively,

$$h_{j_1 \dots j_\lambda k} = \frac{\partial h_{j_1 \dots j_\lambda}}{\partial \overline{z}_k},$$

and $\sum_{j_1,...,j_\lambda}^{*}$ means summation over combinations of different indices $j_1, ..., j_\lambda$.

We prove formula (56) for the case $n = 2$ where formula (56) reads

$$w = T_{\Omega_1} h_1 + T_{\Omega_2} h_2 - T_{\Omega_1} T_{\Omega_2} h_{12}.$$

This implies

$$\frac{\partial w}{\partial \overline{z}_1} = h_1 + T_{\Omega_2} h_{21} - T_{\Omega_2} h_{12} = h_1.$$

Similarly, we can prove the second equation $\partial w / \partial \overline{z}_2 = h_2$. The proof for the general case (56) works in the same way.

Now consider a generalized analytic function in several complex variables which satisfies a system of the form

$$\frac{\partial w}{\partial \overline{z}_j} = f_j(z_1, ..., z_n, w). \tag{57}$$

Under suitable conditions on the right-hand sides one can show that a solution $w = w(z_1, ..., z_n)$ can be factorized in the form (51) where Φ is now a holomorphic function in n complex variables $z_1, ..., z_n$. This shows that the set of all zeros is an analytic set in $\mathbf{C^n}$ provided suitable conditions on the right-hand sides of the system (57) are satisfied (see [20]).

Example 3: Proof of Hartogs Continuation Theorem via the inhomogeneous Cauchy-Riemann system

The Hartogs Continuation Theorem is the following statement:

Let Ω be a domain in $\mathbf{C^n}$ with $n \geq 2$ and K a compact subset of Ω such that $\Omega \setminus K$ is connected. Then each holomorphic function in $\Omega \setminus K$ can be extended holomorphically into K.

An easy proof of this theorem can be given with the help of the inhomogeneous Cauchy-Riemann equation in several complex variables (see L. Hörmander [12]). For this purpose choose any infinitely differentiable function λ which is identically equal to 1 in a neighbourhood of K, and which vanishes identically in a neighbourhood of the boundary $\partial\Omega$ of Ω.

Suppose h is the given holomorphic function in Ω. Define $h_0 = (1 - \lambda)h$. Then we have $h_0 \equiv h$ in a neighbourhood of $\partial\Omega$, while h_0 vanishes identically in a neighbourhood of K. Therefore h_0 is defined and infinitely often differentiable everywhere in Ω if we define $h_0 \equiv 0$ in K.

Now we look for a function g in Ω such that

$$H = h_0 - g$$

is holomorphic everywhere in Ω. This is the case if g satisfies the inhomogeneous Cauchy-Riemann system

$$\frac{\partial g}{\partial \overline{z}_j} = f_j \tag{58}$$

where

$$f_j = \begin{cases} -h\frac{\partial \lambda}{\partial \overline{z}_j} & \text{in } \Omega \setminus K \\ 0 & \text{in } K. \end{cases}$$

Since the f_j vanish identically in a neighbourhood of Ω, they are infinitely differentiable in the whole of $\mathbf{C^n}$ if one defines $f_j \equiv 0$ outside Ω. Moreover, they have compact supports and thus the solution of (58) is given by

$$g(z_1, z_2, ..., z_n) = -\frac{1}{\pi} \iint \frac{f_1(\zeta_1, z_2, ..., z_n)}{\zeta_1 - z_1} d\xi_1 d\eta_1$$

where $\zeta_1 = \xi_1 + i\eta_1$ (cf. formula (54)). If, for instance, $|z_2|$ is large enough, then f_1 vanishes identically and thus g vanishes identically, too. Since there are such points of Ω which do not belong to the support of g, we have

$$H \equiv h_0 \equiv h$$

in an open subset of Ω. In view of the Unique Continuation Theorem the functions H and h coincide in the whole of $\Omega \setminus K$, and H is the desired holomorphic extension.

Example 4: The iterated Vekua equation

Recently a combination of poly-analytic and generalized analytic functions has been investigated by P. Berglez who considered the differential equation

$$D^n w = 0$$

where $Dw = \dfrac{\partial w}{\partial \overline{z}} - A(z)w - B(z)\overline{w}$. Details can be found in P. Berglez' paper [5] in the Proceedings [8] of the Graz Workshop which took place in February 2001.

4.2 Boundary value problems

Using complex methods, boundary value problems can be solved not only for first order systems but also for equations and systems of higher order (see the book [31] written by Wen Guo Chun and H. Begehr).

Boundary value problems can also be solved for poly-analytic functions. In other words, these are boundary value problems for the differential equation (50) which is of order n.

In connexion with the general boundary value problem

$$\begin{aligned} \mathcal{L}u &= \mathcal{F}(.,u) \quad \text{in} \quad \Omega \\ \mathcal{B}u &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

(see Theorem 2 in Section 1.6) the following two problems have to be answered:

- Find a fundamental solution of the operator $\mathcal{L}$.
- Which boundary value problem is well-posed for $\mathcal{L}u = 0$?

Generally speaking, a well-posed boundary value problem for $\mathcal{L}u = 0$ is well-posed for $\mathcal{L}u = \mathcal{F}(.,u)$, too.

Note, finally, that fundamental solutions can also be used for getting interior estimates (see, for instance, [27]).

4.3 Initial value problems

Concerning the application of Complex Analysis to initial value problems, mainly the following questions are to be answered:

- Proof of interior estimates, especially it is to be investigated how the constant in the estimate depends on the distance of a compact subset from the distance from the boundary.
- Construction of associated differential operators. Example: Which differential operators map monogenic functions into themselves? Note that associated spaces in connexion with initial value problems lead to *conservation laws.*
- Formulation and proof of uniqueness theorems for general initial value problems (generalized Holmgren type theorems).
- Investigation of the behaviour of the integral operators of Complex Analysis in weighted function spaces.

4.4 A unified approach to Mathematical Analysis

Calculus should be taught within the framework of Banach spaces (simultaneous investigation of real-valued functions of one real variable, systems of real-valued functions depending on several real variables, complex-valued functions of one complex variable, mappings between Banach spaces such as integral operators mapping the space of continuous functions into itself).

Of course, concerning Complex Analysis, Generalized Analytic Functions cannot be a subject compulsory of all students of Mathematics. However, the spirit of that theory should be used when teaching Complex Analysis. That means, especially,

- Complex differentiation has to be introduced in the framework of partial complex differentiation.
- Instead of starting from the classical Cauchy Integral Formula, the starting point should be the generalized Cauchy Integral Formula

$$w(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{w(z)}{z-\zeta} dz - \frac{1}{\pi} \iint\limits_{\Omega} \frac{\partial w}{\partial \overline{z}}(z) \frac{1}{z-\zeta} dx dy$$

  (Cauchy-Pompeiu Integral Representation, see Section 2, formula (18)).
- Weak (distributional) derivatives in connexion with the concept of distributional solutions of Partial Differential Equations should be compulsory for each student of Mathematics.

As far as possible, existence theorems should be proved using methods of high generality such as the contraction-mapping principle. E.g., in analogy to the integral rewriting of ordinary differential equations, partial differential equations with a linear elliptic principal part should be reduced to fixed-point problems in suitable function spaces (example: W. Walter's proof [30] of the Cauchy-Kovalevskaya Theorem by the contraction-mapping principle).

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While references to special research papers are given at corresponding places, trend-setting monographs whose results are used at various places throughout the present report will be listed in the sequel:

Theory and applications of generalized analytic functions can be found in I. N. Vekua's book [29]. Complex methods for non-linear systems are discussed in the booklet [21]. M. B. Balk's book [3] deals with polyanalytic functions. An approach to holomorphic functions in several complex variables making use of the inhomogeneous Cauchy-Riemann equation is given in L. Hörmander's book [12]. Monogenic functions are discussed in F. Brackx', R. Delanghe's and F. Sommen's Research Notes [7]. Mollifiers (which are in use in order to approximate weak solutions by smooth functions) are introduced in R. A. Adam's monograph [1].

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