# Homology and homotopy groups of the complement of certain family of fibers in a critical point problem

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**Abstract**

In this paper we improve the results of [5] by showing that the collection of fibers over a closed countable subset of the base space of a differentiable fibration is not $CS^{\infty}$-critical, under some different topological conditions on the involved spaces. The example we always have in mind is that of Brieskorn manifolds as total spaces of certain principal fiber bundles.

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## 1 Basic results.

We start this section by proving that the complement of a closed countable subset of a given $n$-dimensional manifold has large $n-1$ rational homology group. The manifold $M$ will be with empty boundary all along the paper.

Let $M$ be an $n$-dimensional differentiable manifold and $A \subseteq M$ be a closed countable subset of $M$. Recall that $A$ has uncountably many isolated points, all the other of its points being accumulation points. Assume that $A = I \cup A'$ where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set, namely its set of accumulation points. If $H_{n-1}(M) \simeq 0$, then for each $k \geq 1$ there exists, according to [5, Proposition 2.1], a surjective group homomorphism

$$\delta_k : H_{n-1}(M\backslash A) \to \mathbf{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k),$$

where $A_k = \{a_{k+1}, a_{k+2}, \ldots\} \cup A'$. Moreover, if $M$ is either not compact or compact but not orientable, then $H_{n-1}(M\backslash A) \simeq \mathbf{Z}^k \oplus H_{n-1}(M\backslash A_k)$, for each $k \geq 1$, that is $H_{n-1}(M\backslash A)$ has free abelian subgroups of arbitrarily large rank.

**Proposition 1.1** *Let $M$ be an $n$-dimensional differential manifold $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of $M$, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set. If $H_{n-1}(M) \simeq 0$, then the group homomorphism*

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$$\delta_k \otimes id_{\mathbf{Q}} : H_{n-1}(M\backslash A) \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathbf{Q}^{k-1} \oplus \big(H_{n-1}(M\backslash A_k) \otimes_{\mathbf{Z}} \mathbf{Q}\big)$$

*is surjective, it actually being a* $\mathbf{Q}$*-linear mapping.*

**Corollary 1.2** *Let $M$ be an $n$-dimensional differentiable manifold with $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of $M$, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set. If $V$ is a $\mathbf{Q}$-vector space and $\varphi : V \to H_{n-1}(M\backslash A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a surjective $\mathbf{Q}$-linear mapping, then $V$ is not finite dimensional. In particular $H_{n-1}(M\backslash A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is not finite dimensional.*

*Proof.* Assume that $V$ is finite dimensional and $\varphi : V \to H_{n-1}(M\backslash A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a surjective $\mathbf{Q}$-linear mapping. It follows that $\big(\delta_k \otimes id_{\mathbf{Q}}\big) \circ \varphi : V \to \mathbf{Q}^{k-1} \oplus \big(H_{n-1}(M\backslash A_k) \otimes_{\mathbf{Z}} \mathbf{Q}\big)$ is also a surjective $\mathbf{Q}$-linear mapping for each $k \geq 1$. Consider $q = \dim_{\mathbf{Q}} V$ and observe that $\big(\delta_{q+2} \otimes id_{\mathbf{Q}}\big) \circ \varphi$ acts surjectively from $V$ to $\mathbf{Q}^{q+1} \oplus \big(H_{n-1}(M\backslash A_k) \otimes_{\mathbf{Z}} \mathbf{Q}\big)$. Therefore we have successively

$$q = \dim_{\mathbf{Q}} V = \dim_{\mathbf{Q}} \ker\big[\big(\delta_{q+2} \otimes id_{\mathbf{Q}}\big) \circ \varphi\big] + \dim_{\mathbf{Q}} Im\big[\big(\delta_{q+2} \otimes id_{\mathbf{Q}}\big) \circ \varphi\big] \geq$$

$$\geq \dim_{\mathbf{Q}} Im\big[\big(\delta_{q+2} \otimes id_{\mathbf{Q}}\big) \circ \varphi\big] = \dim_{\mathbf{Q}} \big[\mathbf{Q}^{q+1} \oplus \big(H_{n-1}(M\backslash A_k) \otimes_{\mathbf{Z}} \mathbf{Q}\big)\big] \geq$$

$$\geq \dim_{\mathbf{Q}} \mathbf{Q}^{q+1} = q + 1. \square$$

**Proposition 1.3** *([5]) If $p : E \to M$ is a fibration whose base space $M$ is an $n$-dimensional differentiable manifold and $A$ is a closed countable subset of $M$, then the pair $(E, E\backslash p^{-1}(A))$ is (n-1)-connected, that is $\pi_q(E, E\backslash p^{-1}(A)) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$. In particular we get that $H_q(E, E\backslash p^{-1}(A)) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$ and that the natural Hurewitz group homomorphism $\chi_n : \pi_n(E, E\backslash p^{-1}(A)) \to H_n(E, E\backslash p^{-1}(A))$ is surjective. On the other hand the inclusion $i_{E\backslash p^{-1}(A)} : E\backslash p^{-1}(A) \hookrightarrow E$ is $(n-1)$-connected, that is the induced group homomorphisms $\pi_q(i_{E\backslash p^{-1}(A)}) : \pi_q(E\backslash p^{-1}(A)) \to \pi_q(E)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism $\chi_n$ is an isomorphism if $E$ is simply connected and $n \geq 3$.*

**Remark 1.4** If $M$ is an $n$-dimensional differentiable manifold and $A$ is a closed countable subset of $M$, then, by considering in Proposition 1.3 the particular fibration $id_M : M \to M$, the pair $(M, M\backslash A)$ is $(n-1)$-connected, that is $\pi_q(M, M\backslash A) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$. In particular we get that $H_q(M, M\backslash A) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$ and the natural group homomorphism $\chi_n : \pi_n(M, M\backslash A) \to H_n(M, M\backslash A)$ is surjective. On the other hand the inclusion $i_{M\backslash A} : M\backslash A \hookrightarrow M$ is $(n-1)$-connected, that is the induced group homomorphism $\pi_q(i_{M\backslash A}) : \pi_q(M\backslash A) \to \pi_q(M)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism $\chi_n$ is an isomorphism if $M$ is simply connected and $n \geq 3$.

**Corollary 1.5** *Let $p : E \to M$ be a fibration whose base space $M$ is an $n$-dimensional differentiable manifold and $A$ be a closed countable subset of $M$. If the total space $E$ is simply connected, $H_n(E) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq 0 \simeq \pi_{n-1}(E) \otimes_{\mathbf{Z}} \mathbf{Q}$, then the group homomorphism $h_{n-1} \otimes id_{\mathbf{Q}} : \pi_{n-1}(E\backslash p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q} \to H_{n-1}(E\backslash p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$, which is actually $\mathbf{Q}$-linear, is injective, where $h_{n-1}$ is the natural group homomorphism.*

*Proof.* Indeed, the following ladder with exact rows and commutative rectangles:

$$\begin{array}{ccccccccc}
\pi_n(E) & \to & \pi_n(E, E\backslash p^{-1}(A)) & \to & \pi_{n-1}(E\backslash p^{-1}(A)) & \to & \pi_{n-1}(E) & \to & \pi_{n-1}(E, E\backslash p^{-1}(A)) \\
\downarrow h_n^E & & \downarrow \chi_n & & \downarrow h_{n-1} & & \downarrow h_{n-1}^E & & \downarrow \chi_{n-1} \\
H_n(E) & \to & H_n(E, E\backslash p^{-1}(A)) & \to & H_{n-1}(E\backslash p^{-1}(A)) & \to & H_{n-1}(E) & \to & H_{n-1}(E, E\backslash p^{-1}(A))
\end{array}$$

leads us, simply by tensorizing with $\mathbf{Q}$ on the right, to the new one

$$\begin{array}{ccccccccc}
\pi_n(E)\otimes\mathbf{Q} & \to & \pi_n(E, E\backslash p^{-1}(A))\otimes\mathbf{Q} & \to & \pi_{n-1}(E\backslash p^{-1}(A))\otimes\mathbf{Q} & \to & \pi_{n-1}(E)\otimes\mathbf{Q} & \to & \pi_{n-1}(E, E\backslash p^{-1}(A))\otimes\mathbf{Q} \\
\downarrow h_n^E\otimes id_{\mathbf{Q}} & & \downarrow \chi_n\otimes id_{\mathbf{Q}} & & \downarrow h_{n-1}\otimes id_{\mathbf{Q}} & & \downarrow h_{n-1}^E\otimes id_{\mathbf{Q}} & & \downarrow \chi_{n-1}\otimes id_{\mathbf{Q}} \\
H_n(E)\otimes\mathbf{Q} & \to & H_n(E, E\backslash p^{-1}(A))\otimes\mathbf{Q} & \to & H_{n-1}(E\backslash p^{-1}(A))\otimes\mathbf{Q} & \to & H_{n-1}(E)\otimes\mathbf{Q} & \to & H_{n-1}(E, E\backslash p^{-1}(A))\otimes\mathbf{Q}
\end{array}$$

with exact rows and commutative rectangles as well. The exactness of the rows follows by using the universal-coefficient theorem [7, Theorem 5.2.14]. In the above diagram we have ommitted, for simplicity, the notation $\otimes_{\mathbf{Z}}$. Because $H_n(E)\otimes_{\mathbf{Z}}\mathbf{Q} \simeq 0 \simeq \pi_{n-1}(E)\otimes_{\mathbf{Z}}\mathbf{Q}$, it follows that $h_n^E\otimes id_{\mathbf{Q}}$ is surjective and $h_{n-1}^E\otimes id_{\mathbf{Q}}$ is injective. On the other hand $\chi_n\otimes id_{\mathbf{Q}}$ is, according to corollary 1.3, an isomorphism such that, by combining all these facts with the fifth Lemma in [4, pp. 46], one can conclude that $h_{n-1}\otimes id_{\mathbf{Q}}$ is indeed injective.□

**Remark 1.6** Let $M$ be an $n$-dimensional differentiable manifold and $A$ be a closed countable subset of $M$. If the natural group homomorphism $h_{n-1}^M : \pi_{n-1}(M) \to H_{n-1}(M)$ is surjective, then the natural group homomorphism $h_{n-1}^{M\backslash A} : \pi_{n-1}(M\backslash A) \to H_{n-1}(M\backslash A)$ is also surjective and the $\mathbf{Q}$-linear mapping $h_{n-1}^{M\backslash A}\otimes id_{\mathbf{Q}} : \pi_{n-1}(M\backslash A)\otimes_{\mathbf{Z}}\mathbf{Q} \to H_{n-1}(M\backslash A)\otimes_{\mathbf{Z}}\mathbf{Q}$ is surjective as well. In particular, according to corollary 1.2, $\pi_{n-1}(M\backslash A)\otimes_{\mathbf{Z}}\mathbf{Q}$ is not finite dimensional.

# 2 Application to critical points

In this section we will show that, under some topological conditions on the total space, on the base spaces and on the fiber of a fibration, the collection of fibers over a closed countable subset of the base space is not critical with respect to certain class of special real functions.

Let $M$, $N$ be differentiable manifolds and $f : M \to N$ be a differentiable mapping. Denote by $C(f)$ its critical set and by $R(f)$ its set of regular points, while the set of its critical values $f(C(f))$ will be denoted by $B(f)$.

**Definition 2.1** We say that a differentiable mapping $f \in C^\infty(M,N)$ *separates the critical values by the regular ones* if $B(f)\cap f(R(f)) = \emptyset$. Denote by $CS^\infty(M,N)$ the set of all of these mappings. A closed subset $C$ of $M$ is said to be $CS^\infty(M,N)$-*(properly) critical* if $C(f) = C$ for some (proper) mapping $f \in CS^\infty(M,N)$.

**Remark 2.2** Let $p : \tilde{N} \to N$ be a covering mapping and $\tilde{f} : M \to \tilde{N}$ be a differentiable mapping. If $\tilde{f} \notin CS^\infty(M,\tilde{N})$, then $p\circ\tilde{f} \notin CS^\infty(M,N)$. Therefore if $g \in CS^\infty(M,N)$ and $\tilde{g} \in C^\infty(M,\tilde{N})$ is a lifting of $g$, then $\tilde{g} \in CS^\infty(M,\tilde{N})$.

Indeed, since $\tilde{f} \notin CS^\infty(M,\tilde{N})$, it follows that there exist $x_0 \in C(f)$, $x_1 \in R(f)$ such that $\tilde{f}(x_0) = \tilde{f}(x_1)$. Because $p$ is a local diffeomorphism, it implies that $x_0 \in$

$C(p\circ\tilde{f})$, $x_1 \in R(p\circ\tilde{f})$ and obviously $(p\circ\tilde{f})(x_0) = (p\circ\tilde{f})(x_1) \in B(p\circ\tilde{f})\cap(p\circ\tilde{f})(R(p\circ\tilde{f}))$, that is $p\circ\tilde{f} \notin CS^\infty(M,N)$.

The next Proposition can be easily proved and it actually appears in [5].

**Proposition 2.3** *(i)* $f \in CS^\infty(M,N)$ *iff* $C(f) = f^{-1}(B(f))$.

*(ii) If* $M$ *is a connected differentiable manifold and* $f \in CS^\infty(M,\mathbf{R})$ *is such that* $R(f) = M\backslash C(f)$ *is also connected, then* $f(R(f)) = (m_f, M_f)$*, where* $m_f = \inf\limits_{x\in M} f(x), M_f = \sup\limits_{x\in M} f(x)$ *and* $B(f) \subseteq \{m_f, M_f\}\cap\mathbf{R}$*. Moreover, if* $M$ *is compact, then* $m_f, M_f \in \mathbf{R}$ *and* $B(f) = \{m_f, M_f\}$.

**Theorem 2.4** *Let* $F \hookrightarrow E \xrightarrow{p} M^n$ *be a differential fibration with compact total space and commutative fundamental group of the fiber* $F$*. If* $A$ *is a closed countable subset of* $M$*,* $n \geq 3$*,* $E$ *is simply connected and* $H_n(E)$*,* $H_{n-1}(M)$*,* $\pi_{n-1}(E)\otimes_{\mathbf{Z}}\mathbf{Q}$ *are trivial and* $\pi_{n-2}(F)$ *is finitely generated, then* $p^{-1}(A)$ *is neither* $CS^\infty(E,\mathbf{R})$*-critical nor* $CS^\infty(E,S^1)$*-critical.*

*Proof.* Assume that there exists a mapping $f \in CS^\infty(E,\mathbf{R})$ such that $C(f) = p^{-1}(A)$. This means that $B(f) = \{m_f, M_f\}$ and that its restriction

$$E\backslash C(f) \to Imf\backslash B(f) = (m_f, M_f),\ p \mapsto f(p)$$

is a proper submersion, that is, via Ehresmann's theorem, a locally trivial fibration whose compact fiber we are denoting by $\mathcal{F}$. Its base space $(m_f, M_f)$ being contractible, it follows that the inclusion $i_{\mathcal{F}} : \mathcal{F} \hookrightarrow E\backslash C(f)$ is a weak homotopy equivalence, namely the induced group homomorphisms $\pi_q(i_{\mathcal{F}}) : \pi_q(\mathcal{F}) \to \pi_q(E\backslash C(f))$ are all isomorphisms. Consequently, using the Whitehead theorem [3, pp. 167] or [7, pp. 399], it follows that the induced group homomorphisms $H_q(i_{\mathcal{F}}) : H_q(\mathcal{F}) \to H_q(E\backslash C(f)) = H_q(E\backslash p^{-1}(A))$ are also isomorphisms. Consequently $H_q(i_{\mathcal{F}})\otimes id_{\mathbf{Q}} : H_q(\mathcal{F})\otimes_{\mathbf{Z}}\mathbf{Q} \to H_q(E\backslash C(f))\otimes_{\mathbf{Z}}\mathbf{Q} = H_q(E\backslash p^{-1}(A))\otimes_{\mathbf{Z}}\mathbf{Q}$ is a $\mathbf{Q}$-linear isomorphism. Therefore $H_q(E\backslash p^{-1}(A))\otimes_{\mathbf{Z}}\mathbf{Q}$ is finite dimensional for all $q$ since $H_q(\mathcal{F})$ is finitely generated as homology group of a compact manifold. The hypothesis $H_{n-1}(M) \simeq 0$ ensures us that $h^M_{n-1}\otimes id_{\mathbf{Q}}$ is obviously surjective and implicitely that $\pi_{n-1}(M\backslash A)\otimes_{\mathbf{Z}}\mathbf{Q}$ is not finite dimensional while the hypothesis $H_n(E) \simeq 0$ and the triviality of $\pi_{n-1}(E)\otimes_{\mathbf{Z}}\mathbf{Q}$ ensure us that the $\mathbf{Q}$-linear mapping $h_{n-1}\otimes id_{\mathbf{Q}} : \pi_{n-1}(E\backslash p^{-1}(A))\otimes_{\mathbf{Z}}\mathbf{Q} \to H_{n-1}(E\backslash p^{-1}(A))\otimes_{\mathbf{Z}}\mathbf{Q}$ is injective. Consequently $\pi_{n-1}(E\backslash p^{-1}(A))\otimes_{\mathbf{Z}}\mathbf{Q}$ is a finite dimensional $\mathbf{Q}$-vector space as a subspace of the finite dimensional $\mathbf{Q}$-vector space $H_{n-1}(E\backslash p^{-1}(A))\otimes_{\mathbf{Z}}\mathbf{Q}$.

The exact homotopy sequence of the fibration $F \hookrightarrow E\backslash p^{-1}(A) \xrightarrow{p} M\backslash A$

$$\cdots \to \pi_{n-1}\big(E\backslash p^{-1}(A)\big) \longrightarrow \pi_{n-1}(M\backslash A) \xrightarrow{\partial_{n-1}} \pi_{n-2}(F) \to \cdots$$

produces the exact sequence

$$\cdots \to \pi_{n-1}\big(E\backslash p^{-1}(A)\big)\otimes_{\mathbf{Z}}\mathbf{Q} \longrightarrow \pi_{n-1}(M\backslash A)\otimes_{\mathbf{Z}}\mathbf{Q} \longrightarrow \pi_{n-2}(F)\otimes_{\mathbf{Z}}\mathbf{Q} \to \cdots$$

simply by tensoring with $\mathbf{Q}$. Its exactness follows by using the universal-coefficient theorem. But since $\pi_{n-1}(E\backslash p^{-1}(A))\otimes_{\mathbf{Z}}\mathbf{Q}$ and $\pi_{n-2}(F)\otimes_{\mathbf{Z}}\mathbf{Q}$ are finite dimensional they force $\pi_{n-1}\big(E\backslash p^{-1}(A)\big)\otimes_{\mathbf{Z}}$ to be finite dimensional too, not being the case as follows from remark 1.6. In order to prove the $CS^\infty(M,S^1)$-non-criticality of $p^{-1}(A)$ we assume that there exists a mapping $f \in CS^\infty(M,S^1)$ such that $C(f)=p^{-1}(A)$. Consider a lifting $\tilde{f}: M \to \mathbf{R}$ with respect to the covering mapping $exp: \mathbf{R} \to S^1$, recall that $\tilde{f} \in CS^\infty(M,\mathbf{R})$ an observe that $C(\tilde{f}) = C(f) = p^{-1}(A)$, such that we have got a contradiction with the $CS^\infty(M,\mathbf{R})$-non-criticality of $p^{-1}(A)$.□

**Example 2.5.** Consider the integers $n \geq 2$ and $d \geq 1$ and the *Brieskorn manifolds* $W_d^{2n-1}$ as the $(2n-1)$-dimensional real algebraic submanifolds of $\mathbf{C}^{n+1}$ defined by the equations

$$z_0^d + z_1^2 + \cdots + z_n^2 = 0 \text{ and } |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1.$$

The topology of Brieskorn manifolds is mostly known it being simply connected for $n \geq 3$ and a homotopy sphere for both $n \geq 3$, $d \geq 1$ odd. On the other hand, if $n = 2m$ is even, then $W_d^{4m-1}$ is a rational homology sphere whose only nontrivial integral homology groups are given by $H_{2m-1}(W_d^{4m-1}) \simeq \mathbf{Z}_d$ and $H_0(W_d^{4m-1}) \simeq H_{4m-1}(W_d^{4m-1}) \simeq \mathbf{Z}$, [1, Corollary 5.3, pp. 275], [6].

All manifolds $W_d^{2n-1}$ are invariant under the standard linear action of $O(n)$ on the $(z_1,\ldots,z_n)$-coordinates. If $n=2m$ is even there is a free circle action on $W_d^{4m-1}$ given by the action of the circle group $S^1 = Z(U(m)) \subset O(2m)$ where $Z$ denotes the center. Moreover if $n = 4m$, then $Sp(1)$ realised as subgroup of $O(4m)$ by the scalar multiplication on $\mathbf{R}^{4m} \simeq \mathbf{H}^m$, acts also freely on $W_d^{8m-1}$. The quotient manifolds $N_d^{4m-2} := W_d^{4m-1}/S^1$ and $\tilde{N}_d^{8m-4} := W_d^{8m-1}/Sp(1)$ are simply connected and their integral cohomology groups are given by

$$H^k\big(N_d^{4m-2}\big) \simeq H^k\big(\mathbf{CP}^{2m-1}\big),\ H^k\big(\tilde{N}_d^{8m-4}\big) \simeq H^k\big(\mathbf{HP}^{2m-1}\big), \text{ see [6].}$$

Therefore if $F \hookrightarrow E \xrightarrow{p} M^n$ is one of the fibrations

$$S^1 \hookrightarrow W_d^{4m-1} \to N_d^{4m-2},\ Sp(1) \hookrightarrow W_d^{8m-1} \to \tilde{N}_d^{8m-4},$$

then it satisfies the conditions of theorem 2.4. Consequently for a closed countable subset $A$ of the base space, $p^{-1}(A)$ is neither $CS^\infty(E,\mathbf{R})$-critical nor $CS^\infty(E,S^1)$-critical. Indeed we have successively

$$H_{4m-2}(W_d^{4m-1}) \simeq 0 \simeq H^1\big(\mathbf{CP}^{2m-1}\big) \simeq H^1\big(N_d^{4m-2}\big) \overset{*}{\simeq} H_{4m-3}\big(N_d^{4m-2}\big),$$

$$H_{8m-4}(W_d^{8m-1}) \simeq 0 \simeq H^1\big(\mathbf{HP}^{2m-1}\big) \simeq H^1\big(\tilde{N}_d^{8m-4}\big) \overset{**}{\simeq} H_{8m-5}\big(\tilde{N}_d^{8m-4}\big),$$

the last isomorphisms $(*)$ and $(**)$ being ensured by the Poincaré duality. Finaly, the triviality of $\pi_{4m-3}\big(W_d^{4m-1}\big) \otimes_{\mathbf{Z}} \mathbf{Q}$ and of $\pi_{8m-5}\big(W_d^{8m-1}\big) \otimes_{\mathbf{Z}} \mathbf{Q}$ follows from [2, pp. 123], while the triviality of $\pi_{4m-4}(S^1) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\pi_{8m-6}(Sp(1)) \otimes_{\mathbf{Z}} \mathbf{Q} = \pi_{8m-6}(S^3) \otimes_{\mathbf{Z}} \mathbf{Q}$ is obvious since $\pi_{4m-4}(S^1) \simeq 0$ and $\pi_{8m-6}(S^3)$ is finite [3, 318].

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