Ramsey theory for structures: Nešetřil's result on finite metric spaces

Carlos Augusto Di Prisco

Contents

1 Introduction

The main objective of these notes is to give a mostly self contained presentation of J. Nešetřil's recent proof of the Ramsey property for the class of ordered finite metric spaces [6]. This result was motivated by a question posed in [3], where the connections between Ramsey theory and the dynamics of groups of automorphisms are explored (see also [5]. These notes were written for a course on Ramsey Theory given by the author in Caracas during the first term of 2005.

A class of finite ordered structures is a Ramsey class if given structures A, B in the class, and a positive integer t, there is another structure C in the class such that for every partition of the set of substructures of C which are isomorphic to A into t pieces, there is a substructure of C isomorphic to B which is homogeneous, in the sense that all of its substructures isomorphic to A are in the same piece. Often, a partition into t pieces is seen as a t-coloring, and a homogeneous set for the partition is then said to be monochromatic.

Finite ordered metric spaces can be seen as labelled binary relational structures of a particular kind. For such a structure the triangular inequality can be obtained using the notion of l-metric system (see section 3); we will see that a finite binary relational structure of the appropriate kind is a metric space if it is l-metric for a sufficiently large number l. The Ramsey property is proved by induction on l for the class of l- metric systems (Main Lemma). The first step of the induction (the case $l = 1$) follows from the fact that the class of finite ordered relational structures has the Ramsey property (Theorem 4 of section 2), which is a result of [8].

Two of the most emblematic results of Ramsey Theory are Ramsey's theorem about partitions, or colorings, of the k element subsets of a finite set, and the Hales -Jewett theorem about colorings of the n^{th} power of a finite set. Ramsey's theorem can be considered the starting point of the theory, and has been extended in many directions. The Hales-Jewett theorem is a powerful result which contains the combinatorial essence of the famous result of van der Waerden about arithmetic progressions. Both results will be used in the following sections.

We introduce some notation in order to state these two theorems. Every natural number n is identified with the set of its predecessors $\{0, 1, \ldots, n-1\}$. Given a set A and $k \in \mathbb{N}$, $A^{[k]}$ denotes the set $\{s \subseteq A : |s| = k\}$. Given positive integers n, m, k, t, the partition symbol

$$n \to (m)^k_t$$

is used to express that for every coloring $c : n^{[k]} \to t$ there is $H \subseteq n$ with $|H| = m$ such that c is constant on $H^{[k]}$.

Theorem 1 *(Ramsey's Theorem) Given positive integers k, r and m there is a positive integer n such that*

$$n \to (m)^r_k.$$

Before stating the Hales-Jewett theorem we need some definitions. Let $k \in \mathbb{N}$ and let $\Lambda_k = \{1, 2, \ldots, k\}$. Given $n \in \mathbb{N}$, Λ_k^n is the set of n-tuples of elements of Λ_k, or words of length n in the alphabet Λ_k.

Definition 2 *A combinatorial line in Λ_k^n is a set $\{x_1, x_2, \ldots, x_k\}$ of elements of Λ_k^n such that for each coordinate j, $1 \leq j \leq n$, either*

$$x_1(j) = x_2(j) = \cdots = x_k(j)$$

or

$$x_i(j) = i \text{ for every } i = 1, \ldots, k,$$

and the second possibility occurs at least once.

Another way to define combinatorial lines is by variable words. A variable word is a word in the alphabet $\{1, \ldots, k, x\}$ where x appears at least once. The symbol x acts as a variable. Given a variable word $w(x)$, we write $w(i)$ to denote the word (in the alphabet Λ_k) resulting from substituting i for x in $w(x)$. If $w(x)$ is a variable word, the combinatorial line associated to $w(x)$ is $\{w(1), w(2), \ldots, w(k)\}$.

Theorem 3 *(Hales-Jewett)*
Given positive integers $k, r \in \mathcal{N}$, there is a number $n = n(k, r)$ such that for every r-coloring Λ_k^n there is a monochromátic combinatorial line.

The proofs of these two theorems can be found in [2, 4].

2 Ramsey properties for relational structures.

We consider finite relational structures defined in the following way. A type is a sequence $\Delta = (\delta_i : i \in I)$ of natural numbers, where I is a finite set. Given a type Δ, a structure of type Δ is a pair $(X, \mathcal{M})$ such that

(i) X is a linearly ordered set, and

(ii) $\mathcal{M} = (\mathcal{M}_i : i \in I)$, and $\mathcal{M}_i \subseteq X^{[\delta_i]}$

The linear order of X is called the standard order.

Rel(Δ) denotes the class of finite structures of type Δ. Note that these relational structures are labelled hypergraphs.

Given structures $A = (X, \mathcal{M})$ and $B = (Y, \mathcal{N})$ of type Δ, a function $f : X \to Y$ is an embedding if

(i) f is one-one and monotone with respect to the standard linear orderings of X and Y, and

(ii) for every $i \in I$ and every subset M of $X^{[\delta_i]}$, $M \in \mathcal{M}_i$ if and only if $\{f(x) : x \in M\} \in \mathcal{N}_i$.

We write $A \leq B$ to express that there is an embedding from A to B, and $A \cong B$ when A and B are isomorphic.

Given structures A, B, $\binom{B}{A}$ denotes the set of all substructures of B which are isomorphic to A.

If $A \leq B \leq C$, the partition symbol

$$C \to (B)_t^A$$

expresses that for every coloring $c : \binom{C}{A} \to t$, there is a $B' \in \binom{C}{B}$ such that the collection $\binom{B'}{A}$ is monochromatic.

Theorem 4 *For any given type Δ, the class Rel(Δ) is a Ramsey class. In other words, given structures A, B in Rel(Δ) with $A \leq B$, and a positive integer t, there is a structure C in Rel(Δ) such that $B \leq C$ and $C \to (B)_t^A$.*

To prove this theorem, we define partite systems and use the amalgamation technique, following [4]. This is done in the next two sections.

2.1 Partite systems.

Definition 5 *Given a type $\Delta = (\delta_i : i \in I)$ and $a \in \mathbb{N}$, an a-partite system of type Δ is a pair $((X_j)_{j=1}^a, \mathcal{M})$ where*

(a) $X = \bigcup_{i=1}^a X_i$ is a linearly ordered set satisfying $X_1 < X_2 < \cdots < X_a$, i.e., for every $i, j \in \{1, \ldots, a\}$ with $i < j$, if $x \in X_i$ and $j \in X_j$, then $x < y$.

(b) $\mathcal{M} = (\mathcal{M}_i : i \in I)$, and $\mathcal{M}_i \subseteq X^{[\delta_i]}$

(c) $|M \cap X_j| \leq 1$ for every $M \in \mathcal{M}_i$, $j = 1, \ldots, a$, $i \in I$.

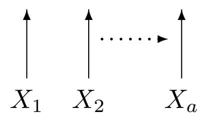

Fig. 1 Partite system

Given a subset $Y \subseteq X$, we denote by $tr(Y)$ the trace of Y, i.e. the set $\{j : X_j \cap Y \neq \emptyset\}$.

A system A is transversal if $|X_j| = 1$ for every $j = 1, \ldots, a$.

The system A is a subsystem of $B = ((Y_k)_{k=1}^b, \mathcal{N})$ if there exists a monotone injection $h : \{1, \ldots, a\} \to \{1, \ldots, b\}$ such that $X_j \subseteq Y_{h(j)}$ for every $j = 1, \ldots, a$ and $\mathcal{M}_i = \mathcal{N}_i \cap X^{[\delta_i]}$ for $i \in I$. An isomorphism is an order preserving isomorphism of structures which also preserves parts.

Lemma 6 *(The Partite Lemma) Let A and B be a-partite systems of type Δ, A transversal, and let t be a positive integer, then there exists an a-partite system C of type Δ such that*

$$C \to (B)_t^A.$$

Proof. Set $A = ((X_j)_{j=1}^a, \mathcal{M})$ and $B = ((Y_j)_{j=1}^a, \mathcal{N})$. Since A is transversal, we may assume without loss of generality that $\bigcup_{i \in I} \mathcal{M}_i$ is the set of all subsets of X. We also can assume that every vertex $y \in Y$ belongs to a copy of A. This is so because otherwise we can work with B^*, the subsystem of B induced by $\binom{B}{A}$, which satisfies this property, and if $C^*$is such that $C^* \to (B^*)_t^A$, then we can obtain C such that $C \to (B)_t^A$ enlarging each copy of B^* in C^* to a copy of B.

We fix a sufficiently large positive integer N, and define an a- partite system $C = ((Z_j)_{j=1}^a, \mathcal{O})$, $\mathcal{O} = (\mathcal{O}_i : i \in I)$ where $Z_j = Y_j \times \cdots \times Y_j$ (N times). Thus, every element of Z_j has the form $(x_l : l = 1, \dots, N)$ with each $x_l \in Y_j$. We will say more about the number N later on.

Set $Z = \bigcup_{j=1}^{a} Z_j$. For each $l = 1, \dots, N$, the projection $\pi_l : Z \to Y$ is defined by $\pi_l(x_k : k = 1, \dots, N) = x_l$. For every l, π_l maps Z_l into Y_l.

We now define $\mathcal{O} = (\mathcal{O}_i : i \in I)$. Put first $\mathcal{N}_i = \mathcal{N}_i' \cup \mathcal{N}_i''$, where $\mathcal{N}_i'$ is the set of edges of $\mathcal{N}_i$ which belong to a copy of A in B, and $\mathcal{N}_i'' = \mathcal{N}_i \setminus \mathcal{N}_i'$.

We put

$$\{(x_1^k, \dots x_N^k) : k = 1, \dots, n_i\} \in \mathcal{O}_i$$

if $tr(\{x_j^k : k = 1, \dots, n_i\}) = tr(\{x_{j'}^k : k = 1, \dots, n_i\})$ for all $j, j' \leq N$, and one of the following possibilities occur:

1. $\{x_j^k : k = 1, \dots, n_i\} \in \mathcal{N}_i'$ for every $j = 1, \dots, N$,

2. there exists a non-empty set $\Gamma \subseteq \{1, \dots, N\}$ such that

$\{x_j^k : k = 1, \dots, n_i\} = \{x_{j'}^k : k = 1, \dots, n_i\} \in \mathcal{N}_i''$ for all $j, j' \in \Gamma$, and
$\{x_j^k : k = 1, \dots, n_i\} \in \mathcal{N}_m'$ for all $j \notin \Gamma$

In general, $m \neq i$, but m is uniquely determined by $tr(x_j^k : k = 1, \dots, n_i)$.

We now prove that $C \to (B)_t^A$ provided N is large enough. This will follow from the two facts stated below.

Fact 1. $A' \in \binom{C}{A}$ if and only if $\pi_l(A') \in \binom{B}{A}$ for every $l = 1, \dots, N$. This is an immediate consequence of the definition of $\mathcal{O}$. If $\pi_l(A') \in \binom{B}{A}$ for every $l = 1, \dots, N$, then clearly $A' \in \binom{C}{A}$. Conversely, let $A' = \{(x_1^k, \dots, x_N^k) : k = 1, \dots, a\}$ be a substructure of C which forms a copy of A, and suppose that $\{(x_1^k, \dots, x_N^k) : k = k_{m_1}, \dots, k_{m_{n_i}}\} \in \mathcal{O}_i$, then for every $j = 1, \dots, N$, the projection $\{x_j^k : k = k_{m_1}, \dots, k_{m_{n_i}}\} \in \mathcal{N}_i$. This is so because by the definition of $\mathcal{O}_i$, this projection is always in $\mathcal{N}_i$: if the second case of the definition occurs, then either $\{x_j^k : k = k_{m_1}, \dots, k_{m_{n_i}}\}$ belongs to $\mathcal{N}_i''$ and thus to $\mathcal{N}_i$, or to $\mathcal{N}_m'$ for some m, but since this edge forms part of a copy of A (because it is in $\mathcal{N}_m'$), it is also in $\mathcal{N}_i$. Note that an edge of A' in $\mathcal{O}_i$ must come then from the first clause of the definition of $\mathcal{O}_i$.

Let $\binom{B}{A} = \{A_1, \dots, A_r\}$, and put $R = \{1, \dots, r\}$. Given $\alpha = (\alpha_1, \dots, \alpha_N) \in R^N$, denote by $V(\alpha)$ the set of all the vertices $x \in Z$ which satisfy $\pi_j(x) \in A_{\alpha_j}$. If L is a combinatorial line in R^N, set $V(L) = \bigcup_{\alpha \in L} V(\alpha)$. By Fact 1, the set $\binom{C}{A}$ is in $1-1$ correspondence with R^N.

Fact 2. Let L be a combinatorial line of R^N. Then, $V(L)$ induces a copy of B in C.

Clear from the definition of C, since B is the union of the r copies of A it contains. Notice that the second option in the definition of $\mathcal{O}_i$ is important to obtain a copy of B in C from the union of all these copies of A; notice also that our assumption that every subset of X is an edge of A is used here.

Now, by the Hales-Jewett Theorem, if N was chosen large enough, for every partition of R^N into t classes, there is a combinatorial line contained in one of the classes. This implies $C \to (B)^A_t$. In fact, if $\binom{C}{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_t$ is a partition, it induces a partition $R^N = \mathcal{A}'_1 \cup \cdots \cup \mathcal{A}'_t$ by $\alpha \in \mathcal{A}'_i$ if $V(\alpha)$ induces a copy of A which is in $\mathcal{A}_i$. By the Hales-Jewett Theorem, there is a monochromatic line L which, by Fact 2, induces a $B'' \in \binom{C}{B}$, such that $\binom{B''}{A}$ is contained in a single class $\mathcal{A}_i$. □

2.2 The partite construction.

To prove Theorem 4, we use an amalgamation technique called the partite construction first used by Nešetřil and R ödl (see [8] , [4]).

Proof of Theorem 4. Let t, and A, B be given as in the statement of Theorem 4. We consider A as a transversal a-partite system and B as a transversal b-partite system. Put $B = ((y_1, \ldots, y_b), \mathcal{N})$ Let p be the minimal n such that $n \to (b)^a_t$, and let $q = \binom{p}{a}$, and put $\binom{\{1,\ldots,p\}}{\{1,\ldots,a\}} = \{M^1, \ldots, M^q\}$.

We will define a sequence $P^0, P^1, \ldots, P^q$ of "pictures", the last of which, P^q, will be the desired system C.

Let $P^0 = ((X^0_i)^p_{i=1}, \mathcal{O})$ be a p-partite system such that for each choice of b parts of P^0, $X^0_{i_1}, \ldots, X^0_{i_b}$, the subsystem of P^0 induced by them contains a copy of B. This can be obtained taking a disjoint union of copies of B.

If the picture $P^k = ((X^k_i)^p_{i=1}, \mathcal{O}^k)$ has been defined, consider M^{k+1} and the a-partite system D^{k+1} induced in P^k by the parts X^k_i for which i belongs to M^{k+1}. By the Partite Lemma 6, there is an a-partite system E^{k+1} such that

$$E^{k+1} \to (D^{k+1})^A_t.$$

Extend each copy of D^{k+1} in E^{k+1} to a copy of P^k in such a way that the distinct copies of P^k intersect only in vertices of E^{k+1}. The resulting a-partite system is P^{k+1}. Finally $C = P^q$. We claim that C has the desired properties.

By a backward induction we verify that

$$C \to (B)^A_t.$$

In the inductive step from $k+1$ to k, by the use of the partite lemma in the construction of P^{k+1}, we can find a copy of P^k in P^{k+1} in which all copies of A with trace M^k have the same color.

We end up with a copy P of P^0 such that the color of a copy of A in P depends only on its trace. This induces a t-coloring of $p^{[a]}$, the collection of a-element subsets of p: the color of s is defined as the color of any copy of A whose trace is s. Since $p \to (b)^a_t$, there is a monochromatic subset of p of size b.

By construction, the subsystem of P_0 induced by any b elements of p contains a copy of B, and therefore there is a monochromatic copy of B in P. □

Given a type Δ, if for every pair of structures A, B of type Δ such that B has substructures isomorphic to A, there is a structure C of type Δ such that $C \to (B)^A_2$, then for every positive integer r, and every pair A, B of structures with the same properties as above, there exists C such that $C \to (B)^A_r$.

3 Finite metric spaces.

In this section we present a proof due to J. Nešetřil of the Ramsey property for the class of finite ordered metric spaces. This result answers a question of [3], and gives information about the group of automorphisms of the Urysohn space.

A finite metric space can be viewed as a labelled complete finite graph: a pair of elements forms an edge labelled by the distance between them. These graphs are, in turn, special cases of relational structures.

We denote by *Rel* the class of all finite ordered relational structures of all possible finite types. Given $d, D \in \mathbb{R}$, with $d < D$, $Rel(d, D)$ is the subclass of *Rel* of all systems $A = (X, (R_i; i \in I))$ where I is a finite subset of the interval $[d.D]$, and for every $i \in I$, $R_i \subseteq X^{[2]}$.

Given structures $A = (X, (R_i; i \in I))$ and $B = (Y, (S_i; i \in J))$, a function $f : X \to Y$ is an embedding if

(i) f is one-one and monotone with respect to the standard linear orderings of X and Y, and

(ii) for every $i \in I$ and every pair $\{x, y\}$ of elements of X, $\{x, y\} \in R_i$ if and only if $\{f(x), f(y)\} \in S_i$ (thus, $I \subseteq J$).

If the embedding f is a bijection,, we say it is an isomorphism. Given structures A, B, $\binom{B}{A}$ denotes the set of all substructures of B which are isomorphic to A.

As a consequence of Theorem 4 we have the following theorem, which will be used in the proof of the result for finite ordered metric spaces.

Theorem 7 *(Nešetřil, [8]) For every pair of real numbers d, D, $0 < d < D$, the class $Rel(d, D)$ is Ramsey.*

Let $0 < d < D$ be real numbers, and let l be a positive integer. Consider a structure $A = (X, (R_i : i \in I))$ where I is a finite subset of the interval $[d, D]$ and each R_i is a symmetric binary relation. An edge of $\{x, y\} \in R_i$ of A is l- metric if for every path $x = x_0, x_1, \ldots, x_t = y$, with $t \leq l$ such that $\{x_{k-1}, x_k\} \in R_{i_k}$ (i.e. the distance between x_{k-1} and x_k is i_k) it holds that $i \leq i_1 + i_2 + \cdots + i_t$.

For every positive integer l, and every pair of real numbers $0 < d < D$, the class $Rel_l(d, D)$ is defined as follows. The class $Rel_l(d, D)$ is the subclass of $Rel(d, D)$ formed by the structures $A = (X, (R_i : i \in I))$ that satisfy:

(i) for every $i \in I$, $R_i \subseteq X^{[2]}$ for every $i \in I$, in particular every R_i is symmetric and anti-reflexive, as before, and the following additional properties

(ii) $R_i \cap R_j = \emptyset$ whenever $i \neq j$ for $i, j \in I$,

(iii) every edge of A is l-metric.

The objects of $Rel_l(d, D)$ are relational structures of type $\Delta = (\delta_i : i \in I)$, where for each $i \in I$, $\delta_i = 2$. For a pair $\{x, y\} \in R_i$, the index $i \in I$ is a real number which is called the length, or the weight, of the pair, and sometimes this is expressed writing $\rho(x, y) = i$.

Note that $Rel_1(d, D)$ is the sub-collection of $Rel(d, D)$ formed by the structures with pairwise disjoint binary relations.

If an edge (x, y) is l-metric for every l, then we say it is a metric edge. If for a system A every pair (x, y) of vertices is an edge and it is a metric edge then A is just a metric space (A, ρ).

Note that in case every pair of vertices of A is an edge, if every edge is 2-metric then every edge is metric.

The objects of $Rel_l(d, D)$ need not be metric spaces, but since an edge (x, y) cannot be shortened by paths of length $\leq l$, then the larger l is the better an approximation to a metric we have.

For $l = 1$, the notion of l-metric system coincides thus with the notion of relational structure with pairwise disjoint binary relations

The following lemma generalizes Theorem 4

Lemma 8 *(Main Lemma) For every positive integer l, and every pair of real numbers $0 < d < D$, if A is metric in $Rel(d, D)$, then the class $Rel_l(d, D)$ is A- Ramsey, i.e. for every $B \in Rel_l(d, D)$ such that $A \leq B$, there exists $C \in Rel_l(d, D)$ such that $B \leq C$ and in $Rel_l(d, D)$ the following partition relation holds,*

$$C \to (B)^A_2.$$

Before we give the proof we need to consider partite l-metric systems and their amalgamation. That will be done in the next section. Now we show that from lemma 8 we can derive that the class of ordered finite metric spaces is a Ramsey class.

Theorem 9 *The class of finite ordered metric spaces is a Ramsey class.*

Proof. Let (X, ρ) and (Y, σ) be finite ordered metric spaces, and assume that (Y, σ) contains an isometric copy of (X, ρ). Let $d = \min\{\sigma(x, y) : x, y \in Y\}$, and

$D = \max\{\sigma(x, y) : x, y \in Y\}$, and let $l \geq D/d$. Consider the binary relational systems $A = (X, (R_i : i \in I))$ and $B = (Y, (S_j : j \in J))$ corresponding to the metric spaces (X, ρ) and (Y, σ). Clearly, all edges in A, and B are metric.

By lemma 8 there is a binary relational system $C = (Z, (T_k : k \in K))$ such that $C \to (B)_2^A$ in the class $Rel_l(d, D)$.

Define a metric θ on Z by $\theta(x, y) = \min\{D, SP\}$, where $SP(x, y)$ (shortest path from x to y) is the minimum value of $i_1 + \cdots + i_t$ where $x = x_0, x_1, \ldots, x_t = y$ is a path such that for every $r \leq t$, $(x_{r-1}, x_r) \in T_{i_r}$. All the values taken by θ lie in the interval $[d, D]$, and since $ld \geq D$, for every edge (x, y) of C, $(x, y) \in T_i$ if and only if $\theta(x, y) = i$.

(Suppose $(x, y) \in T_i$, and $x = x_0, x_1, \ldots, x_t = y$ is a path from x to y. If $t \leq l$, then, since (x, y) is l-metric, $i \leq i_1 + i_2 + \cdots + i_t$. And if $t > l$, $i_1 + i_2 + \cdots + i_t > ld \geq D$. Therefore i is the length of the shortest path. Conversely, if $\theta(x, y) = i$, since (x, y) is an edge of C, $(x, y) \in T_j$ for some $j \in K$, and $i \leq j$. If $i < j$, it is because there is a path $x = x_0, x_1, \ldots, x_t = y$ from x to y of length i, but, as before, any path $x = x_0, x_1, \ldots, x_t = y$ must have length $\geq j$, and thus $j = i$.)

From this follows that any embedding from A into C (in $Rel_l(d, D)$) is an isometry (an isometric embedding) of (X, ρ) into (Z, θ), and similarly any embedding from B into C is an isometry from (Y, σ) into (Z, θ). From this we conclude that $Z \to (Y)_2^X$. □

3.1 Partite *l*-metric systems and their amalgamation

We define now the partite approximation classes $PartiRel_l(d, D)$. An object in $PartiRel_l(d, D)$ is a triple (B, A, ι) where A and B are ordered binary relational structures, $A \in Rel_{l-1}(d, D)$ and $B \in Rel_l(d, D)$. More explicitly, $A = (X, (R_i : i \in I))$ and $B = (Y, (S_j : j \in J))$, I, J finite sets of reals contained in the interval $[d, D]$, and $\iota : B \to A$ is a monotone homomorphism satisfying:

(i) If $(x, y) \in S_j$, then $(\iota(x), \iota(y)) \in R_j$ (thus, $J \subseteq I$),

(ii) for every $x \in A$, the set $\iota^{-1}(x)$ is an interval in the ordering of Y.

An embedding from (B, A, ι) into (B', A', ι') is a pair (f, α) such that

(i) $\alpha : A \to A'$ is an embedding in the class $Rel_{l-1}(d, D)$

(ii) $f : B \to B'$ is an embedding in the class $Rel_l(d, D)$

(iii) $\iota' \circ f = \alpha \circ \iota$

If for (B, A, ι), the ι is an injective mapping, we say that (B, A, ι) is a transversal system.

Any $B \in Rel_l(d, D)$ can be viewed as a transversal system(B, B, ι) in $PartiRel_l(d, D)$ where ι is the identity function.

Lemma 10 *(Amalgamation lemma)*

Let $C \in Rel_l(d, D)$, and A a metric subsystem of C (in $Rel_l(d, D)$), with $1 : A \to C$ the inclusion map. For $i = 1, 2$, let (B_i, C, ι_i) be systems in $PartiRel_{l+1}(d, D)$. Let (B_0, A, ι_0) be a system in $PartiRel_l(d, D)$, with embeddings $(f_i, 1) : (B_0, A, \iota_0) \to (B_i, A, \iota_i)$ in $PartiRel_l(d, D)$, for $i = 1, 2$.

Then, there exists $(B_3, C, \iota_3) \in PartiRel_{l+1}(d, D)$, and embeddings $(g_i, 1) : (B_i, c, \iota_i) \to (B_3, C, \iota_3)$ in $PartiRel_{l+1}(d, D)$ such that $g_1 \circ f_1 = g_2 \circ f_2$, and $\iota_3 \circ g_2 = \iota_2$ and $\iota_3 \circ g_1 = \iota_1$. In other words, (B_3, C, ι_3) is an amalgam of the systems (B_i, C, ι_i)

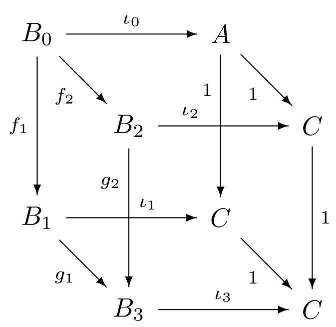

Fig.2 Amalgamation

Proof. We are given the systems (B_1, C, ι_1) and (B_2, C, ι_2), which can be represented as in Fig.3, where the two lines marked C should be identified; the partite subsystem (B_0, A, ι_0) is embedded in both (B_1, C, ι_1) and (B_2, C, ι_2).

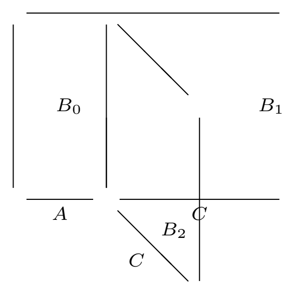

Fig. 3

Let (B_3, C, ι_3) be the free amalgamation of (B_1, C, ι_1) and (B_2, C, ι_2). We have to show that (B_3, C, ι_3) belongs to $PartiRel_{l+1}(d, D)$. Let $\{x, y\}$ be an edge in B_3, and $P = \{x_0 = x, x_1, \ldots, x_t = y\}$ be a path in B_3 from x to y of length $\leq l+1$. We want to prove that the length $\rho(x, y)$ of the edge $\{x, y\}$ is at most $\rho(P) = \sum_{i=1}^{t} \rho(x_{i-1}, x_i)$.

Consider the projection of P, $i_3(P) = \{i_3(x_0), i_3(x_1), \ldots, i_3(x_t)\}$. For each $j = 1, \ldots, t$, $\rho(x_{j-1}, x_j) = \rho(i_3(x_{j-1}), i_3(x_j))$.

$\iota_3(P)$ is a sequence in C, in which some vertices and edges of P might be identified by ι_3. If this in fact occurs, then the length of $i_3(P)$, $\rho(\iota_3(P)) = \rho(P)$ is bounded by the length of a sub-path P' of $\iota_3(P)$ of length $\leq l$. and thus, since $C \in Rel_l(d, D)$, we have that $\rho(x, y) = \rho(i_3(x), i_3(y)) \leq \rho(P') \leq \rho(P)$.

We may thus assume that $\iota_3(P)$ is a path in C of length $l+1$.

If $\iota_3(P)$ is a path in A, then $(\iota_3(x), \iota_3(y))$ is a metric edge, since A is metric, and then $\rho(x, y) = \rho(\iota_3(x), \iota_3(y)) \leq \rho(P)$.

If P is a subset of B_1 or B_2, then also $\rho(x, y) = \rho(\iota_3(x), \iota_3(y)) \leq \rho(P)$, since (B_1, C, ι_1) and (B_2, C, ι_2) are in $PartiRel_{l+1}(d, D)$.

So we have to examine the case in which there are $x_{j_1} \in B_1 \setminus A$ and $x_{j_2} \in B_2 \setminus A$. Since B_3 is a free amalgamation, there are no edges with one vertex in $B_1 \setminus A$ and the other in $B_2 \setminus A$, and so there are at least two vertices x_{k_1} and x_{k_2} for which $\iota_3(x_{k_1})$ and $i_3(x_{k_2})$ lie in A and $\{x_{k_1}, x_{k_2}\}$ are not consecutive in the path P.

Any path in A between $\iota_3(x_{k_1})$ and $\iota_3(x_{k_2})$ adds up to at least $\rho(\iota_3(x_{k_1}), \iota_3(x_{k_2}))$, since A is metric. Now, $\rho(P) \geq \rho(P')$ where P' is the path from $\iota_3(x)$ to $\iota_3(y)$ which goes through $\{x_{k_1}, x_{k_2}\}$, i.e.

$$P' = \{\iota_3(x) = \iota_3(x_0), \iota_3(x_1), \ldots, \iota_3(x_{k_1}), \iota_3(x_{k_2}), \ldots, \iota_3(x_t) = \iota_3(y)\},$$

and $\rho(P') \geq \rho(\iota_3(x), \iota_3(y)) = \rho(x, y)$, since P' is of length at most l and C is in $Rel_l(d, D)$. $\square$

3.2 Proof of the main Lemma

Proof of lemma 8: the proof is by induction on l. For $l = 1$ the lemma follows from Theorem 7. Recall that $Rel_1(d, D)$ is the subclass of $Rel(d, D)$ of structures for which the binary relations R_i are pairwise disjoint. Theorem 7 gives us a structure C in $Rel(d, D)$, but from it we can extract one in $Rel_1(d, D)$ by taking the substructure induced by the copies of B in C. More precisely, we take only the vertices which belong to a copy of B in C, and the edges which lie within a copy of B. By the definition of the embeddings, a copy of B cannot have a pair belonging to two different relations (i.e. no pair has more than one label).

Assume the lemma holds for l, and let $B \in Rel_{(l+1)}(d, D)$. Consider A, B as transversal systems in $PartiRel_{(l+1)}(d, D)$, and let $R \in Rel_{(l)}(d, D)$ be a system satisfying $R \to (B)^A_2$ in $Rel_{(l)}(d, D)$. Fix R, and consider it as a transversal system in $PartiRel_l(d, D)$. We construct now a sequence $P^0, P^1, \ldots, P^a$, of R-partite systems, where $a = |\binom{R}{A}|$. The system P^a will satisfy the required properties.

(P^0, R, ι_0) is the lifting of R obtained by separating all the copies of B contained in R. In other words, P^0 is the disjoint union of $\binom{R}{B} = \{B^1, B^2, \ldots, B^b\}$ with the natural projection to R. Notice that $P^0 \in PartiRel_{l+1}(d.D)$, since $B \in Rel_{l+1}(d, D)$.

Let $\{A^1, \ldots, A^a\}$ list the elements of $\binom{R}{A}$. For the inductive step from i to $i+1$, let (P^i, R, ι^i) be an R-partite system in $PartiRel_{l+1}(d, D)$, and let (D^i, A, ι^i) be the subsystem of (P^i, R, ι^i) induced by the set $(\iota^i)^{-1}(A^i)$. Clearly $(D^i, A, \iota^i) \in PartiRel_{l+1}(d, D)$, and by the inductive hypothesis, there is a system (E^i, A, λ^i) such that

$$E^i \to (D^i)^A_2.$$

Let $(P^{i+1}, R, \iota^{i+1})$ be a free amalgamation of copies of (P^i, R, ι^i) such that every copy of (D^i, A, ι^i) in (E^i, A, λ^i) is extended to a unique copy of (P^i, R, ι^i). According to lemma 10, we know that $(P^{i+1}, R, \iota^{i+1}) \in PartiRel_{l+1}(d, D)$.

Put $(C, R, \iota) = (P^a, R, \iota^a) \in PartiRel_{l+1}(d, D)$. It remains to show that

$$C \to (B)^A_2.$$

This is done by reverse induction from a to 0 in the same fashion as in the end of the proof of Theorem 4 in 2.2 . □

References

[1] Fraïsse, R., Theory of relations. Springer Verlag, 2000.

[2] Graham, R. L., B. L. Rothschild and J. H. Spencer, Ramsey Theory. Wiley, 1990.

[3] Kechris, A., V. Pestov and S. Todorcevic, Fraïssé limits, Ramsey Theory, and Topological Dynamics of Automophism Groups. *Geometric and Functional Analysis*, 15 (2005) 106-189.

[4] Nešetřil, J., Ramsey Theory. In, Handbook of Combinatorics (R. Graham, M. Grötschel and L. Lovász, eds.) Elsevier Science BV and MIT Press, 1995.

[5] Nešetřil, J., Ramsey classes and homogeneous structures. *Combinatorics Probability and Computing*, 14 (2005) 171-189.

[6] Nešetřil, J., Metric spaces are Ramsey. *European Journal of Combinatorics*, 28 (2007) 457-468.

[7] J. Nešetril and V. Rödl, Mathematics of Ramsey Theory. Springer Verlag, 1990.

[8] Nešetřil, J. and V. Rödl, Partitions of finite relational and set systems. *Journal of Combinatorial Theory Ser. A*, 22 (1978) 289-312.

[9] Ramsey, F. P. , On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30 (1930), 264-286.

Carlos A. Di Prisco
Instituto Venezolano de Investigaciones Científicas
Venezuela
cdiprisc@ivic.ve